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Fundamental solutions of a wave operator in problems of field continuation*

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Abstract. In this paper, fundamental solutions of a wave operator in an inhomogeneous medium having the properties of advanced (anti-causal) type are analyzed. These are such functions that act for the future when the field is calculated at the present time. It is shown that the use of this anti-causal property when the wave field is continued from some observation surface allows, in principle, "to look" into the domain containing unknown wave field sources. On the observation surface, the trace of the field and its normal derivative are considered to be known. It is shown that for a certain structure of sources, in particular, for those instantaneously acting in time (Cauchy data), the solution to the inverse problem of reconstructing the sources is given simply by fixing the continued field at the time $t = 0$.

1. Introduction

Continuation of wave fields is an effective instrument for an approximate solution to inverse problems of practical value. In particular, in seismology and seismic prospecting, in which a wave process is generated by a pulse and develops in time, a time-reversed field continuation means that the field is calculated from the observation surface into the medium being investigated and back in time. This procedure of data processing has a clear analogy with the methods of continuation of potential fields in gravimetry, where the process of analytical continuation of the field of gravity has been well studied, but in contrast to gravimetry, in this case appears one more independent variable (time), and the physical process of wave propagation is described by the hyperbolic type equation. The physical basis of this approach is the Huygens principle as well as the possibility of its use for a field continuation both in the direct and in the opposite direction of time.

It should be noted that Yu.V. Timoshin, in his attempts to improve the methods of reflection survey data processing, was the first to indicate to this possibility as applied to seismic prospecting problems [1]. Only later, procedures of a similar type called "wave migration" came into use abroad.

However, despite a wide use of the algorithms based on a time-reversed field continuation, in the practice of processing of seismic prospecting reflection survey data, the literature still lacks a clear description of the functional relations existing between the true and the continued fields in an arbitrary

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inhomogeneous medium. Without a clear understanding of the foregoing, one cannot expect essential advances in the development of seismic methods to investigate complicated media.

In this paper, such functional relations between the true and the continued fields in an arbitrary inhomogeneous medium are established and analyzed from the standpoint of solving inverse problems of wave propagation theory. In other words, this paper gives the answer to the following question: what sort of information about the wave field source can be obtained if a trace of the wave field and its normal derivative on a certain observation surface covering this source are considered to be known.

2. Statement of the problem

A most adequate instrument for considering and analyzing such problems is the Green identity, which is well known in theoretical physics. The only feature that distinguishes the use of this apparatus in the problem of a wave field continuation from the conventional scheme used in problems of mathematical physics is the employment of the Green function (and, more precisely, a fundamental solution) of the *advanced* (*anti-causal*) type (see [2–4]):

$$\begin{aligned} &\Box G_+(\boldsymbol{x},\boldsymbol{\xi},t-\tau) = \delta(\boldsymbol{x}-\boldsymbol{\xi})\delta(t-\tau) \\ &\qquad\qquad \text{for } \boldsymbol{x}\in\mathbb{R}^3,\ t\in\mathbb{R}^1;\ \boldsymbol{\xi}\in D_0,\ \tau>0; \\ &G_+(\boldsymbol{x},\boldsymbol{\xi},t-\tau)\equiv 0 \qquad \text{for } \boldsymbol{x}\in\mathbb{R}^3,\ t>\tau. \end{aligned} \tag{1}$$

Here $\Box \equiv \Delta - c_0^{-2}\partial_{tt}^2$ denotes the wave operator acting on the field point $(\boldsymbol{x}, t)$, $(\boldsymbol{\xi}, \tau)$ are parameters; D_0 is a finite domain bounded by the curvilinear surface S_0 in $\mathbb{R}^3$, which is of interest for the investigation.

If we reverse the inequality sign in the second condition from (1), we will be able to determine a well-known (*causal*) *Green function of retarded type*:

$$G_-(\boldsymbol{x},\boldsymbol{\xi},t-\tau)\equiv 0 \qquad \text{for } t<\tau. \tag{2}$$

It should be noted that in the general case, the wave velocity in (1) and (2) is a known and specified variable, i.e., $c_0 \equiv c_0(\boldsymbol{x})$ is the velocity in the so-called reference model. Thus, everywhere in the text (unless otherwise specified) G is a fundamental solution to the wave equation in an inhomogeneous medium.

When $c_0 = \text{const}$ the geometrical meaning of these Green functions is illustrated in Figures 1 and 2. The function G_- is shown in Figure 1 and the function G_+ is shown in Figure 2.

In Figure 1, the inner sphere appears earlier in time and serves as a source of secondary Huygens waves, whose envelope causes the appearance

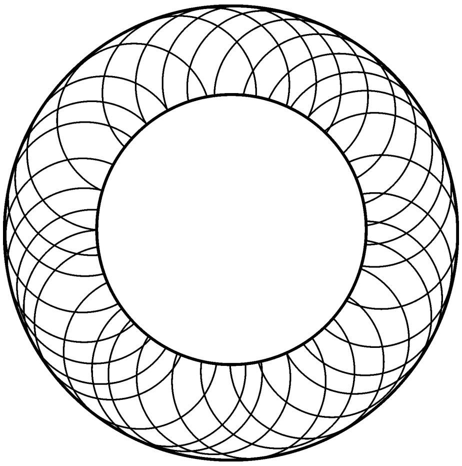

Figure 1

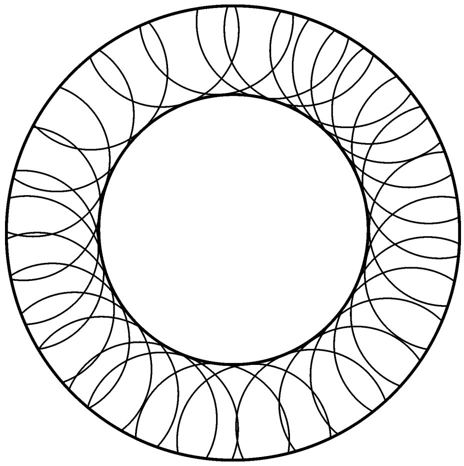

Figure 2

of the external sphere: thus, the wave process propagates from the source (in the sphere center) to infinity.

In Figure 2, the external sphere appears earlier in time and serves as a source of secondary Huygens waves whose envelope causes the appearance of the inner sphere at a later time: thus, the wave process propagates from infinity to the sphere center, in which there is an energy sink. After this "source" is triggered for $t > \tau$, the medium is at rest (*the source acting in reversed time is an energy sink acting in the direct (natural) direction of time*) [6].

From the computational standpoint, it is better to use a linear combination from (1) and (2), since the field has no jumps when it is continued through the observation surface:

$$G_*(\boldsymbol{x}, \boldsymbol{\xi}, t-\tau) = G_-(\boldsymbol{x}, \boldsymbol{\xi}, t-\tau) - G_+(\boldsymbol{x}, \boldsymbol{\xi}, t-\tau). \tag{3}$$

The fundamental solution G_*, which preserves properties of both the advanced and the retarded types, is determined as solution to the following Cauchy problem in the entire 4D space $\mathbb{R}^{3+1}$ [4, 5]:

$$\begin{aligned} &\Box G_*(\boldsymbol{x}, \boldsymbol{\xi}, t-\tau) = 0 \quad \text{for } \boldsymbol{x} \in \mathbb{R}^3,\ t \in \mathbb{R}^1;\ \boldsymbol{\xi} \in D_0,\ \tau > 0; \\ &G_*(\boldsymbol{x}, \boldsymbol{\xi}, 0) = 0, \quad \frac{\partial}{\partial t} G_*(\boldsymbol{x}, \boldsymbol{\xi}, 0) = -c_0^2 \delta(\boldsymbol{x} - \boldsymbol{\xi}) \quad \text{for } \boldsymbol{x} \in \mathbb{R}^3,\ t = \tau. \end{aligned} \tag{4}$$

The following statement of the wave radiation and propagation problem in $\mathbb{R}^{3+1}$ serves as basic model:

$$\begin{aligned} \left(\Delta - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\right) u(\boldsymbol{x}, t) &= f(\boldsymbol{x}, t) \qquad \text{for } \boldsymbol{x} \in \mathbb{R}^3,\ t > 0; \\ u(\boldsymbol{x}, t) &\equiv 0 \qquad\qquad \text{for } \boldsymbol{x} \in \mathbb{R}^3,\ t < 0. \end{aligned} \tag{5}$$

Here, the velocity, $c_0 \equiv c_0(\boldsymbol{x})$, as mentioned above, is considered to be a given function (in the general case, a medium is not assumed to be homogeneous).

The inverse problem is to determine the source function $f(\boldsymbol{x}, t)$ using the field $u_0(\boldsymbol{x}, t)$ and its normal derivative $\mu(\boldsymbol{x}, t) \equiv \partial_n u(\boldsymbol{x}, t)$ known on the closed surface S_0 containing a compact domain of a source $\overline{D} \Subset D_0$ (Figure 3).

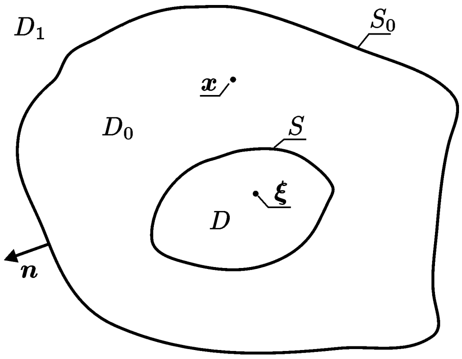

Figure 3

It should be noted that the union $D_0 \cup S_0 \cup D_1$ forms $\mathbb{R}^3$, problem (5) is stated for the entire space $\mathbb{R}^3$, and the observation surface S_0 in Figure 3 is a fictitious surface. On the latter, only the field trace u_0 and its normal derivative μ are "recorded", no boundary conditions are formulated. This refined statement was chosen deliberately to simplify further manipulations and, mainly, to call attention to the conceptual part of the problem of wave equation inversion in the idealized situation of maximally possible information about the source wave field. Actually, before passing on to more realistic statements, the following question should be answered: what can be said about the wave field source if this field, together with the normal derivative, is known at all points of the closed surface surrounding the source?

The Green identity (22), (23) from Appendix applied to a pair of functions, namely, u from (5) and G from (1), (2), or (4), gives a full answer to this question.

If G_- from (2) is taken as G, the Green formula determines the following functional relations:

$$w_-(\boldsymbol{x}, t) = \begin{cases} 0 & \text{for } \boldsymbol{x} \in D_0,\ t \in \mathbb{R}^1; \\ -u(\boldsymbol{x}, t) & \text{for } \boldsymbol{x} \in D_1,\ t \in \mathbb{R}^1. \end{cases} \tag{6}$$

If G_+ from (1) is taken as G, the Green formula determines the following functional relations:

$$w_+(\boldsymbol{x}, t) = \begin{cases} u(\boldsymbol{x}, t) - u_+(\boldsymbol{x}, t) & \text{for } \boldsymbol{x} \in D_0,\ t \in \mathbb{R}^1; \\ \frac{1}{2} u_0(\boldsymbol{x}, t) - u_+(\boldsymbol{x}, t) & \text{for } \boldsymbol{x} \in S_0,\ t \in \mathbb{R}^1; \\ -u_+(\boldsymbol{x}, t) & \text{for } \boldsymbol{x} \in D_1,\ t \in \mathbb{R}^1. \end{cases} \tag{7}$$

If G_* from (4) is taken as G, the Green formula gives a functional relation between the true and the continued fields in the following form:

$$w_*(\boldsymbol{x},t) = u(\boldsymbol{x},t) - u_+(\boldsymbol{x},t) \quad \text{for } \boldsymbol{x} \in \mathbb{R}^3,\ t \in \mathbb{R}^1. \tag{8}$$

In expressions (6)–(8), the field $u(\boldsymbol{x},t)$ is a true field, existing in the medium. It is determined as solution to problem (5) in the form of a volume potential distributed along the domain of sources D (see Figure 3):

$$u(\boldsymbol{x},t) = \int_D f(\boldsymbol{\xi},t) * G_-(\boldsymbol{x},\boldsymbol{\xi},t)\, dV_\xi. \tag{9}$$

Here and below the symbol $*$ is the time convolution operation.

The field $u_+(\boldsymbol{x},t)$ describes the contribution of the volume "sources" $f(\boldsymbol{x},t)$ radiating inverse in time, or energy sinks

$$u_+(\boldsymbol{x},t) = \int_D f(\boldsymbol{\xi},t) * G_+(\boldsymbol{x},\boldsymbol{\xi},t)\, dV_\xi. \tag{10}$$

3. Analysis of the solution

If we add a zero to the function $f(\boldsymbol{x},t)$ to the entire space $\mathbb{R}^{3+1}$ and take into account (3), (9), (10), functional relation (8) allows a representation in the form of an integral equation of the first kind:

$$\text{for } \boldsymbol{x} \in \mathbb{R}^3,\ t \in \mathbb{R}^1 \qquad w_*(\boldsymbol{x},t) = \int_{\mathbb{R}^3} f(\boldsymbol{\xi},t) * G_*(\boldsymbol{x},\boldsymbol{\xi},t)\, dV_\xi. \tag{11}$$

In a homogeneous medium $c_0 = \text{const}$ equation (11) takes the form of a 4D convolution in the space $\mathbb{R}^{3+1}$:

$$f(\boldsymbol{x},t) * G_*(\boldsymbol{x},t) = w_*(\boldsymbol{x},t),$$

or

$$\int_{-\infty}^{\infty} d\tau \int_{\mathbb{R}^3} f(\boldsymbol{\xi},\tau) G_*(\boldsymbol{x}-\boldsymbol{\xi}, t-\tau)\, d\boldsymbol{\xi} = w_*(\boldsymbol{x},t).$$

Here the kernel G_* is known in the explicit form [2–4] as solution to problem (4):

$$G_*(\boldsymbol{x},\boldsymbol{\xi},t-\tau) = -\frac{1}{4\pi}\left\{ |\boldsymbol{x}-\boldsymbol{\xi}|^{-1}\delta\left(t-\tau-\frac{|\boldsymbol{x}-\boldsymbol{\xi}|}{c_0}\right) - |\boldsymbol{x}-\boldsymbol{\xi}|^{-1}\delta\left(t-\tau+\frac{|\boldsymbol{x}-\boldsymbol{\xi}}{c_0}\right)\right\}.$$

In the case of a homogeneous medium, the fields (9)–(11) can have the following physical interpretation (Figures 4–6). In these figures, $\overline{B} \equiv \operatorname{supp} f(\boldsymbol{x},t)$ denotes a set in $\mathbb{R}^{3+1}$, where the source of waves is concentrated.

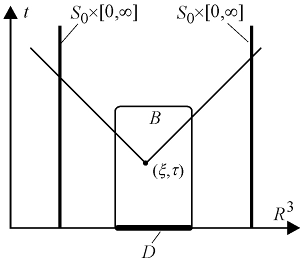

Figure 4

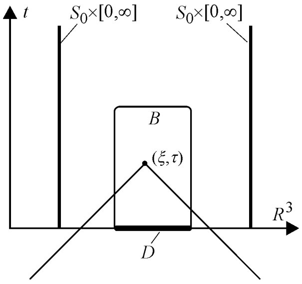

Figure 5

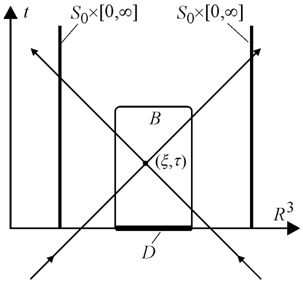

Figure 6

In Figure 4, each source point $(\boldsymbol{\xi}, \tau)$ radiates waves along the lateral surface of the "future" characteristic cone, which is the solution carrier $G_-(\boldsymbol{x}, \boldsymbol{\xi}, t-\tau)$.

In Figure 5, each "source" point $(\boldsymbol{\xi}, \tau)$ radiates waves along the lateral surface of the "past" characteristic cone which is the solution carrier $G_+(\boldsymbol{x}, \boldsymbol{\xi}, t-\tau)$. However, *the source acting back in time is an energy sink acting in the direct* (*natural*) *direction of time* (see Figure 2). Therefore, Figure 5 allows the following interpretation: the waves for $t < \tau$ are radiated from infinity, gather at the point $(\boldsymbol{\xi}, \tau)$, at which they subsequently vanish [6]. After such a "source" for $t > \tau$ is "triggered", the medium is at rest.

Figure 6 shows the choice of the fundamental solution $G_*(\boldsymbol{x}, \boldsymbol{\xi}, t-\tau)$. In this case, both parts of the characteristic cone, directed to the "past" and to the "future", are acting. On the basis of the above-said for energy sinks, Figure 6 demonstrates the field focusing process: *waves for $t < \tau$ are radiated from infinity and focused at the point $\boldsymbol{x} = \boldsymbol{\xi}$ for $t = \tau$. Then they again diverge in space* [6].

A full picture of the influence of the energy source (sink) can be obtained by uniting the corresponding cone boundaries with the help of the point $(\boldsymbol{\xi}, \tau)$ running through the set B.

In the case of an inhomogeneous medium, $c_0 \equiv c_0(\boldsymbol{x})$, the pictures presented in Figures 4–6 should be improved, since in this case, the characteristic cone becomes deformed (rays in an inhomogeneous medium become curvilinear) and, what is most important, in addition to the lateral surface, the inside of the corresponding cones will also act.

The field $w(\boldsymbol{x}, t)$ with the corresponding subscript in the left-hand side of equalities (6)–(8), (11) is considered to be known in $\mathbb{R}^{3+1}$ (since u_0 and μ are known) and is a continued field formed by a surface integral in the Green formula (see Appendix):

$$w_-(\boldsymbol{x}, t) = \int_{S_0} \left\{ u_0(\boldsymbol{\xi}, t) * \frac{\partial G_-}{\partial n_\xi}(\boldsymbol{x}, \boldsymbol{\xi}, t) - \mu(\boldsymbol{\xi}, t) * G_-(\boldsymbol{x}, \boldsymbol{\xi}, t) \right\} dS_\xi, \quad (12)$$

$$w_+(\boldsymbol{x},t) = \int_{S_0} \left\{ u_0(\boldsymbol{\xi},t) * \frac{\partial G_+}{\partial n_\xi}(\boldsymbol{x},\boldsymbol{\xi},t) - \mu(\boldsymbol{\xi},t) * G_+(\boldsymbol{x},\boldsymbol{\xi},t) \right\} dS_\xi, \quad (13)$$

$$w_*(\boldsymbol{x},t) = -\int_{S_0} \left\{ u_0(\boldsymbol{\xi},t) * \frac{\partial G_*}{\partial n_\xi}(\boldsymbol{x},\boldsymbol{\xi},t) - \mu(\boldsymbol{\xi},t) * G_*(\boldsymbol{x},\boldsymbol{\xi},t) \right\} dS_\xi. \quad (14)$$

The smoothness properties of the above functions considerably differ. Thus, $w_-(\boldsymbol{x},t)$, $w_+(\boldsymbol{x},t)$ are potentials of the double and the simple layers spread on the observation surface S_0. The properties of these potentials have been well studied in the classical theory of the Laplace equation. Now they are generalized to the case of the wave equation [7, 8]. It follows from these properties that the potentials are smooth, infinitely differentiable functions everywhere outside the surface S_0 on which they (when passing through it) have a discontinuity. The jumps of the field and its normal derivative determined from (6), (7),

$$\begin{aligned} &[w_-](\boldsymbol{x},t) = [w_+](\boldsymbol{x},t) = -u(\boldsymbol{x},t) \\ &[\partial_n w_-](\boldsymbol{x},t) = [\partial_n w_+] = -\mu(\boldsymbol{x},t), \end{aligned} \quad (15)$$

serve as a right-hand side for the wave equation of the "dipole" and "monopole" types distributed on the observation surface S_0 [9]. These fields satisfy the following conditions:

$$\begin{aligned} &\Box w_-(\boldsymbol{x},t) = -\frac{\partial}{\partial n}(u_0\delta(S_0)) - \mu\delta(S_0) \qquad \text{for } (\boldsymbol{x},t) \in \mathbb{R}^{3+1}; \\ &w_-(\boldsymbol{x},t) = 0 \qquad \text{for } \boldsymbol{x} \in \mathbb{R}^3,\ t < 0; \\ &w_-(\boldsymbol{x},t) = 0 \qquad \text{for } \boldsymbol{x} \in D_0,\ t \in \mathbb{R}^1; \end{aligned} \quad (16)$$

$$\Box w_+(\boldsymbol{x},t) = -\frac{\partial}{\partial n}(u_0\delta(S_0)) - \mu\delta(S_0) \qquad \text{for } (\boldsymbol{x},t) \in \mathbb{R}^{3+1}. \quad (17)$$

Here $\delta(S_0)$ denotes the surface delta-function [9].

The continued field $w_*(\boldsymbol{x},t)$ being their difference:

$$w_*(\boldsymbol{x},t) = w_+(\boldsymbol{x},t) - w_-(\boldsymbol{x},t). \quad (18)$$

It does not have any discontinuities (the field jumps are cancelled out), is everywhere in $\mathbb{R}^{3+1}$ a smooth function satisfying the following homogeneous wave equation:

$$\Box w_*(\boldsymbol{x},t) = 0 \quad \text{for } (\boldsymbol{x},t) \in \mathbb{R}^{3+1}. \quad (19)$$

Therefore, the use of $w_*(\boldsymbol{x},t)$ instead of $w_+(\boldsymbol{x},t)$ is attractive for a field continuation, although both algorithms contain the necessary field focusing, which, in principle, makes possible "to look" into the domain D_0 containing the sought for wave sources, in contrast to the field $w_-(\boldsymbol{x},t)$ which gives an identical zero in this case.

In the general case of an arbitrary $f(\boldsymbol{x},t)$, the solution to the basic integral equation of the first kind (11) is non-unique. A set of the sources distributed inside D_0 and not changing the field trace u_0 and its normal derivative μ (and, hence, the continued field w_* from (14)) are the so-called "non-radiating" sources: they do not radiate in terms of an observer on the closed surface S_0 but are different inside D_0 [10, 11].

For sources with a special structure, in particular, those acting instantaneously in time (Cauchy data)

$$f(\boldsymbol{x},t) = -c_0^{-2} f(\boldsymbol{x})\delta'(t), \tag{20}$$

the solution to the inverse problem is unique [12]. In this case, the additional field $u_+(\boldsymbol{x},t)$ from (10) is different from zero only for $t < 0$ and is an odd continuation of the true field $u(\boldsymbol{x},t)$ to negative time: $u_+(\boldsymbol{x},t) = -u(\boldsymbol{x},-t)$. As a result, functional relation (8) between the true and the continued fields takes the following form:

$$w_*(\boldsymbol{x},t) = \begin{cases} u(\boldsymbol{x},t) & \text{for } \boldsymbol{x} \in \mathbb{R}^3, t > 0 \\ u(\boldsymbol{x},-t) & \text{for } \boldsymbol{x} \in \mathbb{R}^3, t < 0, \end{cases}$$

which determines the function $w_*(\boldsymbol{x},t)$ that is even with respect to t.

The sought for solution to the inverse problem is given by fixing the continued field at the time $t = 0$:

$$w_*(\boldsymbol{x},0) = u(\boldsymbol{x},0) = f(\boldsymbol{x}), \qquad \partial_t w_*(\boldsymbol{x},0) = \partial_t u(\boldsymbol{x},0) = 0. \tag{21}$$

Appendix. The Green identity

The Green identity is valid for functions u and G satisfying the wave equation in $\mathbb{R}^{3+1}$. One of these functions, namely, the function G from (1),)2), or (4) is a fundamental solution. Let us present the Green identity in the following form [2, 3, 7, 9]:

$$\begin{aligned} \kappa u(\boldsymbol{x},t) &- \int_{D_0} \Box u(\boldsymbol{\xi},t) * G(\boldsymbol{x},\boldsymbol{\xi},t)\, dV_\xi \\ &= \int_{S_0} \Big\{ u(\boldsymbol{\xi},t) * \frac{\partial G}{\partial n_\xi}(\boldsymbol{x},\boldsymbol{\xi},t) - \partial_n u(\boldsymbol{\xi},t) * G(\boldsymbol{x},\boldsymbol{\xi},t) \Big\}\, dS_\xi, \end{aligned} \tag{22}$$

where (see Figure 3)

$$\kappa = \begin{cases} 1 & \text{for } \boldsymbol{x} \in D_0, \\ 0 & \text{for } \boldsymbol{x} \in D_1, \\ \frac{1}{2} & \text{for } \boldsymbol{x} \in S_0. \end{cases}$$

Any of the solutions (G_- or G_+) can be taken as the function G in (22).

If we use G_* as the function G, $\kappa = 0$ everywhere for $\boldsymbol{x} \in \mathbb{R}^3$, the Green formula becomes simpler:

$$
-\int_{D_0} \Box u(\boldsymbol{\xi}, t) * G_*(\boldsymbol{x}, \boldsymbol{\xi}, t)\, dV_\xi
$$

$$
= \int_{S_0} \left\{ u(\boldsymbol{\xi}, t) * \frac{\partial G_*}{\partial n_\xi}(\boldsymbol{x}, \boldsymbol{\xi}, t) - \partial_n u(\boldsymbol{\xi}, t) * G_*(\boldsymbol{x}, \boldsymbol{\xi}, t) \right\} dS_\xi. \quad (23)
$$

It should be noted that the Green formulas (22), (23) are written down in the form that includes the initial conditions of the function u: they are transferred to the right-hand side of the wave operator $\Box u$ in the form of instantaneously acting sources (which can always be done [9]). The symbol $*$ in (22), (23) denotes, as usual, the time convolution operation.

Expressions (22), (23) clearly show the discontinuous character of the field determined with the use of the fundamental solutions G_- and G_+ and the smooth properties of the field formed with the help of the fundamental solution G_*.

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