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# Reconstruction of tsunami initial form via water level oscillation

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The paper is devoted to reconstruction of the movement of the ocean bottom when the water elevation is known at the finite set of points. For this ill-posed problem the technique using $r$-solutions is suggested; it is based on the singular value decomposition of the compact operator. The dependence of results on the number and disposition of receiving stations is studied in numerical experiments.

## 1. Statement of the problem

The paper deals with the problem of tsunami source reconstruction using water level records at the set of stations. The process of tsunami propagation is considered within the scope of the shallow water theory, when the water elevation $\eta(x, y; t)$ satisfies the following Cauchy problem for the scalar wave equation:

$$\frac{\partial^2 \eta}{\partial t^2} = \operatorname{div}\,(h(x,y)\,\operatorname{grad}\eta) + \frac{\partial^2 f}{\partial t^2}(t,x,y); \tag{1}$$

$$\eta|_{t=0} = \eta_t|_{t=0} = 0. \tag{2}$$

In equation (1), $h(x, y)$ is the known function that describes the depth of the ocean. The function

$$f(x,y;t) = \theta(t)\phi(x,y). \tag{3}$$

describes the movement of the ocean bottom (see, for example, [1]) and is supposed to be unknown except of the Heaviside function $\theta(t)$. The inverse problem is to recover the function $\phi(x, y)$ by input data being water level oscillation given at the set of the points (stations) $\mathcal{M} = \{M_j = (x_j, y_j), j = \overline{1, N}\}$:

$$\eta(x,y;t)|_{M_j} = \eta_j(t), \quad 0 \le t \le T. \tag{4}$$

The function $\phi(x, y)$ is supposed to be supported over rectangle $\Pi = \{(x, y) : 0 \le x \le X;\ 0 \le y \le Y\}$ and being from $L_2(\Pi)$.

The inverse problem (1), (4) can be formulated as the problem of resolution of the linear operator equation of the first kind:

$$\mathcal{A}\langle\phi(x,y)\rangle = U(t) = \begin{pmatrix} \eta_1(t) \\ \eta_2(t) \\ \vdots \\ \eta_N(t). \end{pmatrix}. \tag{5}$$

Its solution will be searched for in the least-squares formulation[1]:

$$\phi_*(x,y) = \arg\min \|\mathcal{A}\langle\phi(x,y)\rangle - U(t)\|_{L_2(\mathcal{M}\times(0,T))}.$$

The linear operator $\mathcal{A} : L_2(\Pi) \to L_2(\mathcal{M}\times(0,T))$ is defined by the following way: for each given $\phi(x,y)$ we should resolve the Cauchy problem (1) and take patterns of its solution at the points from the set $\mathcal{M}$.

The "data space" is defined as follows:

$$L_2(\mathcal{M}\times(0,T)) = \left\{U(t)\} : \|U(t)\| = \left(\sum_{j=1}^{N}\int_0^T \eta_j^2(t)\right)^{1/2} < \infty\right\}$$

and can be treated as the Hilbert space with the scalar product

$$(U_1, U_2) = \sum_{j=1}^{N}\int_0^T \eta_{1,j}(t)\eta_{2,j}(t)\,dt.$$

The "model space" is usual $L_2(\Pi)$.

By means of the standard technique of integral inequalities (see [2]) we can conclude that the operator $\mathcal{A} : L_2(\Pi) \to L_2(\mathcal{M})$ is the compact one and, so, does not possess bounded inverse. It follows that every attempt to resolve equation (5) numerically should be followed by some regularization procedure. In the paper, this regularization is performed by means of truncated singular value decomposition (SVD) that leads to the notion of $r$-solution (see [3]). Shortly the notion of $r$-solution can be described as follows.

Let us consider the linear operator equation (5) with the compact operator. Each compact operator possesses the singular system $\{s_j, u_j, v_j\}$, i.e. singular values $s_j \geq 0$ $(s_1 \geq s_2 \geq \ldots \geq s_j \geq \ldots)$ and left $(v_j)$ and right $(u_j)$ singular vectors:

$$\mathcal{A}u_j = s_j v_j, \qquad \mathcal{A}^* v_j = s_j u_j.$$

The very important property of singular vectors is that they form bases in the model and data spaces, that is each functions $\phi(x,y) \in L_2(\Pi)$ and $U(t) \in L_2(\mathcal{M}\times(0,T))$ can be presented as the Fourier series

[1]The most significant reason to search for the solution to (5) in least-squares formulation is possibility to deal forever with consistent system of linear algebraic equations

$$\phi(x,y)=\sum_{j=1}^{\infty}\phi_j u_j(x,y), \qquad U(t)=\sum_{j=1}^{\infty}U_j v_j,$$

with $\phi_j=(\phi(x,y),u_j(x,y))$, $U_j=(U(t),v_j(t))$. Taking into account these properties we can rewrite equation (5) in the "diagonal" form:

$$\mathcal{A}\Big(\sum_{j=1}^{\infty}\phi_j u_j\Big)\equiv\sum_{j=1}^{\infty}s_j\phi_j v_j=\sum_{j=1}^{\infty}(U,v_j)v_j\equiv\sum_{j=1}^{\infty}U_j v_j,$$

or, finally,

$$\phi(x,y)=\sum_{j=1}^{\infty}\frac{(U,v_j)}{s_j}u_j(x,y). \tag{6}$$

The solution given by (6) is nothing else but the "normal general" solution and operator, given by the right-hand side of (6), is the normal general pseudoinverse for $\mathcal{A}$ (see [4]).

As we can see from (6) the ill-poseness of the operator equation of the first kind with the compact operator is due to the fact that $s_j\to 0$ with $j\to\infty$, so, we can perturb the right-hand side $U(t)$ in such a way that some its vanishing perturbation $\varepsilon(t)$ can lead to rather large perturbation of the solution[2]. It should be noted that operator perturbation also leads to the solution instability.

The regularization procedure based on truncated SVD leads to the notion of $r$-solution given by the relation

$$\phi_r(x,y)=\sum_{j=1}^{r}\frac{(U,v_j)}{s_j}u_j(x,y). \tag{7}$$

This truncated series is stable for each fixed parameter $r$ with respect to perturbations of the right-hand side and operator itself (see [3]).

Any numerical method to resolve (5) should prevent its finite–dimensional approximation. The usual way to do this is the projective methods. Let us suppose $\{\psi_j(x,y)\}$ and $\{e_k(t)\}$ to be bases in the model and data spaces respectively. As the operator $\mathcal{A}$ is the compact one, it possesses matrix presentation and the operator equation (5) can be rewritten as an infinite system of linear algebraic equations

$$\sum_{j=1}^{\infty}(\mathcal{A}\psi_j,e_k)\,(\phi,\psi_j)=(U,e_k)\,,\quad k=\overline{1,\infty},$$

with respect to unknown coefficients $(\phi,\psi_j)$ with the matrix

[2]For example, $\varepsilon(t)=\varepsilon_j U_j(t)$, with $\varepsilon_j\to 0$ for $j\to\infty$ in such a way that $s_j/\varepsilon_j\to 0$.

$$A = \begin{pmatrix} (\mathcal{A}\psi_1, e_1) & (\mathcal{A}\psi_2, e_1) & \dots & (\mathcal{A}\psi_N, e_1) & \dots \\ (\mathcal{A}\psi_1, e_2) & (\mathcal{A}\psi_2, e_2) & \dots & (\mathcal{A}\psi_N, e_2) & \dots \\ \vdots & \vdots & & \vdots & \\ (\mathcal{A}\psi_1, e_M) & (\mathcal{A}\psi_2, e_M) & \dots & (\mathcal{A}\psi_N, e_M) & \dots \\ \vdots & \vdots & & \vdots & \end{pmatrix}.$$

Obviously, numerically we can resolve only its finite–dimensional subsystem with $N \times M$ submatrix. As the operator $\mathcal{A}$ is the compact one, its every finite–dimensional approximation by $N \times M$ matrix will converge to the operator itself for $N, M \to \infty$. So, we should search for $r$-solution to finite-dimensional system of linear algebraic equations. Its convergence to the $r$-solution of operator equation is carefully investigated in [3].

## 2. Numerical experiments: description and discussion

### 2.1. Finite–dimensional approximation

The main goals of presented below numerical experiments were to analyze the influence of the observation system on the quality of the recovering of the tsunami initial form. In order to avoid influence of other factors we supposed that ocean depth is constant: $h(x, y) = h_0$, so the tsunami wave propagation velocity is constant as well and is equal to $c_0 = \sqrt{gh_0}$. Next, after the Fourier transformation with respect to time, we have the following inverse problems: to recover the function $\phi(x, y) \in L_2(\Pi)$ by the data

$$\hat{\eta}(x, y; \omega)|_{M_j} = V_j(\omega), \quad \omega_1 \le \omega \le \omega_2, \tag{8}$$

with $\hat{\eta}(x, y; \omega)$ being a solution to the Helmholtz equation

$$\Delta\hat{\eta} + \frac{\omega^2}{h_0}\hat{\eta} = \frac{i\omega}{h_0}\phi(x, y); \tag{9}$$

satisfying the Sommerfeld radiation condition.

Solution to equation (9) can be presented as

$$\hat{\eta}(x, y, \omega) = -\frac{i\omega}{h_0}\int\limits_{\Pi} \phi(\xi, \zeta) H_0^{(1)}\left(\frac{\omega}{c_0}\sqrt{(x-\xi)^2 + (y-\zeta)^2}\right) d\xi d\zeta, \tag{10}$$

and inverse problem is to recover $\phi(x, y)$ from the equations

$$-\frac{i\omega}{h_0}\int\limits_{\Pi} \phi(\xi, \zeta) H_0^{(1)}\left(\frac{\omega}{c_0}\sqrt{(x_j-\xi)^2+(y_j-\zeta)^2}\right) d\xi d\zeta = V_j(\omega), \ j = \overline{1, N}, \tag{11}$$

where $N$ is the number of stations.

In order to obtain the system of linear algebraic equations by means of the projective method, in the model space was chosen the trigonometric basis, i. e. the unknown function $\phi(x,y)$ was searched for as

$$\phi(x,y) = \sum_{m=1}^{N_x} \sum_{n=1}^{N_y} a_{mn} \sin \frac{m\pi}{X} x \sin \frac{n\pi}{Y} y \tag{12}$$

while in the data space with respect to frequency $\omega$ the steps were chosen for the basis. This leads to the following system of linear algebraic equations with respect to the coefficients $a_{mn}$:

$$\sum_{m=1}^{N_x} \sum_{n=1}^{N_y} a_{mn} \int_{\Pi} H_0^{(1)} \left( \frac{\omega_j}{c_0} \sqrt{(x_k - \xi)^2 + (y_i - \zeta)^2} \right) \sin \frac{m\pi}{X} \xi \sin \frac{n\pi}{Y} \zeta \, d\xi d\zeta$$
$$= -\frac{i}{\omega_j} V_k(\omega_j), \quad j = \overline{1,M}, \quad k = \overline{1,N}. \tag{13}$$

It is necessary to pay attention to that $a_{mn}$ in (13) here is a solution to be searched for, while matrix of the system should be computed via numerical integration of integrals with Hankel's function. In order to compute these integrals the uniform grid was introduced over the rectangle $\Pi$ and within each elementary rectangle Hankel's function was bilinearly approximated and explicit integration formulas next applied.

## 2.2. Discussion of numerical results

Synthetic data for numerical experiments presented below were computed for the function

$$\phi(x,y) = \begin{cases} 1 - \dfrac{(x-x_0)^2}{R_x^2} + \dfrac{(y-y_0)^2}{R_y^2}, & \text{if } \dfrac{(x-x_0)^2}{R_1^2} + \dfrac{(y-y_0)^2}{R_2^2} < 1 \\ 0, & \text{if } \dfrac{(x-x_0)^2}{R_1^2} + \dfrac{(y-y_0)^2}{R_2^2} \geq 1. \end{cases}$$

with $R_x = 50\,\text{km}$, $R_y = 25\,\text{km}$. Next this function was searched for over the rectangle $\Pi = \{(x,y) : -100 \leq x \leq 100\,\text{km},\ 100 \leq y \leq 200\,\text{km}\}$ as the series (12) with $N_x = N_y = 21$. Over this rectangle the uniform grid $201 \times 101$ was introduced in order to calculate integrals in 0 (13). The time frequencies were taken within from 0.001 Hz up to 0.01 Hz. Forever the parameter $r$ in (7) was chosen in order to provide the inequality $s_r/s_1 \leq 10^{-6}$.

The first series of numerical experiments dealt with 3 stations placed onto the circle with radius 150 km centered at the ellipse center. These stations were placed uniformly with respect to the central point for a range of angles $(-\alpha, \alpha)$. Some results are shown in Figure 1.

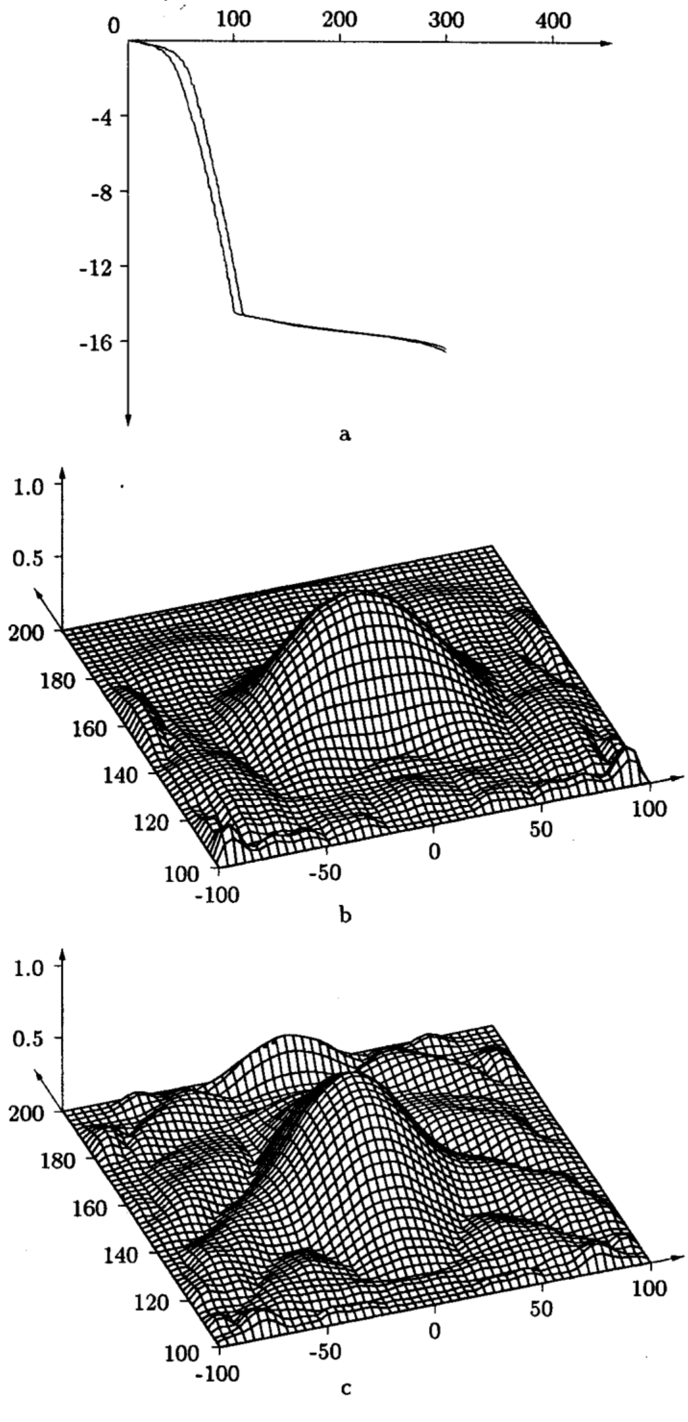



**Figure 1.** 3 stations: a) singular values (log scale). Recovered initial form: b) for $\alpha = \frac{\pi}{10}$, max $= 0.773$; c) for $\alpha = \frac{\pi}{2}$, max $= 0.772$

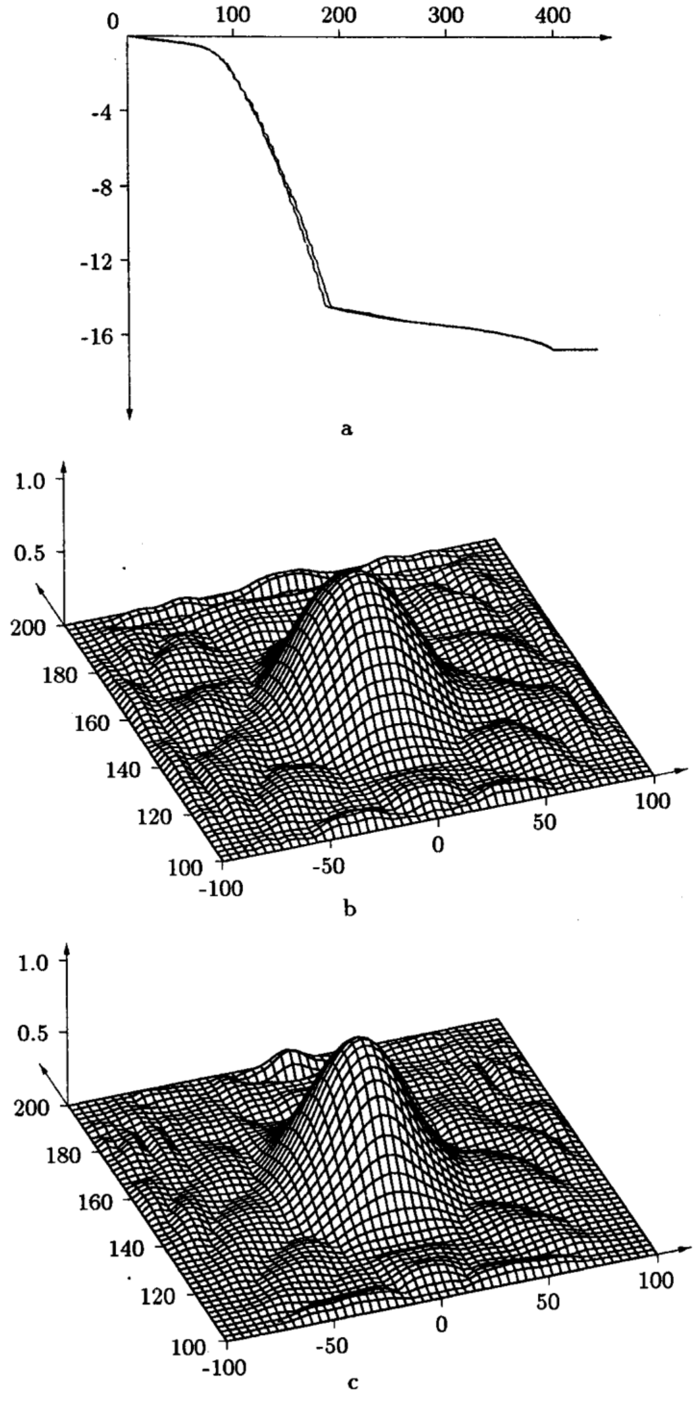
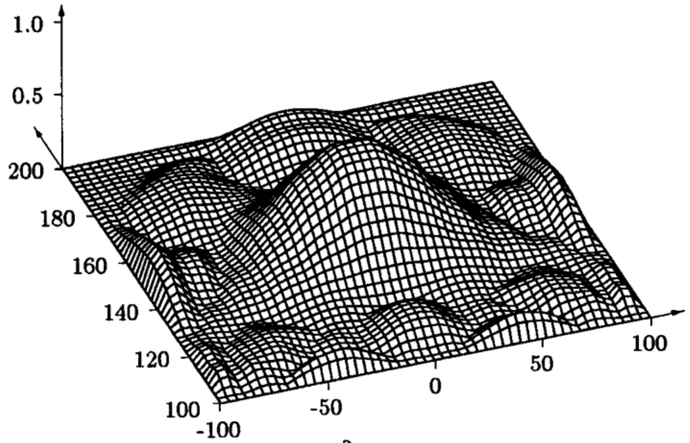
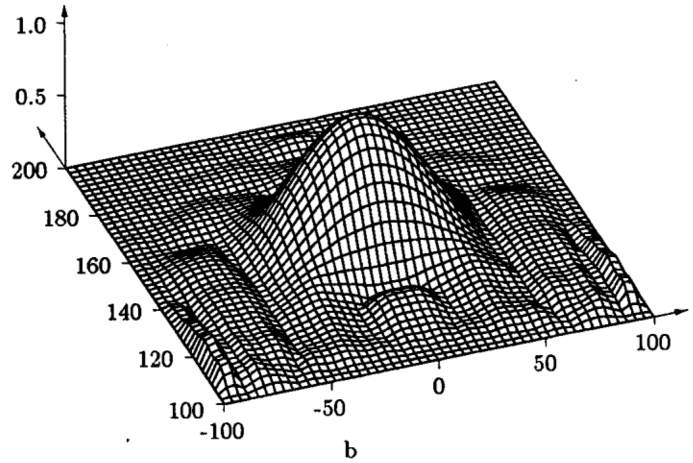


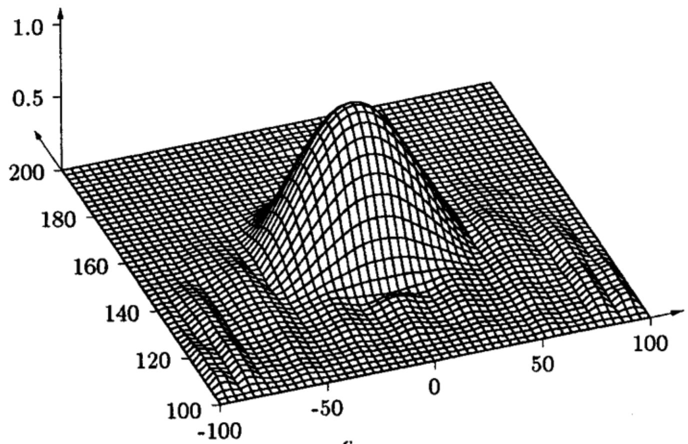

**Figure 2.** 5 stations: a) singular values (log scale). Recovered initial form: b) for $\alpha = \frac{\pi}{2}$, max $= 0.867$; c) for $\alpha = \pi$, max $= 0.976$




a




b




c

**Figure 3.** Stations on the straight line: a) 3 stations, max = 0.669; b) 5 stations, max = 0.864; c) 7 stations, max = 0.960

In the second series, the number of stations was 5; they were placed onto the half-circle ($\alpha = \pi/2$) and the entire circle ($\alpha = \pi$). The results are in Figure 2.

The third series (Figure 3) dealt with 3, 5 or 7 stations placed on the segment ($-100 \leq x \leq 100$) of the straight line $\{y = 0\}$.

Since the maximum of the true function $\phi(x, y)$ equals to 1.0, the maximum of every approximate solution is written under the figures. The difference between them shows the degree of accuracy.

After the experiments we can make the following conclusions:

1. Using the $r$-solutions we get an effective instrument for solving this ill–posed problem.
2. The distance from the stations to the tsunami source does not have any visible influence on the quality of reconstruction. This quality depends mainly on the range of angles of observation. The best range is the entire circle.
3. The quality of a solution is improving when a number of stations increase up to 10–15. The further increase is useless.
4. Since the calculations were really made in the time–spectral domain, we can conclude that for solution being satisfactory the shortest wavelength should be less than a half of a characteristic size of the tsunami source. The number of frequencies used in calculations should not be large; it is enough to use about 15 frequencies.

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