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On the condition number of interpolation and collocation method for Symm's equation for some quasiuniform grids*

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The paper contains the proof of theorem estimating the condition number of matrix after discretization of the integral equation of the first kind with logarithmic singularity on the closed curve by the collocation method with piecewise-linear approximation of unknown function. It is based on a new approach using convex properties of images of the basic functions, so it gives the opportunity to study a quasi-uniform sequence of grids. If the initial integral equation is uniquely solvable, the quasiuniform parameter does not exceed 4.25 and the step is sufficiently small, then the norm of finite-dimensional inverse operator is not greater than a constant divided by the step.

1. Introduction

This article is devoted to theoretical foundation of a direct numerical method for the integral equation of the 1-st kind, known as "Symm's equation". Equations of this type are met very often in boundary equations methods for elliptic problems; particularly, in diffraction problems for elastic and electromagnetic wavefields ([1]). Saying "direct" we mean a method based on a direct discretization of a given equation without any procedures of explicit regularization. In this paper we consider the method using the piecewise-linear interpolation of unknown function and the collocation conditions.

Previous works on the subject ([2–5]) concern the case of uniform grids only; that approach used decomposition into trigonometric functions which are eigenfunctions both of integral and of finite-dimensional operator. But it is true for the uniform grid only. Later some properties of special matrices were used ([5–6]). Now we develop the approach using the convex properties of the images of basic functions. As we hope, this approach is mostly universal and can be generalized onto arbitrary quasiuniform grids and multi-dimensional equations.

In this paper the letter "c" (with or without subscripts) will denote *different* positive constants whose precise value is of no importance. The

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statements of the form "the inequality $\varphi(t) < c\psi(t)$ is valid" will automatically mean that there exists such a positive constant c that this inequality is true, and this constant is independent on the step of the grid (but it may depend on the quasiuniform parameter and other parameters of the problem).

2. Symm's equation and interpolation and collocation method

Let us suppose that after the parametrization Symm's equation defined initially on a smooth closed curve may be written in the form:

$$\int_0^1 [K(t,\tau) + K_1(t,\tau)]f(\tau)\,d\tau = g(t), \quad t \in [0,1], \tag{2.1}$$

where all functions are periodic with period 1, and the kernel consists of a "standard" logarithmic part $K(t,\tau)$ and the "junior" term $K_1(t,\tau)$ without the singularity. As it will be clear lately, we can separate the logarithmic part by different ways, so let us assume that the first term of the kernel is of the form:

$$K(t,\tau) \equiv \log \frac{1}{|\tau - t|'},$$

where

$$|\tau - t|' = \begin{cases} |\tau - t|, & \text{if } -1/2 \le \tau - t \le 1/2, \\ |\tau - t + 1|, & \text{if } \tau - t \le -1/2, \\ |\tau - t - 1|, & \text{if } 1/2 \le \tau - t. \end{cases} \tag{2.2}$$

So, $|\tau - t|'$ denote the quantity, that is equal to $|\tau - t|$, if $|\tau - t| < 1/2$ and periodically continued onto the case $|\tau - t| \le 1$.

Equation (2.1) will be called *general equation*. If we eliminate the junior part, we get *model equation*:

$$(\mathcal{K}f)(t) \equiv \int_0^1 \log \frac{1}{|\tau - t|'} f(\tau)\,d\tau = g(t), \quad t \in [0,1]. \tag{2.3}$$

The numerical method used now, is the following. Let us consider a grid, i.e., an ordered sample of points $\{\tau_j\}$, $j = 1, \dots, N$ on the segment [0,1]. Let $\{\varphi_j(\tau)\}$, $j = 1, \dots, N$ be the basic functions of piecewise-linear interpolation from this grid, that is

$$\varphi_j(\tau) = \begin{cases} (\tau - \tau_{j-1})/(\tau_j - \tau_{j-1}), & \text{if } \tau \in [\tau_{j-1}, \tau_j], \\ (\tau_{j+1} - \tau)/(\tau_{j+1} - \tau_j), & \text{if } \tau \in [\tau_j, \tau_{j+1}], \\ 0, & \text{elsewhere.} \end{cases} \tag{2.4}$$

The approximate solution to the arbitrary linear equation $(\mathcal{K}f)(t) = g(t)$ is presented in the form

$$f(\tau) = \sum_{j=1}^{N} a_j \varphi_j(\tau), \tag{2.5}$$

where the unknown coefficients a_j have to be found from the collocation conditions; the collocation points are chosen coinciding with the interpolation ones, therefore the conditions are:

$$\mathcal{K}f(\tau_k) = g(\tau_k), \qquad k = 1, \dots, N.$$

Since the operator is linear, these conditions are equivalent to the linear algebraic system with respect to $\{a_j\}$:

$$\sum_{j=1}^{N} a_j (\mathcal{K}\varphi_j)(\tau_k) = g(\tau_k), \qquad k = 1, \dots, N. \tag{2.6}$$

If the operator is not degenerate, the images of the basic functions are linearly independent. The main problem is to prove this property for their restrictions onto the grid and, what is desirable, to estimate the condition number of the matrix. We shall do this under the following conditions.

Let us suppose that there is a quasiuniform family of grids. If we denote by $h_{j+1/2}$ the steps of some grid $h_{j+1/2} = \tau_{j+1} - \tau_j$, and h is the minimum of them for the given grid, then let us require

$$\max_j h_{j+1/2} \le Qh,$$

where the *quasiuniform parameter* Q is the same for every grid of the family.

Since we consider the periodical case, the grid and the basic functions are suggested to be periodical with period 1.

We shall estimate from above the norm of finite-dimensional inverse operator of system (2.6), corresponding to uniform norms in vector spaces. It will be firstly done for the model equation without the junior term and then generalized for equation (2.1). The desired result is the following

Theorem 1. *Let $f(\tau)$ be of the form* (2.5), $\max_j |a_j| = 1$, *the quasiuniform parameter satisfies the condition* $Q \le 4.25$, *and* $g = \mathcal{K}f$. *Then for sufficienly small h there exists such a gridpoint τ_j, that* $|g(\tau_j)| \ge ch$.

3. Formulation of the theorem in terms of finite differences

Let us begin the proof with the new formulation of the theorem. If $g(\tau)$ is some function, then the symbol $D_2(g; i, j, k)$ (where $i < j < k$) will denote the second finite difference calculated from the values at the gridpoints τ_i, τ_j, τ_k, namely:

$$\begin{aligned} D_2(g;i,j,k) &\equiv \frac{\tau_k-\tau_j}{\tau_k-\tau_i}g(\tau_i)+\frac{\tau_j-\tau_i}{\tau_k-\tau_i}g(\tau_k)-g(\tau_j) \\ &\equiv \frac{\tau_k-\tau_j}{\tau_k-\tau_i}[g(\tau_i)-g(\tau_j)]+\frac{\tau_j-\tau_i}{\tau_k-\tau_i}[g(\tau_k)-g(\tau_j)]. \end{aligned} \tag{3.1}$$

In this formula the coefficients $\alpha = (\tau_k - \tau_j)/(\tau_k - \tau_i)$ and $\beta = (\tau_j - \tau_i)/(\tau_k - \tau_i)$ satisfy the conditions $\alpha, \beta > 0$; $\alpha + \beta = 1$. Therefore if for some function g the inequality $|D_2(g;i,j,k)| > ch$ is valid, then at least one of modulus of differences $[g(\tau_i) - g(\tau_j)]$ and $[g(\tau_k) - g(\tau_j]$ is greater than ch, and hence at least one of values of $g(\tau)$ at the gridpoints has the modulus greater than $ch/2$.

So, if it is shown that modulus of at least one second difference of the function $g = \mathcal{K}f$ is greater than ch, then the statement of Theorem 1 is true.

The same is valid, if we replace the difference D_2 by the "simplified" second difference

$$\begin{aligned} \tilde{D}_2(g;i,j,k) &\equiv \frac{1}{2}g(\tau_i)+\frac{1}{2}g(\tau_k)-g(\tau_j) \\ &\equiv \frac{1}{2}[g(\tau_i)-g(\tau_j)]+\frac{1}{2}[g(\tau_k)-g(\tau_j)]. \end{aligned} \tag{3.2}$$

As it is known, the second difference D_2 can be expressed in terms of the second derivative

$$D_2(g;i,j,k) = \int_{\tau_i}^{\tau_k} g''(\xi)J(\xi)\,d\xi, \tag{3.3}$$

where

$$J(\xi) = \begin{cases} \dfrac{\tau_k-\tau_j}{\tau_k-\tau_i}(\xi-\tau_i), & \text{if } \xi < \tau_j, \\ \dfrac{\tau_j-\tau_i}{\tau_k-\tau_i}(\tau_k-\xi), & \text{if } \xi > \tau_j. \end{cases}$$

As a consequence of this formula:

$$D_2(g;i,j,k) \geq \min_{[\tau_i,\tau_k]} g''(\xi)\cdot\int_{\tau_i}^{\tau_k} J(\xi)\,d\xi = \min_{[\tau_i,\tau_k]} g''(\xi)\cdot\frac{(\tau_k-\tau_j)(\tau_j-\tau_i)}{2}. \tag{3.4}$$

Now we can transform the formulation of the theorem once more. Let the function $f_0(\tau)$ be identically equal to 1. Since the kernel of the model operator in (2.3) is periodical and invariant with respect to shift, then the image of f_0 under $\mathcal{K}$ is identically equal to the constant

$$\mathcal{K}f_0 \doteq A_0 \equiv \int_0^1 \log \frac{1}{|\tau - t|'} \, d\tau = \int_{-1/2}^{1/2} \log \frac{1}{|\tau|} \, d\tau > 0. \tag{3.5}$$

According to the condition of Theorem 1, $\max |a_j| = 1$, so it is possible to choose one of such numbers j that $|a_j| = 1$. Let us assume that this coefficient a_j is equal to (-1). For our convenience, we change the numbering. The gridpoint corresponding to this chosen coefficient gets the number 0; the rest of gridpoints are numbered in accordance with their order from number $(-N_1)$ to N_2; $N_1 + N_2 = N$. Let us give a shift to the variable τ, namely: the gridpoint τ_0 will correspond to the value $\tau = 0$, and the interval, where τ varies, will be $[-1/2; 1/2]$. After that one of basic functions (number $-N_1$ or N_2) can be broken to pieces: one part of its support is disposed in the beginning of the segment $[-1/2; 1/2]$ and the other – in the end.

Now we replace the function: $f(\tau) = \tilde{f}(\tau) - f_0(\tau)$. Since $\mathcal{K}f_0 \equiv \text{const}$, the second differences of $\mathcal{K}f$ and $\mathcal{K}\tilde{f}$ are the same.

The new function $\tilde{f}$ has the same structure

$$\tilde{f}(\tau) = \sum_{j=-N_1}^{N_2} b_j \varphi_j(\tau), \tag{3.6}$$

where $b_j = a_j + 1$; hence $0 \le b_j \le 2$; $b_0 = 0$.

As it will be shown later (Lemma 2), if integral of f differs from 0 (or integral of $\tilde{f}$ differs from 1) by more than ch, then the average value of $\mathcal{K}f$ in the gridpoints exceeds ch too, so the statement of Theorem 1 is obviously true. So, looking forward, we formulate the following statement, whose validity will imply the validity of Theorem 1.

Theorem 1′. *Let the function $\tilde{f}$ be of the form* (3.6), *where* $b_0 = 0$, $0 \le b_j \le 2$, *and*

$$\int_{-1/2}^{1/2} \tilde{f}(\tau) \, d\tau \ge 1 - C_* h. \tag{3.7}$$

If the quasiuniform parameter Q does not exceed 4.25, *then for sufficiently small h there exist two gridpoints τ_{-i}, τ_j, $(i, j > 0)$ such that*

$$|D_2(\mathcal{K}\tilde{f}; -i, 0, j)| > ch \quad \text{or} \quad |\tilde{D}_2(\mathcal{K}\tilde{f}; -i, 0, j)| > ch. \tag{3.8}$$

4. Some properties of the images of the basic functions

For further exposition we need some information about the images of the piecewise-linear basic functions φ_j. Denoting $\psi_j \equiv \mathcal{K}\varphi_j$, we can write

$$\psi_j(t) = \int_{\tau_{j-1}}^{\tau_{j+1}} \varphi_j(\tau) \log \frac{1}{|\tau - t|'} d\tau. \tag{4.1}$$

Assuming that $|t - \tau| < 1/2$ we write this function in the explicit form:

$$\psi_j(\tau_j + \eta) = \alpha_j + h_{j-1/2} \times \mu\Big(1 + \frac{\eta}{h_{j-1/2}}\Big) + h_{j+1/2} \times \mu\Big(1 - \frac{\eta}{h_{j+1/2}}\Big), \tag{4.2}$$

where

$$\begin{aligned} \alpha_j &= -\frac{1}{2}(h_{j-1/2} \log h_{j-1/2} + h_{j+1/2} \log h_{j+1/2}), \\ \mu(\xi) &= -\frac{1}{2}\xi^2 \log|\xi| + \frac{1}{2}(\xi^2 - 1)\log|\xi - 1| + \frac{1 + 2\xi}{4}. \end{aligned} \tag{4.3}$$

The key point in our consideration is a convexity of the functions ψ_j themselves and also of their restrictions onto the grid. We understand the latter property as a positiveness of second differences; it may be called *discrete convexity*. The following lemma describes these properties.

Lemma 1.

1. *If* $t \notin \operatorname{supp} \varphi_j$, *then*

$$\psi_j''(t) \geq \frac{ch}{|t - \tau_j|'^2}. \tag{4.4}$$

2. *If* $t \notin [\tau_{j-2}; \tau_{j+2}]$, *then*

$$\psi_j''(t) \leq \frac{ch}{|t - \tau_j|'^2}. \tag{4.5}$$

3. *The difference between values of* ψ_j *at two points from one of the segments* $[\tau_k; \tau_{k+1}]$ *does not exceed* ch.
4. *If* $m \neq j-1, j, j+1$, *then*

$$D_2(\psi_j; m-1, m, m+1) \geq \frac{ch^3}{|\tau_m - \tau_j|'^2}. \tag{4.6}$$

5. *If* $m = j - 1$ *or* $m = j + 1$ *and quasiuniform parameter* $Q < 4.25$, *then*

$$D_2(\psi_j; m-1, m, m+1) \geq ch. \tag{4.7}$$

Proof. The integration with respect to τ and calculation of derivatives or finite differences with respect to t are commutative, so these latter operations can be applied to the logarithmic kernel under the integral sign in (4.1). So, the first two statements of Lemma can be easily deduced from (4.1). Indeed, under these conditions (taking into account the quasiuniformity of the grid) the second derivative of the kernel is of order $|t - \tau_j|^{r-2}$, the basic function φ_j is positive and its integral is of order ch.

Statement 3 follows from the explicit formula (4.2), because function $\mu(\xi)$ is bounded.

Statement 4 is a consequence of estimate (3.4) and the reasons written above. The break of the derivative of the kernel at points $t = \tau \pm 1/2$ may only increase the value of the second difference.

Property 5 is the most complicated. Moreover, it is valid under significant restriction for Q; if the ratio of steps is arbitrary then the second difference may become negative for the gridpoint neighbouring to τ_j. Let us assume that $m = 0$ and j is a neighbouring number, namely, $j = 1$. Denoting the ratios of steps

$$a = h_{-1/2}/h_{1/2}; \qquad b = h_{3/2}/h_{1/2}, \tag{4.8}$$

we can conclude that $a, b \geq 1/Q \geq 1/4.25 > 0.235$. Under these conditions we have to verify a validity of estimate (4.7) for the quantity written in the form:

$$\begin{aligned} D_2(\psi_1; -1, 0, 1) &= \psi_1(\tau_{-1})\frac{h_{1/2}}{h_{1/2}+h_{-1/2}} + \psi_1(\tau_1)\frac{h_{-1/2}}{h_{1/2}+h_{-1/2}} - \psi_1(\tau_0) \\ &= \frac{h_{1/2}}{2b(1+a)}\nu(a,b), \end{aligned}$$

where

$$\begin{aligned} \nu(a,b) &= -(1+a+b)^2\log(1+a+b) + (1+a)^2(1+b)\log(1+a) + \\ &\quad (1+a)(1+b)^2\log(1+b) - a^2b\log a - ab^2\log b. \end{aligned}$$

Let us verify the positiveness of $\nu(a,b)$. Its derivatives are:

$$\begin{aligned} \frac{\partial \nu}{\partial a} &= -2(1+a+b)\log(1+a+b) + 2(1+a)(1+b)\log(1+a) + \\ &\quad (1+b)\log(1+b) - 2ab\log a + b^2\log b; \\ \frac{\partial^2 \nu}{\partial a^2} &= -2\log\frac{1+a+b}{1+a} + 2b\log\frac{1+a}{a}; \\ \frac{\partial^3 \nu}{\partial a \partial b^2} &= 2\Big[\log\frac{1+b}{b} - \frac{1}{1+a+b}\Big]. \end{aligned} \tag{4.9}$$

Since the inequality $\log((1+x)/x) > 1/(1+x)$ can be verified by the usual way, the latter of this expressions is positive. Estimating $\log((1+a+b)/(1+a) < b/(1+a)$, we can also apply the same inequality to the second quantity (4.8), therefore it is positive too.

Let us denote $a_0 = b_0 = 0.235$. The direct calculation gives

$$\nu(a_0,b_0) \approx 2.4369 \times 10^{-2} > 0; \quad \frac{\partial \nu}{\partial a}(a_0,b_0) = \frac{\partial \nu}{\partial b}(a_0,b_0) \approx 0.07305 > 0;$$

$$\frac{\partial^2 \nu}{\partial a \partial b}(a_0,b_0) \approx 0.63344 > 0.$$

As it can easily be deduced from facts written above, all first and second partial derivatives of ν are positive for all $a, b > 0.235$, so $\nu(a,b) \geq \nu(a_0,b_0) \geq \text{const}$. Lemma 1 is proved. □

Let us prove the following important fact based on the properties of ψ_j:

Lemma 2. *There exists a positive constant C_* independent on h such that the following statement is valid: if $f(\tau)$ is a function of the form* (1.5), *where* $\max|a_j| = 1$, *and the inequality*

$$\Big| \int_{-1/2}^{1/2} f(\tau)\, d\tau \Big| > C_* h$$

is true, then under sufficiently small h there exists a gridpoint τ_j such that

$$|g(\tau_j)| \equiv |\mathcal{K} f(\tau_j)| > ch.$$

Proof. Since the kernel of operator $\mathcal{K}$ is periodical and invariant with respect to shift, we have

$$\begin{aligned} \Big| \int_{-1/2}^{1/2} g(\tau)\, d\tau \Big| &= \Big| \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} \log \frac{1}{|t-\tau|'} f(t)\, dt\, d\tau \Big| \\ &= \Big| \int_{-1/2}^{1/2} dt\, f(t) \int_{-1/2}^{1/2} \log \frac{1}{|t-\tau|'}\, d\tau \Big| \\ &= \Big| A_0 \int_{-1/2}^{1/2} f(\tau)\, d\tau \Big| > A_0 C_* h, \end{aligned} \tag{4.10}$$

where A_0 is the constant defined in (3.5). Let us now consider the difference E between the precise value of integral of g and its approximation by trapezoidal quadrature formula on the grid $\{\tau_j\}$.

Let us denote $\delta_k = (h_{k-1/2} + h_{k+1/2})/2$. Since $g(\tau) = \sum a_j \psi_j(\tau)$, this difference is

$$E = \int_{-1/2}^{1/2} g(\tau)\,d\tau - \sum_{k=-N_1}^{N_2} g(\tau_k)\delta_k$$

$$= \sum_{j=-N_1}^{N_2} a_j \Big[\int_{-1/2}^{1/2} \psi_j(\tau)\,d\tau - \sum_{k=-N_1}^{N_2} \psi_j(\tau_k)\delta_k \Big] \equiv \sum_{j=-N_1}^{N_2} a_j E_j. \quad (4.11)$$

The bracket denoted by E_j is exactly the error of trapezoidal quadrature formula for the integral of $\psi_j(\tau)$. This error is equal to a sum of errors E_{jk} for segments $[\tau_{k-1}, \tau_k]$ $(k = -N_1, \ldots, N_2)$. If $|k - j| \geq 2$, then this error has the estimate

$$|E_{jk}| \leq \frac{h^3_{k-1/2}}{12} |\psi_j''(\xi_k^*)|,$$

where ξ_k^* is any point in $[\tau_{k-1}, \tau_k]$.

As one can see from formula (4.1),

$$\psi_j''(\xi_k^*) \leq \int_{\tau_{k-1}}^{\tau_k} \frac{1}{|\tau_j - \tau|'^{2}}\,d\tau.$$

Therefore the sum of these errors E_{jk} for $|k - j| \geq 2$ can be estimated

$$\sum_{k:|k-j|\geq 2} |E_{jk}| \leq ch^3 \int_{2h}^{1/2} \frac{d\xi}{\xi^2} \leq ch^2. \quad (4.12)$$

As to segments neighbouring to τ_j, the variaton of $\psi_j(\tau)$ on each of them does not exceed ch, according to Statement 3 of Lemma 1, and the lengths of these segments do not exceed Qh, hence the sum of these errors is less than ch^2. So,

$$|E_j| \leq ch^2.$$

Returning to (4.11), we see that

$$|E| \leq \sum_j |a_j| \cdot |E_j| \leq ch^2 \sum_j |a_j| \leq ch^2 N \leq c_0 h.$$

Consequently, because of (4.10), if we choose $C_* > c_0/A_0$, then

$$\Big|\sum_k g(\tau_k)\delta_k\Big| \geq \Big| \int_{-1/2}^{1/2} g(\tau)\,d\tau\Big| - c_0 h > (A_0 C_* - c_0)h > ch.$$

But the quantities δ_k are positive and their sum equals to the length of $[-1/2, 1/2]$, i.e., it equals to 1; so the sum in the left-hand side of the inequality is some weighted average of $g(\tau_k)$, therefore modulus of at least one of them is greater than ch. Lemma 2 is proved. □

Corollary. *In the view of Lemma 2, in order to prove Theorem 1 it is sufficient now to consider the situation when*

$$\left|\int f(\tau)\,d\tau\right| < C_* h.$$

Taking into account that

$$\int \tilde{f}(\tau)\,d\tau = \int f(\tau)\,d\tau + 1,$$

we have to prove Theorem 1′ *under condition* (3.7).

5. Proof of the continual analogue of Theorem 1′

First of all, let us note that

$$D_2(\mathcal{K}\tilde{f}; -1, 0, 1) = \sum_{j=-N_1}^{N_2} b_j D_2(\psi_j; -1, 0, 1). \tag{5.1}$$

Since $b_0 = 0$, in the view of Lemma 1, all members in the right-hand side are positive. If there exist two constants $k_* > 0$ and $\gamma > 0$ independent on h such that the following statement is valid:

> *at least one of coefficients b_j corresponding to gridpoints $\tau_j \in [-k_* h, k_* h]$ is greater than γ,*

then, according to Statements 4, 5 of Lemma 1, the sum in (5.1) contains members that are greater than ch. Indeed, if this gridpoint is neighbouring to τ_0, Statement 5 provides the desired estimate, else, Statement 4 gives

$$b_j D_2(\psi_j; -1, 0, 1) \geq \gamma \cdot \frac{ch^3}{|\tau_j|^2} \geq \gamma \cdot \frac{ch^3}{|k_* Q h|^2} \geq ch.$$

Hence the statement of Theorem 1 is true. So, it is sufficient to consider the situation when for gridpoints from $[-k_* h, k_* h]$ all coefficients b_j are less than γ. Then the second difference for the three neighbouring points τ_{-1}, τ_0, τ_1 does not satisfy the desired estimate. Now we shall consider the second differences for far distant gridpoints. Namely, it will be the "simplified" second difference $\tilde{D}_2$, defined by (3.2).

We shall firstly prove the "continual" statement, when endpoints in a second difference do not belong to grid. Namely, we shall study the second difference

$$d_2(\tilde{f},\theta) \equiv \frac{1}{2}\mathcal{K}\tilde{f}(\theta) + \frac{1}{2}\mathcal{K}\tilde{f}(-\theta) - \mathcal{K}\tilde{f}(0). \tag{5.2}$$

Theorem 1″. *Let $\tilde{f}(\tau)$ be of the form* (3.6), *where $0 \le b_j \le 2$; $b_0 = 0$, and condition* (4.7) *is valid. Then there exist constants k_* and γ independent on h such that if for $|\tau_j| < k_* h$ quantities $b_j < \gamma$ and h are sufficiently small, then for any θ*

$$d_2(\tilde{f},\theta) > ch. \tag{5.3}$$

Beginning the proof, let us take the quantity $H \ge 2\cdot\max h_{j+1/2}$ and study the weighted average value of function $d_2(\tilde{f},\theta)$ on the segment $[H, 1/2]$; the weight function is $1/\theta^2$:

$$M(\tilde{f}) \equiv \frac{1}{W}\int\limits_H^{1/2}\Big[\frac{1}{2}\mathcal{K}\tilde{f}(\theta) + \frac{1}{2}\mathcal{K}\tilde{f}(-\theta) - \mathcal{K}\tilde{f}(0)\Big]\frac{1}{\theta^2}\,d\theta, \tag{5.4}$$

where

$$W = \int\limits_H^{1/2}\frac{1}{\theta^2}\,d\theta = \frac{1-2H}{H}.$$

Substituting the expression of $\mathcal{K}\tilde{f}(\theta)$ and changing the order of integration, we get

$$M(\tilde{f}) = \int\limits_{-1/2}^{1/2}\tilde{f}(\tau)\Phi(\tau)\,d\tau, \tag{5.5}$$

where

$$\Phi(\tau) = \frac{1}{W}\int\limits_H^{1/2}\Big[\frac{1}{2}\log\frac{1}{|\theta-\tau|'} + \frac{1}{2}\log\frac{1}{|-\theta-\tau|'} - \log\frac{1}{|\tau|'}\Big]\frac{1}{\theta^2}\,d\theta. \tag{5.6}$$

The function $\Phi(\tau)$ is obviously even, therefore integral (5.5) may be written in the form

$$M(\tilde{f}) = \int\limits_0^{1/2}F(\tau)\Phi(\tau)\,d\tau, \tag{5.7}$$

where $F(\tau) = \tilde{f}(\tau)+\tilde{f}(-\tau)$. As it follows from the conditions of Theorem 1′, $0 \le F(\tau) \le 4$, and the integral of $F(\tau)$ on the segment $[-1/2, 1/2]$ is not less than $1 - C_* h$.

Let us study now the function $\Phi(\tau)$. Calculating integral (5.6), we write the result in the following form:

For $0 < \tau < \dfrac{1}{2} - H$:

$$\Phi(\tau) = \frac{1}{1-2H}\{\Phi_0(\tau) + \Phi_1(\tau) + \Phi_2(\tau)\},$$

where

$$\Phi_0(\tau) = 2H \log \frac{1}{\tau\left(\frac{1}{2} - \tau\right)};$$

$$\Phi_1(\tau) = \frac{H}{\tau(1-\tau)}\left[-\log 2 - 4\left(\frac{1}{2} - \tau\right)^2 \log\left|\frac{1}{2} - \tau\right|\right];$$

$$\Phi_2(\tau) = \frac{H}{2\tau}\log\left|\frac{1 - H/\tau}{1 + H/\tau}\right| - \frac{1}{2}\log\left|1 - \frac{H^2}{\tau^2}\right|. \tag{5.8}$$

For $\dfrac{1}{2} - H \le \tau \le \dfrac{1}{2}$:

$$\Phi(\tau) = \frac{1}{1-2H}\left\{-H\log H \cdot \frac{1}{2\tau(1-\tau)} + \frac{2H}{\tau(\tau-1)}\left(\frac{1}{2} - \tau\right)^2 \log\left|\frac{1}{2} - \tau\right| + \right.$$
$$\frac{H}{2\tau}\log|\tau - H| + \frac{1}{2}\log\left|1 + \frac{2\tau(\tau - \frac{1}{2}) + H(1-H)}{(\tau - H)(1 - \tau - H)}\right| -$$
$$\left.\frac{H}{2\tau(1-\tau)}\log 2 - \frac{H}{2(\tau-1)}\log|\tau + H - 1|\right\}. \tag{5.9}$$

Let us now verify some properties of $\Phi(\tau)$.

Lemma 3.

$$\textit{If} \quad \tau \in \left[\frac{1}{2} - H, \frac{1}{2}\right], \quad \textit{then} \quad \Phi(\tau) = O(H|\log H|). \tag{5.10}$$

The proof can be fulfilled by the trivial consideration of every member in (5.9).

The behaviour of functions (5.8) will be studied more attentively.

Lemma 4. *The function* $\Phi_0(\tau)$

- *is even with respect to point* $\tau = 1/4$*;*
- *decreases on the segment* $[0, 1/4]$*;*
- *increases on the segment* $[1/4, 1/2]$*;*
- *for* $\tau \in [H, 1/2 - H]$ *satisfies the inequlities*

$$H \cdot 8\log 2 = \Phi_0(1/4) \le \Phi_0(\tau) \le 2H|\log H| + O(H). \tag{5.11}$$

The proof is obviuos in the view of the first formula (5.8).

Lemma 5. *The following inequalities are valid:*

$$-4\log 2\cdot H \leq \Phi_1(\tau) \leq 0. \tag{5.12}$$

Proof. Verifying the second inequality in (5.12), let us show that the bracket in the second formula (5.8) is not positive. Denoting $1/2-\tau=x$, let us study the function $\varphi(x)=-\log 2-4x^2\log x$ on the segment $x\in[0,1/2]$.

Since $\varphi'(x)=-4x(2\log x+1)$, for $x<1/2$ we have $\log x<\log(1/2)<-1/2$. Therefore $\varphi'(x)>0$, hence, $\varphi(x)$ has a maximum at $x=1/2$. But $\varphi(1/2)=0$, so the second inequality (5.12) is true.

Verifying the first inequality (5.12), we rewrite the second formula (5.8) in the form

$$\begin{aligned}\Phi_1(\tau) &= H\Big[4\log\frac{1}{2}+4\Big(\frac{1}{2}-\tau\Big)^2\cdot\frac{\log(1-2\tau)}{\tau}\cdot\frac{1}{\tau-1}\Big]\\ &= H\Big[4\log\frac{1}{2}+\frac{\psi(\tau)}{\tau-1}\Big],\end{aligned} \tag{5.13}$$

where

$$\psi(\tau)=4\Big(\frac{1}{2}-\tau\Big)^2\frac{\log(1-2\tau)}{\tau}.$$

At the end of the segment $[0,1/2]$ we have $\psi(\tau)\to 0$ under $\tau\to 1/2$. Let us show that $\psi(\tau)$ increases on $[0,1/2]$. Indeed,

$$\psi'(\tau)=\frac{(1-2\tau)[\log(1-2\tau)(-2\tau-1)-2\tau]}{\tau^2}.$$

But $|\log(1-2\tau)|>2\tau$; $|-2\tau-1|>1$, so the bracket in the latter expression is positive, therefore $\psi'(\tau)>0$. Then $\psi(\tau)$ is not positive everywhere in $[0,1/2]$. Since $\tau-1<0$, we get the first inequality (5.12) from (5.13). □

The properties of the function $\Phi_2(\tau)$ are formulated in the next lemma.

Lemma 6. *The function $\Phi_2(\tau)$ is negative elsewhere on $[0,1/2]$, and for $\tau>\sqrt{H}$ satifties the inequalities*

$$-\frac{2}{3}H\leq\Phi_2(\tau)\leq 0. \tag{5.14}$$

Proof. Denoting $H/\tau=x$, we rewrite formula (5.8) for Φ_2 in the form

$$\Phi_2(\tau)=\phi(x)\equiv\frac{x}{2}\log\Big|\frac{1-x}{1+x}\Big|-\frac{1}{2}\log|1-x^2|,\qquad 0<x<\sqrt{H}.$$

Hence

$$\phi(0) = 0; \quad \phi'(x) = \frac{1}{2}\log\left|\frac{1-x}{1+x}\right|; \quad \phi'(0) = 0; \quad \phi''(x) = -1/(1-x^2).$$

According to Tailor's formula with the residual in the Lagrange form

$$\phi(x) = \frac{1}{2}\phi''(\xi)\cdot x^2 = \frac{-x^2}{2}\cdot\frac{1}{1-\xi^2}, \text{ where } \xi \in [0, x].$$

Since $|\xi| < 1/2$, then $1/(1-\xi^2) < 4/3$, i.e.,

$$-\frac{2}{3}x^2 \le \phi(x) \le 0.$$

Since $x < \sqrt{H}$, this is equivalent to the statement of the lemma. □

And finally let us note the following trivial property of $\Phi(\tau)$.

Lemma 7.

$$\int_0^{1/2} \Phi(\tau)\, d\tau = 0. \tag{5.15}$$

Proof. If $\tilde{f}(\tau) \equiv$ const, then also $\mathcal{K}\tilde{f} \equiv$ const, so all second differences equal zero; because of that the integral (5.4), is zero, together with integrals (5.5), (5.7). □

Now we have got the following. There is the function $\Phi(\tau)$ which is positive on the segment $[\sqrt{H}, 1/2]$, as it follows from the lower estimates of Lemmas 4–6:

$$\Phi_0 + \Phi_1 + \Phi_2 \ge H(8\log 2 - 4\log 2 - 2/3) \ge H(4\log 2 - 2/3) > 2H \tag{5.16}$$

(because $\log 2 > 2/3$). But the integral of Φ on the entire segment $[-1/2, 1/2]$ equals to zero, therefore the integral on the beginning segment $[0, \sqrt{H}]$ must be negative and neutralize the positive part. We shall show later that, in fact, the negative part of integral is "almost completely" concentrated on the segment, whose length is $l_* H$, where l_* is a constant independent on H.

After that, comparing integral (5.7) with integral (5.15), we can show that, if $F(\tau) \le 2\gamma$ on the beginning segment $[0, l_* H]$, then the negative part of (5.15) is multiplied by the factor less than 2γ, and its positive part – by the quantity which is greater than some constant. So, integral (5.7) will be positive and it can be estimated from below in desired manner.

Let us fulfil this study. Let us firstly represent $\Phi(\tau)$ in the form $\Phi(\tau) = \Phi_+(\tau) + \Phi_-(\tau)$, where

$$\Phi_+(\tau) = \frac{1}{1-2H}(\Phi_0(\tau) + \Phi_1(\tau)); \qquad \Phi_-(\tau) = \frac{1}{1-2H} \cdot \Phi_2(\tau). \quad (5.17)$$

Lemma 8. *For every constant $m \in (0,1)$ there exists such a constant l_* independent on H that*

$$\left| \int_0^{l_* H} \Phi_-(\tau)\, d\tau \right| > (1-m) \cdot \left| \int_0^{\infty} \Phi_-(\tau)\, d\tau \right|. \quad (5.18)$$

Proof. Omitting the factor, we can replace $\Phi_-(\tau)$ by $\Phi_2(\tau)$ in (5.18). The desired inequality is obviously equivalent to the inequality

$$\left| \int_{l_* H}^{\infty} \Phi_2(\tau)\, d\tau \right| < m \left| \int_0^{\infty} \Phi_2(\tau)\, d\tau \right|.$$

Let us make the same substitution $H/\tau = x$, as it was done in Lemma 6, and write

$$\int_0^{\infty} \Phi_2(\tau)\, d\tau = \frac{H}{2} \left[\int_0^{\infty} \frac{1}{x} \log\left| \frac{1-x}{1+x} \right| dx - \int_0^{\infty} \frac{\log|1-x^2|}{x^2}\, dx \right] = -\frac{H\pi^2}{4}.$$

Denoting by $\phi(x)$ the same function, as in Lemma 6, and taking into account that $|\phi(x)| < x^2 \cdot 2/3$, we get

$$\left| \int_{l_* H}^{\infty} \Phi_2(\tau)\, d\tau \right| = H \int_0^{1/l_*} \frac{\phi(x)}{x^2}\, dx \le H \cdot \frac{2}{3} \cdot \frac{1}{x^2}.$$

Under choice $l_* > 8/(3\pi^2 m)$ the statement of the lemma is true. □

Corollary. *The following inequalities are true, as they are more weak than* (5.18)

$$\left| \int_0^{l_* H} \Phi_-(\tau)\, d\tau \right| > (1-m) \cdot \left| \int_0^{1/2} \Phi_-(\tau)\, d\tau \right|; \quad (5.19)$$

$$\left| \int_{l_* H}^{1/2} \Phi_-(\tau)\, d\tau \right| < m \cdot \left| \int_0^{1/2} \Phi_-(\tau)\, d\tau \right|. \quad (5.20)$$

To estimate the integral (5.7) from below, let us divide it into several members

$$\int_0^{1/2} F(\tau)\Phi(\tau)\,d\tau = \int_0^{\sqrt{H}} F(\tau)\Phi_+(\tau)\,d\tau + \int_{\sqrt{H}}^{1/2} F(\tau)\Phi_+(\tau)\,d\tau + \int_0^{l_*H} F(\tau)\Phi_-(\tau)\,d\tau + \int_{l_*H}^{1/2} F(\tau)\Phi_-(\tau)\,d\tau$$
$$\equiv I_{1+}(F) + I_{2+}(F) + I_{3-}(F) + I_{4-}(F). \tag{5.21}$$

If we replace the function $F(\tau)$ by 1 in these integrals, we get new integrals which will be denoted by $I_{1+}(1)$, $I_{2+}(1)$, $I_{3-}(1)$, $I_{4-}(1)$.

According to Lemma 7

$$I_{1+}(1) + I_{2+}(1) + I_{3-}(1) + I_{4-}(1) = 0. \tag{5.22}$$

In view of the properties

$$0 \le F(\tau) \le 2, \quad \int_0^{1/2} F(\tau)\,d\tau \ge 1 - C_*h, \quad \text{and} \quad F(\tau) \le 2\gamma \ \text{for} \ \tau \in [0, l_*H],$$

we can derive the following statements from Lemmas 4–8:

1. $I_{1+}(F) > 0$.

2.

$$I_{2+}(F) \ge \frac{16}{21}(1 - C_*h - 4\sqrt{H})I_{2+}(1). \tag{5.23}$$

Indeed,

$$I_{2+}(F) \ge \min_{[\sqrt{H},1/2]} \Phi_+(\tau) \cdot \int_{\sqrt{H}}^{1/2} F(\tau)\,d\tau \ge \frac{H}{1-2H}\left(\int_0^{1/2} + \int_0^{\sqrt{H}}\right)$$
$$\ge \frac{H}{1-2H} \cdot 4\log 2(1 - C_*h - 4\sqrt{H}).$$

From the other side, we can estimate

$$I_{2+}(1) = \frac{1}{1-2H}\int_{\sqrt{H}}^{1/2} (\Phi_0(\tau) + \Phi_1(\tau))\,d\tau$$
$$< \frac{1}{1-2H}\int_0^{1/2} \Phi_0(\tau)\,d\tau = \frac{2H}{1-2H}\int_0^{1/2}\left[\log\frac{1}{\tau} + \log\left|\frac{1}{2} - \tau\right|\right] d\tau$$
$$= \frac{2H}{1-2H}(\log 2 + 1). \tag{5.24}$$

Because of that

$$\frac{I_{2+}(F)}{I_2(1)} > \frac{2\log 2}{\log 2 + 1}(1 - C_* h - 4\sqrt{H}).$$

Since

$$\frac{2}{3} < \log 2 < \frac{3}{4}, \quad \text{then} \quad \frac{2\log 2}{\log 2 + 1} < \frac{2 \cdot 2/3}{1 + 3/4} = \frac{16}{21}.$$

3. $|I_{4-}(F)| < 4m|I_{3-}(1)|$, or, because of the negativity of these quantities, $I_{4-}(F) > 4mI_{3-}(1)$. It is the consequence of the inequality $F(\tau) \leq 4$ and inequality (5.20).

4. $I_{3-}(F) \geq 2\gamma I_{3-}(1)$. It is the consequence of inequalities $\Phi_- < 0$ and $F(\tau) \leq 2\gamma$ for $\tau \in [0, l_* h]$.

5. $I_{1+}(1) < C\sqrt{H}|\log H| \cdot I_{2+}(1)$. Indeed, according to (5.11), (5.12)

$$I_{1+}(1) = \int\limits_0^{\sqrt{H}} \Phi_+(\tau)\,d\tau \leq (4H|\log H| + O(H)) \cdot \frac{\sqrt{H}}{1 - 2H};$$

$$I_{2+}(1) = \int\limits_{\sqrt{H}}^{1/2} \Phi_+(\tau)\,d\tau > \frac{1}{1 - 2H}\Big(\frac{1}{2} - \sqrt{H}\Big) \cdot 4H \log 2. \qquad (5.25)$$

These two inequalities imply Statement 5.

Let us substitute all these estimates into (5.21):

$$\int\limits_0^{1/2} F(\tau)\Phi(\tau)\,d\tau \geq \frac{16}{21}(1 - C_* h - 4\sqrt{H})I_{2+}(1) + (2\gamma + 4m)I_{3-}(1).$$

From the other side, (5.22) and Statement 5 give us

$$I_{3-}(1) = -I_{1+}(1) - I_{2+}(1) - I_{4-}(1) > -(1 + C\sqrt{H}|\log H|)I_{2+}(1).$$

So, we can conclude

$$M(\tilde{f}) \equiv \int\limits_0^{1/2} F(\tau)\Phi(\tau)\,d\tau$$

$$\geq I_{2+}(1)\Big[\frac{16}{21}(1 - C_* h - 4\sqrt{H}) - (2\gamma + m)(1 + C\sqrt{H}|\log H|)\Big]. \qquad (5.26)$$

If m and γ are not too greate, the bracket is not less than some constant; in the view of (5.24), now we get: $M(\tilde{f}) > ch$. We would remind that it is true under the following conditions: h is sufficiently small, l_* depends on m according to Lemma 8 and for $|\tau| < l_*H$ the values of $\tilde{f}(\tau)$ are less than γ.

So, we have to choose small constant m, find l_* according to Lemma 8 and then to take γ in order to make the bracket in (5.26) be greater than some constant.

Let us remark that because of the quasiuniformity of the grid there exists k_* such that $k_*h = l_*H$.

The quantity $M(\tilde{f})$ defined by (5.4) is an average of $d_2(\tilde{f};\theta)$, and so our result provides the existence of $\theta \in [k_*h, 1/2]$ such that

$$d_2(\tilde{f};\theta) > ch.$$

Theorem 1″ is proved.

6. The concluding part of the proof of Theorem 1′

As we have seen, the second difference $d_2(\tilde{f},\theta)$ has the desired estimate from below for some three points $\{-\theta, 0, \theta\}$. We should prove now that the simplified second difference for some three gridpoints $\tau_{-i}, 0, \tau_j$ satisfies the same estimate. But we shall not compare the values of $\mathcal{K}\tilde{f}(\theta)$ with its values at nearest gridpoints. Approximately speaking, we shall compare a weighted average (with the weight $1/\theta^2$) of the function $\mathcal{K}\tilde{f}(\theta)$ and its weighted average for the gridpoints.

Let us introduce the following notation. If $\varphi(\psi)$ is an arbitrary function, then $L(\psi;\theta)$ will denote the value at the point θ of the function which interpolates $\psi(\theta)$ linearly from its values at the gridpoints. Namely, if $\theta \in [\tau_i, \tau_{i+1}]$, then

$$L(\psi;\theta) \equiv \psi(\tau_i)\frac{\tau_{i+1} - \theta}{h_{j+1/2}} + \psi(\tau_{i+1})\frac{\theta - \tau_i}{h_{j+1/2}}. \tag{6.1}$$

We shall consider together with the average $M(\tilde{f})$ the similar discretized average: so, let us denote

$$\tilde{M}(\tilde{f}) \equiv \frac{1}{W}\int\limits_H^{1/2} \Big[\frac{1}{2}L(\mathcal{K}\tilde{f};\theta) + \frac{1}{2}L(\mathcal{K}\tilde{f};-\theta) - \mathcal{K}\tilde{f}(0)\Big]\frac{d\theta}{\theta^2}. \tag{6.2}$$

Besides the notation $\psi_j(\theta) = \mathcal{K}\varphi_j(\theta)$, we define $\tilde{\psi}_j(\theta) = L(\mathcal{K}\varphi_j;\theta)$. Since $\tilde{f}(\tau) = \sum b_j\varphi_j(\tau)$, where $b_0 = 0$, and the integration with respect to τ is commutative with the integration or summation with respect to θ, then the difference $\delta = \tilde{M}(\tilde{f}) - M(\tilde{f})$ may be written in the form

$$\delta = \sum_{j\neq 0} b_j \left\{ \frac{1}{2W} \int_H^{1/2} \frac{\tilde{\psi}_j(\theta) - \psi_j(\theta)}{\theta^2}\, d\theta + \right.$$

$$\left. \frac{1}{2W} \int_H^{1/2} \frac{\tilde{\psi}_j(-\theta) - \psi_j(-\theta)}{\theta^2}\, d\theta \right\}. \tag{6.3}$$

Let us assume that $j > 0$, therefore $\tau_j > 0$. We shall study the quantity

$$J_j \equiv \int_{|\theta|>H} \frac{\tilde{\psi}_j(\theta) - \psi_j(\theta)}{\theta^2}\, d\theta.$$

We divide the set $D = \{\theta : \ H < |\theta| < 1/2\}$ into two sets

$$D'_j = D \cap [\tau_{j-2}, \tau_{j+2}]; \qquad D''_j = D \setminus D'_j.$$

Correspondingly, the integral J_j is now represented as a sum $J_j = J'_j + J''_j$.

As it have been noted in Lemma 1, the variation of $\psi_j(\tau)$ on the nearest to τ_j four segments does not exceed $c_0 h$, therefore

$$|\tilde{\psi}_j - \psi_j| \le c_0 h; \quad \frac{1}{\theta^2} \le \frac{c'_0}{\tau_j^2},$$

and lengths of these segments do not exceed Qh. Hence

$$|J'_j| = \left| \int_{D'_j} \frac{\tilde{\psi}_j(\theta) - \psi_j(\theta)}{\theta^2}\, d\theta \right| \le c_0 h \cdot \frac{c'_0}{\tau_j^2} \cdot 4Qh \le c_1 \frac{h^2}{\tau_j^2}. \tag{6.4}$$

Let us estimate the quantity

$$J''_j = \sum_{k:\tau_k \notin D'_j} \int_{\tau_{k-1}}^{\tau_{k+1}} \frac{\tilde{\psi}_j(\theta) - \psi_j(\theta)}{\theta^2}\, d\theta.$$

According to Lemma 1, if $\theta \in [\tau_{k-1}, \tau_k]$, where $k \neq j-1, j, j+1, j+2$, then

$$|\tilde{\psi}_j(\theta) - \psi_j(\theta)| \le \max_{[\tau_{k-1},\tau_k]} \psi''_j(\theta) \cdot \frac{1}{2} h^2_{k-1/2} \le c_2 \frac{h^3}{|\theta - \tau_j^2|'^2}.$$

So,

$$|J''_j| \le c_2 h^2 \int_{D''_j} \frac{d\theta}{\theta^2 |\theta - \tau_j|'^2}. \tag{6.5}$$

This latter integral can be easily estimated.

1. On the segment $[H, \tau_{j-2}/2]$ (which is absent if $\tau_{j-2}/2 < H$) we have $1/|\theta - \tau_j|'^2 \leq 1/|\tau_j/2|^2 = 4/\tau_j^2$, so the integral on this segment does not exceed
$$\frac{4}{\tau_j^2} \int_H^{\tau_{j-2}/2} \frac{d\theta}{\theta^2} \leq \frac{c_3}{\tau_j^2 h}.$$

2. On the segment $[\tau_{j-2}/2, \tau_{j-2}]$ (it can be absent too) we have $1/\theta^2 \leq 4/\tau_{j-2}^2 \leq c_4/\tau_j^2$, and, besides that, $|\theta - \tau_j| \geq ch$, so the integral does not exceed
$$\frac{c_4}{\tau_j^2} \int_{\tau_{j-2}/2}^{\tau_{j-2}} \frac{d\theta}{|\theta - \tau_j|'^2} \leq \frac{c_5}{\tau_j^2 h}.$$

3. On the segment $[\tau_{j+2}, 1/2]$ we have $1/\theta^2 \leq 1/\tau_j^2$, and the estimate is the same. The segment $[-1/2, -H]$ should be considered similarly.

Substituting these latter estimates into (6.5), we get the result similar to (6.4)
$$|J_j''| \leq c_6 \frac{h^2}{\tau_j^2}. \tag{6.6}$$

So, the integral J_j has this estimate too.
Using it in (6.3), we get
$$|\delta| \leq \frac{c_7}{W} \sum_{j \neq 0} b_j \cdot \frac{h^2}{\tau_j^2}. \tag{6.7}$$

Let us divide the set of subscripts j into two subsets: $\mathcal{M}'$ consists of j such that the corresponding gridpoint τ_j belongs to the segment $[-k_* h, k_* h]$, and then $b_j \leq 2\gamma$; the set $\mathcal{M}''$ contains the rest of values of subscript. According to that, the sum (5.7) will be divided into two sums, denoted by δ' and δ''.

Since
$$\frac{h}{\tau_j^2} \leq c \int_{\Delta_j} \frac{d\tau}{\tau^2} \quad \text{and} \quad W \geq \frac{c}{h},$$
we can estimate
$$\begin{aligned} \delta' &\leq c_8 h \sum_{j \in \mathcal{M}'} b_j h \cdot \int_{\tau_{j-1}}^{\tau_j} \frac{d\tau}{\tau^2} \leq c_9 \gamma h^2 \int_{h/2} k_* h \frac{d\tau}{\tau^2} \\ &\leq \gamma c_{10} h \leq \gamma c_{11} I_{2+}(1), \end{aligned} \tag{6.8}$$
because of estimate (5.25) for $I_{2+}(1)$. From the other side,

$$\delta'' \le \frac{c_{12}}{W} \sum_{j\in\mathcal{M}''} b_j h \int_{\tau_{j-1}}^{\tau_j} \frac{d\tau}{\tau^2} \le c_{13}h^2 \int_{k_*h}^{1/2} \frac{d\tau}{\tau^2}$$

$$\le \frac{c_{13}h^2}{k_*h} \le \frac{c_{14}}{k_*} I_{2+}(1). \tag{6.9}$$

And so,

$$|\tilde{M}(\tilde{f}) - M(\tilde{f})| \le \Big(\gamma c_{11} + \frac{c_{14}}{k_*}\Big) I_{2+}(1).$$

This implies together with (5.26)

$$\tilde{M}(\tilde{f}) \ge I_{2+}(1)\Big[\frac{16}{21}(1 - C_*h - 4\sqrt{H}) - (2\gamma + 4m)(1 + C\sqrt{H}|\log H|) - \gamma c_{11} - \frac{c_{14}}{k_*}\Big].$$

It is clear now that after proper choice of k_*, m, γ and for sufficiently small h and H the quantity in the latter bracket will become greater than some constant. So, in the view of (5.25), we have now $\tilde{M}(\tilde{f}) > ch$.

According to (6.2), $\tilde{M}(\tilde{f})$ is the average of second differences of the piecewise-linear function $l(\theta)$, which interpolates the function $\tilde{g} \equiv \mathcal{K}\tilde{f}$. If this average is estimated from below, then there exist some values $\theta \in [H, 1/2]$ such that

$$\tilde{d}_2(l;\theta) = \frac{1}{2}l(\theta) + \frac{1}{2}l(-\theta) - l(0) > ch. \tag{6.10}$$

Let one of these points (θ) be disposed between the gridpoints τ_{n_2-1}, τ_{n_2}, and its opposite point $(-\theta)$– between the gridpoints $\tau_{-n_1}, \tau_{-n_1+1}$. Since $|\theta| > H = 2\max h_{j+1/2}$, no one of these gridpoints does coinside with $\tau_0 = 0$. But every value $l(\theta)$, $l(-\theta)$ is an average of values at its neighbouring gridpoints

$$l(\theta) = \alpha\tilde{g}_{n_2-1} + (1-\alpha)\tilde{g}_{n_2}, \quad l(-\theta) = \beta\tilde{g}_{-n_1} + (1-\beta)\tilde{g}_{-n_1+1},$$

where $\alpha, \beta \in [0,1]$, and $\tilde{g}_k$ denotes the value of $\tilde{g}$ at the gridpoint τ_k. As it can be easily verified, if we take the quantity $\mu = \min(\alpha, \beta)$, then $\tilde{d}_2(l,\theta)$ is the following average of four second differences calculated from gridpoints:

$$\begin{aligned}\tilde{d}_2(l,\theta) = {} & \mu\Big[\frac{1}{2}\tilde{g}_{-n_1} + \frac{1}{2}\tilde{g}_{n_2-1} - \tilde{g}_0\Big] + (\beta - \mu)\Big[\frac{1}{2}\tilde{g}_{-n_1} + \frac{1}{2}\tilde{g}_{n_2} - \tilde{g}_0\Big] + \\ & (\alpha - \mu)\Big[\frac{1}{2}\tilde{g}_{-n_1+1} + \frac{1}{2}\tilde{g}_{n_2-1} - \tilde{g}_0\Big] + \\ & (1 - \alpha - \beta + \mu)\Big[\frac{1}{2}\tilde{g}_{-n_1+1} + \frac{1}{2}\tilde{g}_{n_2} - \tilde{g}_0\Big].\end{aligned}$$

(The coefficients are positive and their sum equals 1; the brackets contain the second differences.) So, it follows from (6.10) that at least one of the brackets is not less than ch.

So, the proof of Theorem 1′ is completed and, hence, the statement of Theorem 1 is true.

7. The proof for general Symm's equation

Finally, let us consider the general case when the integral equation includes also junior terms. We shall assume now that our equation is:

$$(\mathcal{K}+\mathcal{K}_1)f(t) \equiv \int_0^1 \log\frac{1}{|\tau - t|'} f(\tau)\,d\tau + \int_0^1 K_1(t,\tau)f(\tau)\,d\tau = g(t), \qquad (7.1)$$

where the kernel $K_1(t,\tau)$ is of the form

$$K_1(t,\tau) = \log\frac{a|\tau - t|'}{|x(t)-x(\tau)|} + |x(t)-x(\tau)|\cdot\log|x(t)-x(\tau)|\cdot P_1(t,\tau) + \\ \operatorname{sign}(t-\tau)\cdot|x(t)-x(\tau)|\log|x(t)-x(\tau)|\cdot P_2(t,\tau) + P_3(t,\tau). \qquad (7.2)$$

In this formula $x(\tau)$ is a periodical vector-function which describes a parametrization of any curve.

Let us denote by $C^{0,\lambda}$ the Banach space of functions (or vector-functions) of one variable satisfying the Hölder condition of degree λ; $C^{1,\lambda}$ is the space of functions whose derivative belongs to $C^{0,\lambda}$.

Theorem 2. *Let equation* (7.1) *be uniquely solvable for every right-hand side from $C^{0,\lambda}$, vector-function $x(t)$ belongs to $C^{1,\lambda}$, and derivatives of the functions P_1, P_2, P_3 with respect to t belong to $C^{0,\lambda}$ with respect to each variable. If $f(\tau)$ is of the form* (2.5), *where* $\max|a_j| = 1$, *and h is small enough, then there exists such a gridpoint τ_j that $|g(\tau_j)| > ch$.*

Proof. Let X_h be a linear space of functions of the form (2.5). We have to show that if $f \in X_h$ and $\max|f| = 1$, then

$$\max_j |(\mathcal{K}+\mathcal{K}_1)f(\tau_j)| > ch. \qquad (7.3)$$

Let us introduce the operator Π_h that carries out a projection of space $C^{0,\lambda}$ onto X_h by the following way.

Let $\Delta_j \equiv \operatorname{supp}\varphi_j = [\tau_{j-1}, \tau_{j+1}]$, and let δ_j be an integral of φ_j on this set, that is $\delta_j = (h_{j-1/2} + h_{j+1/2})/2$. If $\psi(\tau)$ is any function, let $\bar{\psi}_j$ be the weighted average of ψ on Δ_j:

$$\bar{\psi}_j = \frac{1}{\delta_j}\int_{\Delta_j} \psi(\tau)\varphi_j(\tau)\,d\tau.$$

Then we define

$$\Pi_h\psi(\tau) \equiv \sum_j \bar{\psi}_j\varphi_j(\tau). \qquad (7.4)$$

Let us remark that, by the definition,

$$\int [\psi(\tau) - \bar{\psi}_j]\varphi_j(\tau)\,d\tau = 0. \qquad (7.5)$$

Let $g = (\mathcal{K} + \mathcal{K}_1)f$, where f satisfies the conditions of Theorem. We have to prove that $\max|g(\tau_k)| > ch$. It would be true, if we find a function $\tilde{g}$ such that

$$\max|\tilde{g}(\tau_k)| \geq ch; \qquad \max|g(\tau_k) - \tilde{g}(\tau_k)| = o(h). \qquad (7.6)$$

We shall define this function by the formula

$$\tilde{g} = \mathcal{K}\Pi_h\mathcal{K}^{-1}(\mathcal{K} + \mathcal{K}_1)f. \qquad (7.7)$$

If $f \in X_h$, then $\Pi_h f = f$. So, introducing the notation $\psi = \mathcal{K}^{-1}\mathcal{K}_1 f$ and taking into account that $\mathcal{K}(f + \psi) = g$, we can rewrite (7.7) in the form

$$\tilde{g} = \mathcal{K}\Pi_h(f + \psi) = \mathcal{K}[f + \psi + (\Pi_h\psi - \psi)] \equiv g + \mathcal{K}(\Pi_h\psi - \psi). \qquad (7.8)$$

In order to substantiate this construction and to verify properties (7.6), let us prove two lemmas.

Lemma 9. *If $\psi \in C^{0,\lambda}$, then for every gridpoint τ_k*

$$|\mathcal{K}\Pi_h\psi(\tau_k) - \mathcal{K}\psi(\tau_k)| < ch^{\lambda+1}|\log h|. \qquad (7.9)$$

Proof. To be definite, let us consider the gridpoint τ_0. Since

$$\psi(\tau) = \sum_j \psi(\tau)\varphi_j(\tau),$$

we can write

$$\delta \equiv |\mathcal{K}\Pi_h\psi(\tau_0) - \mathcal{K}\psi(\tau_0)| = \Big|\sum_j \int_{\Delta_j} [\psi(\tau) - \bar{\psi}_j]\varphi_j(\tau)\log\frac{1}{|\tau|}\,d\tau\Big|$$

$$\leq \sum_j \Big|\int_{\Delta_j} [\psi(\tau) - \bar{\psi}_j]\varphi_j(\tau)\log\frac{1}{|\tau|}\,d\tau\Big|.$$

Because of (7.5), if we subtract the constant from logarithm in the latter integral, its value does not vary, so

$$\delta \leq \sum_j \Big|\int [\psi(\tau) - \bar{\psi}_j]\varphi_j(\tau)\Big[\log\frac{1}{|\tau|} - \log\frac{1}{|\tau_j|}\Big]\,d\tau\Big|.$$

(If $j = 0$, we do not subtract anything.) Now, if $j = -1, 0, 1$, then for the function from $C^{0,\lambda}$ we have $|\psi(\tau) - \bar{\psi}_j| \leq ch^\lambda$, therefore these integrals do not exceed the quantity

$$ch^{\lambda}\int_{\tau_{-2}}^{\tau_2}\log\frac{1}{|\tau|}\,d\tau \le ch^{1+\lambda}|\log h|.$$

If $|j|\ge 2$, then the difference of logarithms may be estimated as the derivative of logarithm multyplied by the length of the segment; so it does not exceed $(c/|\tau|)\times|\tau-\tau_j|\le c/|j|$ for $\tau\in\operatorname{supp}\varphi_j$.

So,

$$\delta\le ch^{1+\lambda}|\log h|+\sum_{|j|\ge 2}ch^{\lambda}\frac{c}{|j|}h\le ch^{1+\lambda}|\log h|,$$

because $\sum(1/|j|)\le c\log N\le c|\log h|$. □

Lemma 10. *The operator $\mathcal{K}$ has the inverse operator $\mathcal{K}^{-1}$, which is continuous from $C^{1,\lambda}$ in $C^{0,\lambda}$ for every $\lambda\in(0,1)$.*

Proof. Let us compare the operator $\mathcal{K}$ with the standard Hilbert operator on the unit circle (see [7]); it may be written after parametrization on the segment $[-1/2,1/2]$ in the form

$$\mathcal{L}_0u(t)\equiv\int_{-1/2}^{1/2}\log\frac{a}{|2\sin\pi(\tau-t)|}u(\tau)\,d\tau.$$

Its inverse operator $\mathcal{L}_0^{-1}$ exists under $a\ne 1$ and is continuous from $C^{1,\lambda}$ in $C^{0,\lambda}$; it is defined by the formula

$$\mathcal{L}_0^{-1}v(t)=-\frac{1}{\pi}\int_{-1/2}^{1/2}\cot\pi(\tau-t)v'(\tau)\,d\tau+\frac{1}{\log a}\int_{-1/2}^{1/2}v(\tau)\,d\tau. \tag{7.10}$$

Since the operator $\mathcal{K}$ can be written in the form $\mathcal{K}=\mathcal{L}_0+\mathcal{L}_1$, where $\mathcal{L}_1$ is the following operator:

$$\mathcal{L}_1u(t)\equiv\int_{-1/2}^{1/2}\log\frac{|2\sin\pi(\tau-t)|}{a|\tau-t|'}u(\tau)\,d\tau,$$

then the operator $\mathcal{K}$ is inversible, if and only if the operator $(I+\mathcal{L}_0^{-1}\mathcal{L}_1)$ is inversible; and then

$$\mathcal{K}^{-1}=(I+\mathcal{L}_0^{-1}\mathcal{L}_1)^{-1}\mathcal{L}_0^{-1}. \tag{7.11}$$

The operator $\mathcal{L}_0^{-1}\mathcal{L}_1$ is a compact one from $C^{0,\lambda}$ in $C^{0,\lambda}$, and so, according to the Fredholm theory, if the operator $(I+\mathcal{L}_0^{-1}\mathcal{L}_1)$ is not degenerate (i.e., if no one function is mapped into zero), then its inverse operator exists and is continuous. But this condition is equivalent to the condition that the operator $\mathcal{K}$ is not degenerate. So, the desired fact is true, if there does not exist such a function $\varphi \in C^{0,\lambda}$ that $\mathcal{K}\varphi \equiv 0$.

We can assume that $\max|\varphi| = 1$.

Then $|\Pi_h\varphi(\tau) - \varphi(\tau)| \leq ch^\lambda$, therefore $\max|\Pi_h\varphi(\tau)| \geq 1 - ch^\lambda$. But $\Pi_h\varphi(\tau)$ is a function from X_h, and the maximum of its modulus is not less than a constant; then, according to Theorem 1, $\max_j|\mathcal{K}\Pi_h\varphi(\tau_j)| > ch$.

From the other side, according to Lemma 9, the values of $\mathcal{K}\varphi$ at the gridpoints differ from the values of $\mathcal{K}\Pi_h\varphi$ by the quantity which is $o(h)$. So, $\max|\mathcal{K}\varphi(\tau_k)| > ch$. This implies the propetry of $\mathcal{K}$ to be not degenerate, and therefore, according to (7.11), the operator $\mathcal{K}^{-1}$ exeists and has the same properties as $\mathcal{L}_0^{-1}$, i.e., it is continuous from $C^{1,\lambda}$ in $C^{0,\lambda}$. □

Remark. According to the general theory (see [8]), the operator $(I + \mathcal{L}_0^{-1}\mathcal{L}_1)^{-1}$ is also the integral operator of the second kind with a weak singularity in kernel.

As we can see after this Lemma, the definition of $\tilde{g}$ is correct. It is easy to verify that the operator $\mathcal{K}_1$ with a kernel of the form (7.2) maps the function $f \in X_h$ on the function with the modulus of derivative of order $h|\log h|$, hence $\mathcal{K}_1 f$ belongs to $C^{1,\lambda}$ for every $\lambda \in [0,1]$, and $\psi \equiv \mathcal{K}^{-1}\mathcal{K}_1 f \in C^{0,\lambda}$. As the consequence of (7.8) and Lemma 9, the second inequality in (7.6) is valid.

As to the first inequality (7.6), let us take into account that the operator $(\mathcal{K} + \mathcal{K}_1)$ is inversible according to the conditions of Theorem 2, so the operator $\mathcal{K}^{-1}(\mathcal{K} + \mathcal{K}_1) = (I + \mathcal{K}^{-1}\mathcal{K}_1)$ is inversible too. This latter operator is a Fredholm operator of a second kind, therefore its inversibility in the Hölder spaces implies its inversibility in uniform norms. So, $\max|\mathcal{K}^{-1}(\mathcal{K}+\mathcal{K}_1)f(\tau) \geq const$, and this function is of class $C^{0,\lambda}$. Therefore, its proection after applying Π_h differs from the function itself by the small quantity, according to Lemma 9, hence the uniform norm of $\Pi_h\mathcal{K}^{-1}(\mathcal{K}+\mathcal{K}_1)f$ is not less than a constant. But it is a function from X_h, i.e., a function of a form (2.5), and, applying Theorem 1 for it, in the view of (7.7), we get the first inequality (7.6).

So, Theorem 2 is proved.

Finally we should remark that there was the only point where the restriction for the quasiuniform parameter played its role. Namely, it is a latter statement of Lemma 1, concerning the discrete convexity of the image of the basic function at the neighbouring gridpoint. Indeed, this restriction is not connected with the essence of the problem; this is due only to a technique of our consideration. In order to weaken this restriction or to avoid it completely, we can vary the definition of discrete convexity. Namely, we can use as a criterium of convexity the positiveness of some combinations of second differences. This approach is not fulfilled completely.

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