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The relative strength of topological properties for event structures

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The intention of the paper is to characterize and examine density and crossing properties of prime event structures. We show the coincidence of L-density and L-crossing for this class of event structures. It has turned out that any configuration in an M-dense prime event structure is full (i.e., at least one successor (if it exists) for any event occuring in the configuration must also occur). The structural restrictions which guarantee M-density are determined.

1. Introduction

Behavioral properties of concurrent processes can be formulated and studied in terms of acyclic Petri nets, posets, event structures, e.a., as evidenced in the literature [2, 6, 11]. However, not all instances of these models are suitable for the purpose of an adequate representation of "reasonable" concurrent processes. In [1, 9, 10] and earlier works, a number of properties has been proposed to be satisfied by suitable classes of the models. K-density and crossing properties have been defined very nicely and elegantly for causal nets and posets. L- and M-density as special properties for generalized processes represented by acyclic Petri nets with nondeterministic choices have been introduced in [7].

The strength and significance of these properties are not self-evident for event structures which are reminiscent of many partial ordering models. The advantage of event structures is that the nondeterministic aspects of concurrent processes are explicitly described and the choices can be naturally expressed. Some modifications of N-density for flow event structures in the context of algebraic specifications are developed in [3].

In these notes, we try to convey an understanding and evaluate the power and limitations of density and crossing properties for prime event structures (here, event structures for the sake of brevity).

The notes are organized as follows. In Section 2 we shortly recall some basic notions of event structures. A number of density and crossing properties for event structures is treated in Section 3. As could be assumed the finiteness of event structures guarantees their K- and L-crossing properties. The coincidence of L-density with L-crossing for event structures is established. We show that in M-dense event structure all of the executions are completely "successful", i.e., at least one successor (if it exists) for any event occurring in the execution must also occur. The structural restrictions which imply M-density of event structures are determined. In Section 4 we end with concluding remarks where the future lines of research are pointed out.

2. Event structures

Our framework is event structures introduced by Nielsen, Plotkin and Winskel in [8] as a model for computational processes. Event structures are represented via sets of events with relations expressing causal dependencies and conflicts between them. The subsets of events representing executions in the event structure are called configurations. They have to be conflict-free and left-closed with respect to $\leq$ (all prerequisites for any event occurring in the execution must also occur).

Definition 2.1. An event structure is a triple $S = (E, \leq, \#)$, where

- E is a set of events,
- $\leq \subseteq E \times E$ is a partial order (the causality relation) satisfying the principle of finite causes: $\forall e \in E : \{d \in E \mid d \leq e\}$ is finite,
- $\# \subseteq E \times E$ is a symmetrical and irreflexive relation (the conflict relation) satisfying the principle of conflict heredity: $\forall e_1, e_2, e_3 \in E : e_1 \leq e_2 \;\&\; e_1 \# e_3 \Rightarrow e_2 \# e_3$.

The components of an event structure S will be denoted by $E_S, \leq_S$ and $\#_S$. If clear from the context, the index S will be omitted. Let **S** denote the domain of event structures. An event structure $S = (E, \leq, \#)$ is *finite* iff E is finite, S is *conflict-free* (CFF-structure) iff $\# = \emptyset$, i.e., $\forall e, e' \in E : \neg(e \# e')$, S is *concurrency-free* (COF-structure) iff $co = \emptyset$, i.e., $\forall e, e' \in E : \neg(e \; co \; e')$.

Let $S = (E, \leq, \#)$ be an event struture, then $id = \{(e, e) \mid e \in E\}$; $< \; = \; \leq \setminus id$; $\leq^2 \subseteq \leq$ (transitivity); $\lessdot \; = \; < \setminus <^2$; $li = \leq \cup \geq$; $co =$

$(E \times E) \setminus (< \cup > \cup \#)$; ${}^{\bullet}e = \{e' \in E \mid e' \lhd e\}$ and $e^{\bullet} = \{e' \in E \mid e \lhd e'\}$; $e \#_1 d$ iff $e \# d$ & $\forall e', d' \in E : e' \leq e$ & $d' \leq d$ & $e' \# d' \Rightarrow e' = e$ & $d' = d$.

Definition 2.2. Let $S = (E, \leq, \#)$ be an event structure and $C \subseteq E$. Then

- C is left-closed iff $\forall e, d \in E : e \in C$ & $d \leq e \Rightarrow d \in C$.
- C is conflict-free iff $\forall e, e' \in C : \neg(e \# e')$.
- C is a configuration of S iff C is left-closed and conflict-free.
- A configuration C is complete iff $\forall d \in E : d \notin C \Rightarrow \exists e \in C : d \# e$.
- A configuration C is maximal iff for any configuration C' of S such that $C \subseteq C'$ is valid that $C = C'$. Let $\mathbf{C}(S)$ denote the set of maximal configurations of S.
- A maximal configuration C is full iff $\forall e \in C : e^{\bullet} \neq \emptyset \Rightarrow e^{\bullet} \cap C \neq \emptyset$.

Now we establish the fullness property of maximal configurations of CFF- and COF-structures.

Lemma 2.3. *Let $S = (E, \leq, \#)$ be an event structure and C be a maximal configuration of S. Then C is full if either condition holds:*
(i) S is a CFF-structure,
(ii) S is a COF-structure.

Proof. *(i)* Straightforward. *(ii)* We suppose the contrary. Let $C \in \mathbf{C}(S)$ be not a full configuration. This means that there exists an event $e \in C$ such that $e^{\bullet} \neq \emptyset$ and for any $e' \in e^{\bullet}$ is valid that $e' \notin C$. Since C is a maximal configuration then there exists an event $d \in C$ such that $d \# e'$. Proceeding from the definition of COF-structure we have $d < e$. Since $d \# e'$ and $d < e \lhd e'$ we get a contradiction. □

We introduce now some auxiliary notions which will be useful throughout the paper.

Definition 2.4. Let $S = (E, \leq, \#)$ be an event structure.

- $L \subseteq E$ is a li-set iff $\forall e_1, e_2 \in L : e_1 \; li \; e_2$, L is a li-section iff L is a maximal li-set. The set of li-sections in S is denoted by $LI(S)$.
- $R \subseteq E$ is a co-set iff $\forall e_1, e_2 \in R : e_1 \; co \; e_2$, R is a co-section iff R is a maximal co-set. The set of co-sections is denoted by $CO(S)$.

- $A \subseteq E$ is a cf-set iff $\forall e_1, e_2 \in A : e_1 \# e_2$, A is a cf-section iff A is a maximal cf-set. The set of cf-sections is denoted by $CF(S)$.

For an event structure S we give some informal comments concerning substructures of S. *CFF-substructures* of S contain only concurrent events and events to be in the causal relation. Hence, maximal CFF-substructures of S are closely associated with maximal configurations of S in an obvious way. The set of maximal CFF-substructures of S characterizes the projection of S to a plane formed by li- and co-axes, whereas *COF-substructures* of S contain only conflicting events and events to be in the causal relation. The set of maximal COF-substructures of S represents the projection of S to a plane formed by cf- and co-axes.

Definition 2.5. Let $S = (E, \leq, \#)$, $S' = (E', \leq', \#')$ be event structures.

- S' is a substructure of $S (S' \subseteq S)$ iff $E' \subseteq E$, $\leq' \subseteq \leq \cap E'^2$, $\#' \subseteq \# \cap E'^2$.
- S' is a CFF-substructure (COF-substructure) of S iff S' is a CFF-structure (COF-structure) and a substructure of S.
- S' is a maximal CFF-substructure (maximal COF-substructure) of S if for any CFF-substructure (COF-substructure) S'' of S such that $S' \subseteq S''$ is valid that $S' = S''$.

It is known that in any event structure there is no infinite li-section, be it ascending or descending between any pair of events, i.e., event structures are *discrete* models of processes. The following auxiliary lemma exhibits this fact and connects up any cf-section of an event structure with its maximal COF-substructure.

Lemma 2.6. *Let* $S = (E, \leq, \#)$ *be an event structure. Then*
(i) $\forall e_1, e_2 \in E, \forall L \in LI(S) : |\, [e_1, e_2] \cap L \,| < \infty$,
where $[e_1, e_2] = \{e \in E \mid e_1 \leq e \leq e_2\}$.
(ii) $\forall A \in CF(S)$: *there exists a maximal COF-substructure* $S' = (E', \leq', \#')$ *of* S *such that* $A \subseteq E'$.

Proof. *(i)* follows from the finite causality axiom of an event structure. *(ii)* Straightforward. □

In such a way, we have recalled basic terminology of event structures and defined some additional notions needed to introduce density and crossing concepts for event structures.

3. Density and crossing properties of event structures

First we rephrase the concepts of K-, N-density and, the so-called, K-crossing [10] in terms of event structures. K-density (N-density,K-crossing) of an event structure is defined by K-density (N-density,K-crossing) of every of its maximal CFF-substructure. Before formalizing these notions it will be convenient to adopt the following notations.

Let $S = (E, \leq, \#)$ be an event structure and $X \subseteq E$. Then $\downarrow X = \{e' \mid e' \in E \ \& \ \exists e \in X : e' \leq e\}$, and $\uparrow X = \{e' \mid e' \in E \ \& \ \exists e \in X : e \leq e'\}$.

Definition 3.1. Let $S = (E, \leq, \#)$ be an event structure and $S' = (E', \leq')$ be a maximal CFF-substructure of S. Then

- S' is K-dense iff $\forall L \in LI(S'), \forall R \in CO(S') : \mid L \cap R \mid = 1$.
- S' is N-dense iff $\forall e_0, e_1, e_2, e_3 \in E$: if ($e_0 < e_1$ & e_0 *co* e_2) and ($e_2 < e_3$ & e_1 *co* e_3) then $e_0 < e_3 \Rightarrow e_2 < e_1$.
- S' is K-crossing iff $\forall L \in LI(S'), \forall R \in CO(S') : L \cap \downarrow R \neq \emptyset \ \& \ L \cap \uparrow R \neq \emptyset$.
- S is K-dense (N-dense, K-crossing) iff any maximal CFF-substructure of S is K-dense (N-dense, K-crossing).

The next result states a connection between the properties defined above.

Proposition 3.2. *Let $S = (E, \leq, \#)$ be an N-dense event structure. Then S is K-dense, iff S is K-crossing.*

PROOF. It follows from Theorem 2.3.11 in [2] and part *(i)* of Lemma 2.6. □

The following proposition establishes some basic relations between several finiteness and K-crossing.

Proposition 3.3. *Let $S = (E, \leq, \#)$ be an event structure. Then S is K-crossing, if either condition holds:*
(i) *S is finite,*
(ii) *any li-section in S is finite,*
(iii) *any co-section in S is finite.*

Proof. *(ii)* follows from Proposition 3.2 (5) in [10]. *(iii)* follows from Proposition 3.2 (6,8) in [10]. *(ii)* and *(iii)* imply *(i)*. □

Now we formulate the definitions of L-density and L-crossing properties of event structures as follows.

Definition 3.4. Let $S = (E, \leq, \#)$ be an event structure and $S' = (E', \leq', \#')$ be a maximal COF-substructure of S.

- S' is L-dense iff $\forall L \in LI(S'), \forall A \in CF(S') \mid A \neq \emptyset : \mid L \cap A \mid = 1$.
- S' is L-crossing iff $\forall L \in LI(S'), \forall A \in CF(S') \mid A \neq \emptyset : L \cap \downarrow A \neq \emptyset \ \& \ L \cap \uparrow A \neq \emptyset$.
- S is L-dense (L-crossing) iff any maximal COF-substructure S' of S is L-dense (L-crossing).

Proposition 3.5. *Let $S = (E, \leq, \#)$ be an event structure. Then S is L-dense iff S is L-crossing.*

Proof.
($\Rightarrow$): This is obvious.
($\Leftarrow$): Let $S' = (E', \leq', \#')$ be a maximal COF-substructure of S. Let $L \in LI(S')$, $A \in CF(S')$. According to the L-crossing property there exists a maximal element $e \in L \cap \downarrow A$ and a minimal element $d \in L \cap \uparrow A$. Assume $e \neq d$, then $e < d$.

- If $e \lhd d$, then there exist $a, a' \in A$ such that $e \leq a$ and $a' \leq d$.
  1. If $e = a = a'$ or $d = a = a'$, then the result is proved.
  2. If $a = a'$ and $e \neq a$, $d \neq a'$, then this contradicts the definition of $\lhd$.
  3. If $a \neq a'$, $e \neq a$ and $d \neq a'$, then it contradicts the definition of a COF-structure.
- If $\neg(e \lhd d)$, then there exists $a \in L$ such that $e < a < d$. If $a \in \downarrow A$, then it contradicts the maximality of e. If $a \in \uparrow A$, then it contradicts the minimality of d. But no other cases remain because $\uparrow A \cup \downarrow A = E$. Hence $e = d \in \downarrow A \cap \uparrow A = A$. □

A number of the structural restrictions which imply L-crossing is established by the following proposition.

Proposition 3.6. *Let $S = (E, \leq, \#)$ be an event structure. Then S is L-crossing, if either condition holds:*
(i) S is finite,
(ii) any cf-section in S is finite,
(iii) any li-section in S is finite.

PROOF. *(ii)* follows using the obvious fact that if any cf-section in S is finite, then $\forall L \in LI(S)$, $\forall A \in CF(S) : (L \subseteq \downarrow A \Rightarrow \exists a \in A$ such that $L \subseteq \downarrow\{a\})$ & $(L \subseteq \uparrow A \Rightarrow \exists a \in A$ such that $L \subseteq \uparrow\{a\})$.
(iii) We assume that S is not L-crossing. Then there exists a maximal non-L-crossing COF-substructure $S' = (E', \leq', \#')$ of S. Let $A \neq \emptyset \in CF(S')$, $L \neq \emptyset \in LI(S')$ such that $L \cap \uparrow A = \emptyset$ or $L \cap \downarrow A = \emptyset$. Then $L \subseteq \{\downarrow A\} \setminus A$, or $L \subseteq \{\uparrow A\} \setminus A$, respectively. Let $L = \{x_1, ..., x_n\}$.

- If $L \subseteq \{\downarrow A\} \setminus A$, then according to the definition of $\downarrow A$ there exists $a \in A$ such that $x_n < a$. Hence L is not a li-section.
- If $L \subseteq \{\uparrow A\} \setminus A$, then the proof is analogous.

(i) is a consequence of *(ii)* and *(iii)*. □

The concept of M-density was developed in [5, 7] as a nice property of generalized processes represented by acyclic (possibly infinite) Petri nets with conflicts. M-density is formulated in terms of the intersection of planes formed, on the one hand, by li- and co-sections and, on the other hand, by li- and cf-sections. The definition below rephrases this property by event structures as follows.

Definition 3.7. Let $S = (E, \leq, \#)$ be an event structure. Then S is M-dense iff the intersection of any maximal CFF-substructure of S with any maximal COF-substructure of S results in some (unique) li-section of S.

We establish now fullness of maximal configurations in an M-dense event structure. It has turned out that in L- and M-dense event structure the intersection of every of its maximal configuration with every of its cf-section contains one element.

Proposition 3.8. *Let $S = (E, \leq, \#)$ be an M-dense event structure. Then*
(i) any maximal configuration of S is full.
(ii) S is L-dense $\Rightarrow \forall C \in \mathbf{C}(S)$, $\forall A \in CF(S) : |\, C \cap A \,| = 1$.

PROOF. *(i)* It is straightforward to show that any li-section in any CFF-substructure or in any COF-substructure is a li-section in S as well. Hence

any maximal configuration of S is full.
(ii) We will prove by contradiction. Let C be a maximal configuration and A be a cf-section of S. Two cases are possible.

- If $C \cap A = \emptyset$. From the left-closed principle follows that $C \cap \uparrow A = \emptyset$. Clearly for any maximal configuration C there exists a maximal CFF-substructure $S' = (E', \leq', \#')$ such that $C = E'$. According to part *(ii)* of Lemma 2.6 for any cf-section A in S there exists a maximal COF-substructure $S'' = (E'', \leq'', \#'')$ such that $A \subseteq E''$. From M-density follows that $E'' \cap E'$ is a li-section L in S. Since $E' \cap \uparrow A = \emptyset$, we have $L \cap \uparrow A = \emptyset$. It means that for a maximal COF-substructure S'' is also valid that $L \cap \uparrow A = \emptyset$. This contradicts the L-crossing of S.

- If $\mid C \cap A \mid > 1$. This contradicts the definition of a maximal configuration. □

Now one may characterize the structural restrictions which define plain event structures for which M-density holds as shown below.

Definition 3.9. Let $S = (E, \leq, \#)$ be an event structure. S is called plain iff for all $e, d \in E$ such that $e \# d$ the following is valid:
(i) $\forall e', e'' \in \{d \mid d \# e\} : \neg(e' \; co \; e'')$,
(ii) $\forall e', e'' \in ({}^{\bullet}e \cup e) \cup ({}^{\bullet}d \cup d) : \neg(e' \; co \; e'')$.

Let us see some consequences of Definition 3.9. The first observation is that any event being in conflict with another one has unique immediate predecessor. Another consequence says that any two events being in immediate conflict have the same predecessors. These are formulated as follows.

Lemma 3.10. *Let $S = (E, \leq, \#)$ be a plain event structure. Then for all events $e, d \in E$ the following holds:*
(i) $e \# d \Rightarrow \mid {}^{\bullet}e \mid \leq 1 \; \& \; \mid {}^{\bullet}d \mid \leq 1$,
(ii) $e \#_1 d \Rightarrow ({}^{\bullet}e = {}^{\bullet}d) \; \& \; (\forall e', d' \in \uparrow e \cup \uparrow d : \neg(e' \; co \; d'))$.

PROOF. Straightforward. □

Proposition 3.11. *Let $S = (E, \leq, \#)$ be a plain event structure. Then S is M-dense.*

PROOF. This follows from Lemma 3.10. □

Now we establish the relationship between M-density and some properties introduced in [3].

Definition 3.12. Let $S = (E, \leq, \#)$ be an event structure.

- S is called $\bigtriangledown$-free if it does not contain the following: $\forall e, e', e'' \in E$: $e < e' \# e'' \; co \; e$.
- S is called $N^{\#}$-dense iff $\forall e_0, e_1, e_2, e_3 \in E$: if ($e_0 \# e_1$ & $e_0 \; co \; e_2$) and ($e_2 \# e_3$ & $e_1 \; co \; e_3$) then $e_0 \# e_3 \Rightarrow e_1 \# e_2$

Proposition 3.13. *Let $S = (E, \leq, \#)$ be an event structure.*
(i) If S is $\bigtriangledown$-free then any maximal configuration of S is full.
(ii) If S is $\bigtriangledown$-free and $N^{\#}$-dense then S is M-dense.

Proof. *(i)* Straightforward. *(ii)* Suppose S is not M-dense. Then there exist its maximal CFF-substucture $S' = (E', \leq', \#')$ and maximal COF-substructure $S'' = (E'', \leq'', \#'')$ such that $E' \cap E'' = l$ is not a li-section in S.

- Let $l = 0$. Then there exit at least four events $e_0, e_1, e_2, e_3 \in E$ such that $e_0, e_1 \in E'$ and $e_2, e_3 \in E''$, i.e., $e_0 \# e_2, e_1 \# e3$ and $e_2 \# e_3$. We get a contradition to $N^{\#}$-density of S.
- Let $l = \{e_0, ..., e_n\}$ is a li-set.
    1. If there exits an event e' such that $e' < e_0$ then $e', e_0 \in E'$ by Definition 2.5 and there is an event $e'' \in E''$ such that $e'' \# e_0$ and $e'' \; co \; e'$. This contradicts the $\bigtriangledown$-freeness of S.
    2. If there exits an event e' such that $e_n < e'$ then the reasoning is analogous to the one in the previous case. We get a contradiction to the principle of conflict heredity. □

4. Concluding remarks

In these notes we have tried to present some density and crossing properties of event structures and discuss why these properties might be useful.

The work presented here is by no means complete. Regarding future works, two lines of research may be pointed out. So far we have limited ourselves to prime event structures. We expect that our results may be generalized by the concept of flow event structures introduced in [4]. The second line of research to pursue should be to provide dense and crossing event structures with a concrete interpretation. This would enable us to evaluate the power and the weakness of these properties of event structures. A concrete interpretation would add some insight to the vague explanations of the various concepts which we have provided within the notes.

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