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The first Darboux problem for second order hyperbolic equations with memory

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Abstract. We study the first Darboux problem for hyperbolic equations of second order with memory and consider the solvability of this problem.

1. Statement of the problem

In the plane of the independent variables x and t, we consider the linear hyperbolic equation with memory of the form

$$Lu := u_{tt} - u_{xx} + (\ln\sigma)'(x)\,u_x - b(x,t)\frac{\rho_l(x)}{\rho_s(x)}u_t + b^2(x,t)\frac{\rho_l(x)}{\rho_s(x)}u - b^2(x,t)\frac{\rho_l(x)}{\rho_s(x)}\int_0^t \exp\Big(-\int_s^t b(x,y)\,dy\Big)b(x,s)\,u(x,s)\,ds = f(x,t). \qquad (1)$$

Here u is a desired component of the velocity vector of the particle displacement of the elastic porous body with the partial density $\rho_s(z)$, $\sigma(x) = \sqrt{\mu(x)\,\rho_s(x)}$, $\mu(x)$ and $b(x,t)$ are positive functions, and $f(x,t)$ is a given function. The velocity component of the liquid v with the partial density $\rho_l(z)$ is associated with the function u by the expression

$$v(x,t) = \int_0^t \exp\Big(-\int_s^t b(x,y)\,dy\Big)b(x,s)\,u(x,s)\,ds.$$

Equation (1) arises in poroelasticity theory [1–7].

Following [8], we introduce the triangular domain $D_T = \{(x,t):\ 0 < x < t,\ 0 < t < T\}$, $T < \infty$, bounded by the characteristic segment $\Gamma_{1,T} = \{x = t,\ 0 \le t \le T\}$ and the segments $\Gamma_{2,T} = \{x = 0,\ 0 \le t \le T\}$ and $\Gamma_{3,T} = \{t = T,\ 0 \le x \le T\}$.

We consider the first Darboux problem of finding the solution $u(x,t)$ of equation (1) in the domain D_T satisfying the boundary conditions (see, e. g., [9, p. 228]):

$$u|_{\Gamma_{i,T}} = 0, \quad i = 1,2. \qquad (2)$$

Definition 1 [8]. Let $\rho_s, \mu \in C^1[0,T]$, $\rho_l \in C[0,T]$, $b, f \in C(\overline{D}_T)$. A function $u \in C(\overline{D}_T)$ is called a *strong* generalized solution to problem (1), (2) of the class C in D_T if there exists a sequence of functions $u_n \in \widetilde{C}^2(\overline{D}_T, S_T)$, such that $u_n \to u$ and $Lu_n \to f$ in $C(\overline{D}_T)$ at $n \to \infty$, where $\widetilde{C}^2(\overline{D}_T, S_T) = \{u \in C^2(\overline{D}_T):\ u|_{S_T} = 0\}$, $S_T = \Gamma_{1,T} \cup \Gamma_{2,T}$.

2. Equivalent reduction of problem (1), (2) to the linear Volterra integral equation of the second kind

Let $P = (x,t)$ be an arbitrary point of the domain D_T. Denoted by $D_{x,t}$ a quadrangle with vertices at points $O = (0,0)$, P, and the points $P_1 = (0, t-x)$ and $P_3 = ((x+t)/2, (x+t)/2)$ lying respectively on $\Gamma_{2,T}$ and $\Gamma_{1,T}$. Obviously, the domain $D_{x,t}$ consists of the characteristic rectangle $D_{1;x,t} = PP_1P_2P_3$ and the triangle $D_{2;x,t} = OP_1P_2$, where $P_2 = ((t-x)/2, (t-x)/2)$.

Further assume that ρ_s, $\mu \in C^3[0,T]$, $\rho_l \in C^1[0,T]$, and $b \in C^1(\overline{D}_T)$. It is known that under these conditions, a well-defined Green–Hadamard function $G(x,t;x',t')$ exists, which is bounded and piecewise continuous with its partial derivatives up to the second order, and the discontinuity of the first kind appears only when passing through the singularity manifold $t' + x' - t + x = 0$ (see, e. g., [10; 11, p. 230; 12, p. 38]).

For the classic solution $u \in C^2(\overline{D}_T)$ of problem (1), (2) the following integral equation is valid:

$$u(x,t) - \int_{D_{x,t}} G(x',t';x,t)\, b^2(x',t') \frac{\rho_l(x')}{\rho_s(x')} \times \int_0^{t'} \exp\Big(-\int_s^{t'} b(x',y)\,dy\Big) b(x',s)\, u(x',s)\, ds\, dx'\, dt' = \int_{D_{x,t}} G(x',t';x,t) f(x',t')\, dx'\, dt', \quad (x,t) \in \overline{D}_T. \tag{3}$$

Let $u \in C(\overline{D}_T)$ be the solution of the integral Volterra equation of the second kind (3). Since the function f is continuous on $\overline{D}_T$, and the space $C^2(\overline{D}_T)$ is dense in $C(\overline{D}_T)$, there exists a sequence of functions $f_n \in C^2(\overline{D}_T)$ such that $f_n \to f$ in the space $C(\overline{D}_T)$ at $n \to \infty$. Similarly, since $u \in C(\overline{D}_T)$, there exists a sequence of functions $\tilde{u}_n \in C^2(\overline{D}_T)$ such that $\tilde{u}_n \to u$ in the space $C(\overline{D}_T)$ with $n \to \infty$.

Assuming

$$u_n := M_1 \tilde{u}_n + M_2 f_n, \quad n = 1, 2, \dots$$

Here M_1 and M_2 are linear operators acting according to the formulas

$$M_1 u := \int_{D_{x,t}} G(x',t';x,t)\, b^2(x',t') \frac{\rho_l(x')}{\rho_s(x')} \times \int_0^{t'} \exp\Big(-\int_s^{t'} b(x',y)\,dy\Big) b(x',s)\, u(x',s)\, ds\, dx'\, dt',$$

$$M_2\, u := \int_{D_{x,t}} G(x',t';x,t) u(x',t')\, dx'\, dt', \quad (x,t) \in \overline{D}_T.$$

It is easy to verify that $u_n \in \tilde{C}^2(\overline{D}_T, S_T)$, as M_1, M_2 are continuous linear operators acting in the space $C(\overline{D}_T)$, and

$$\lim_{n\to\infty} \|\tilde{u}_n - u\|_{C(\overline{D}_T)} = 0, \quad \lim_{n\to\infty} \|f_n - f\|_{C(\overline{D}_T)} = 0.$$

Hence we have $u_n \to M_1 u + M_2 f$ in the space $C(\overline{D}_T)$ at $n \to \infty$. However from equation (3) it follows that $M_1 u + M_2 f = u$. In this way, we have proved

Lemma 1. *The function $u_n \in C(\overline{D}_T)$ is the generalized solution of problem* (1), (2) *of the class C in the domain D_T if and only if it is a nonlinear continuous solution to integral equation* (3).

By the linearity and Volterra property we can prove an analogue to Lemma 2 [8]:

Lemma 2. *For the strong generalized solution of problem* (1), (2) *the class C in the domain DT a priori estimate holds:*

$$\|u\|_{C(\overline{D}_T)} \leq c \, \|f\|_{C(\overline{D}_T)}$$

with positive constants $c(T, \rho_l, \rho_s, \mu, b)$, independent of u and f.

Following [8], we introduce

Definition 2. Suppose that the coefficients $\rho_s(z)$, $\mu(x)$ are one time continuously differentiable functions, $\rho_l(x)$ is a continuous function at $[0, T]$, $b(x,t) \in C(\overline{D}_T)$. Let us say that problem (1), (2) is *globally solvable in the class of continuous functions* if for any finite $T > 0$ this problem has a strong generalized solution of the class C in D_T.

Equation (3) can be rewritten in the operator form

$$u = M_1 u + M_2 f. \tag{4}$$

Here the linear operator $M_1 : C(\overline{D}_T) \to C(\overline{D}_T)$ is continuous and compact. At the same time, according to Lemmas 1 and 2, for any parameter $s \in [0,1]$ and for any solution $u \in C(\overline{D}_T)$ of the operator equation

$$u = s M_1 u + M_2 f,$$

an a priori estimate

$$\|u\|_{C(\overline{D}_T)} \leq c \, \|f\|_{C(\overline{D}_T)}$$

takes place with a positive constant c not dependent on u, f, and s. Therefore, according to the Leray–Schauder theorem (see, e.g., [13, p. 375]) equation (4) under the conditions of Lemma 2 has at least one solution $u \in C(\overline{D}_T)$. Thus, applying Lemma 1 we have proved the following

Theorem. *Problem* (1), (2) *is globally solvable in the class of continuous functions in the sense of Definition 2, i.e. if* $f \in C(\overline{D}_T)$*, then, for any* $T > 0$*, problem* (1), (2) *has a strong generalized solution of the class* C *in* D_T*.*

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