An investigation of equivalence notions on some subclasses of Petri nets*

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In this paper a variety of Petri net equivalences is examined. A correlation of all the considered equivalences is established, and a lattice of implications is obtained. In addition, the equivalences are treated for some subclasses of Petri nets: sequential nets, T-nets and nets with strict labelling.

1. Introduction

In recent years, a wide range of semantic equivalences were defined and investigated in concurrency theory. In linear time semantics, where a process is fully determined by the set of its possible (partial) runs, interleaving, step and pomset trace equivalences [3] are known.

In branching time semantics the information is preserved where two courses of actions diverge. Bisimulation is a fundamental behavioural equivalence in this semantics. Interleaving [6], step [5], partial word [11], pomset [4] and process [1] bisimulation equivalences were proposed in the literature.

(Interleaving) ST-bisimulation equivalence [4] respects the duration of transition occurrences. A definition of the equivalence was extended to partial words and pomsets in [11].

(Pomset) history preserving bisimulation equivalence, which respects the “past” of the processes, was first defined in [8] under the name “bisimulation equivalence of behaviour structures”.

In this paper the above mentioned definitions are supplemented by partial word history preserving and by process (ST- and history preserving) bisimulation equivalences. The equivalences are considered in the framework of Petri nets with finite processes. A correlation of all the equivalences is examined on usual Petri nets and their subclasses: sequential nets, T-nets and strictly labelled nets.

In Section 2 the basic definitions are given. Trace equivalences are described in Section 3. Bisimulation equivalences are presented in Section 4. In Section 5 the theorem establishing a correlation of all the introduced equivalences is proved.

Section 6 is devoted to the examination of the equivalences on different net subclasses. The concluding Section 7 contains some ideas about further development of the theme. Most of the proofs are omitted because of absence of space. The early results can be found in [9].

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2. Basic definitions

2.1. Multisets

Let X be some set. A multiset M over X is a mapping M : X → N, where N is a set of natural numbers. For x ∈ X, M(x) is a multiplicity x in M. We write x ∈ M if M(x) > 0.

When ∀x ∈ X : M(x) ≤ 1, M is a proper set. M is finite if M(x) = 0 for all x ∈ X, except maybe a finite number of them. Cardinality of multiset M is defined in such a way: |M| = \(\sum_{x \in X} M(x)\). From now on we will consider only finite multisets. M(X) denotes the set of all finite multisets over X.

Set-theoretic notions are extended to finite multisets in the standard way. If M, M' ∈ M(X), we define M + M' by (M + M')(x) = M(x) + M'(x). We write M ⊆ M', if ∀x ∈ X : M(x) ≤ M'(x). When M' ⊆ M, we define M − M' by (M − M')(x) = M(x) − M'(x). Notation M + x − y is used instead of M + {x} − {y}. We write symbol ∅ for empty multiset.

2.2. Marked nets

Let \(\mathcal{A} = \{a, b, \ldots\}\) be an alphabet of action names (labels) A labelled net is a quadruple N = \(\langle P_N, T_N, F_N, l_N \rangle\), where:

- \(P_N = \{p, q, \ldots\}\) is a set of places;
- \(T_N = \{u, v, \ldots\}\) is a set of transitions;
- \(F_N : (P_N \times T_N) \cup (T_N \times P_N) \rightarrow N\) is the flow relation with weights;
- \(l_N : T_N \rightarrow \mathcal{A}\) is a labelling of transitions with action names.

It is believed that \(P_N \cap T_N = \emptyset\).

Given a labelled net N and some transition u ∈ T_N, the precondition and postcondition u, written respectively *u and u*, are the multisets defined in such a way: \((*u)(p) = F_N(p, u)\) and \((u*)(p) = F_N(u, p)\). Analogous definitions are introduced for places: \((*p)(u) = F_N(u, p)\) and \((p*)(u) = F_N(p, u)\). A transition u is unstable if *u = ∅*. A labelled net is stable if it has no unstable transitions. Further we will deal only with stable labelled nets. A labelled net N is ordinary if ∀p ∈ P_N : *p and p* are proper sets. A labelled net N is finite if \(P_N \cup T_N\) is. Let \(N^0 = \{p ∈ P_N \mid *p = \emptyset\}\) is a set of initial places of N and \(N^* = \{p ∈ P_N \mid p* = \emptyset\}\) is a set of final places of N.

Let N be a labelled net. A marking of N is a multiset M ∈ M(P_N). A marked net is a tuple N = \(\langle P_N, T_N, F_N, l_N, M_N \rangle\) such that \(\langle P_N, T_N, F_N, l_N \rangle\) is a labelled net and \(M_N ∈ M(P_N)\) is an initial marking. We write “net” instead of “marked net”. Let M ∈ M(P_N) be a marking of a net N. A transition u ∈ T_N is firable in M if *u ∈ M. If u is firable in M, firing it yields a new marking \(M' = M − *u + u^*\), written \(M \xrightarrow{u} M'\). We write \(M \rightarrow M'\) if \(M \xrightarrow{u} M'\) for some u. A marking M' of a net N is reachable from marking M of the net, if:

1) \(M' = M\), or
2) there exists a reachable from M marking \(M''\) of a net N, such that \(M'' \rightarrow M'\).
A marking \( M \) of a net \( N \) is reachable, if it is reachable from \( M_N \). \( \text{Mark}(N, M) \) denotes a set of all reachable from \( M \) markings of a net \( N \), and \( \text{Mark}(N) \) denotes a set of all reachable markings of a net \( N \).

An action \( a \in \mathbb{A} \) is autoconcurrent in \( N \) if \( \exists M \in \text{Mark}(N) \exists t, u \in T_N \) such that \( l_N(u) = \pi_N(t) = a \) and \( *t + *u \subseteq M \). A net \( N \) is autoconcurrent if no action is autoconcurrent in \( N \).

2.3. Processes

A causal net is a labelled net \( C = (P_C, T_C, F_C, l_C) \), where:

1) \( \forall \rho \in P_C \ |\rho| \leq 1 \) and \( |\rho^*| \leq 1 \), i.e., places are unbranched and \( C \) is an ordinary labelled net;

2) \( F_C \) is well-founded, i.e., there is no backward infinite chain \( \ldots (r_n, v_n) (r_{n-1}, v_{n-1}) \cdots (r_1, v_1) (r_0, v_0) \) in \( F_C \).

The fundamental property of causal nets is known: if \( C \) is a causal net, then there exists a transition sequence \( *C = L_0 \xrightarrow{a_1} \cdots \xrightarrow{a_n} L_n = C^* \) such that \( L_i \subseteq P_C \ (0 \leq i \leq n) \), \( P_C = \bigcup_{i=0}^{n} L_i \) and \( T_C = \{ v_1, \ldots, v_n \} \). Such a sequence is called a full execution of \( C \).

Given a net \( N \) and a causal net \( C \). A mapping \( f : P_C \cup T_C \rightarrow P_N \cup T_N \) is an embedding \( C \) into \( N \), written \( f : C \rightarrow N \), if:

1) \( f(P_C) \in \mathcal{M}(P_N) \) and \( f(T_C) \in \mathcal{M}(T_N) \);

2) \( \forall v \in T_C \ l_C(v) = l_N(f(v)) \);

3) \( \forall v \in T_C \ *f(v) = f(*v) \) and \( f(v)^* = f(v^*) \).

Point 3 means that embeddings respect the flow relation. Consequently, if \( *C \xrightarrow{a_1} \cdots \xrightarrow{a_n} C^* \) is a full execution of \( C \), then \( M = f(*C) \xrightarrow{f(a_1)} \cdots \xrightarrow{f(a_n)} f(C^*) = M' \) is a transition sequence in \( N \), corresponding to this full execution, written \( M \xrightarrow{C} M' \). Conversely, for any transition sequence \( M \xrightarrow{a_1} \cdots \xrightarrow{a_n} M' \) of a net \( N \) there exists a causal net \( C \) and an embedding \( f : C \rightarrow N \) such that \( M = f(C) \), \( M' = f(C^*) \), \( u_i = f(v_i) \ (0 \leq i \leq n) \) and \( *C \xrightarrow{a_1} \cdots \xrightarrow{a_n} C^* \) is a full execution of \( C \).

A firable in marking \( M \) process of a net \( N \) is a pair \( \pi = (C, f) \), where \( C \) is a causal net and \( f : C \rightarrow N \) is an embedding such that \( M = f(C) \). A firable in \( M_N \) process is a process of \( N \). We write \( \Pi(N, M) \) for a set of all firable in \( M \) processes of \( N \). We write \( \Pi(N) \) for a set of all processes of \( N \). Processes and reachable markings of a net \( N \) are connected in the following way: \( \text{Mark}(N, M) = \{ f(C^*) | \pi = (C, f) \in \Pi(N, M) \} \). Further we will deal only with finite processes, i.e., with processes having finite causal nets.

If \( \pi \in \Pi(N, M) \), then firing of this process transforms a marking \( M \) into \( M' = M - f(*C) + f(C^*) = f(C^*) \), written \( M \xrightarrow{\pi} M' \). A causal net sets an ordering on transitions (the causal dependence relation) \( \prec_C \), defined in such a way: \( \prec_C = F_C^+ \cap T_C \times T_C \), where \( F_C^+ \) is a transitive closure of \( F_C \). The initial process of a net \( N \) is \( \pi_N = (C_N, f) \in \Pi(N) \), where \( T_{C_N} = \emptyset \). Let \( \pi = (C, f) \), \( \tilde{\pi} = (\tilde{C}, \tilde{f}) \in \Pi(N) \), \( \tilde{\pi} = (\tilde{C}, \tilde{f}) \in \Pi(N, f(C^*)) \), \( C = (P_C, T_C, F_C, l_C) \), \( \tilde{C} = (P_C, T_C, F_C, l_C) \), \( C' = (P_{C'}, T_{C'}, F_{C'}, l_{C'}) \).
We write \( \pi \xrightarrow{\delta} \hat{\pi} \), if:

1) \( P_C \cup P_C = P_C \); \( T_C \cup T_C = T_C \); \( F_C \cup F_C = F_C \); \( F_C \cup l_C = l_C \);

2) \( f \cup \hat{f} = \hat{f} \).

In such a case \( \hat{\pi} \) is an extension of \( \pi \) by process \( \hat{\pi} \), and \( \hat{\pi} \) is an extending process for \( \pi \). We write \( \pi \rightarrow \hat{\pi} \), if \( \pi \xrightarrow{\delta} \hat{\pi} \) for some extending process \( \hat{\pi} \).

Let \( \pi \xrightarrow{a} \hat{\pi} \). A process \( \hat{\pi} \) is an extension of \( \pi \) by one action, if \( |T_C| = 1 \). In such a case we write \( \pi \xrightarrow{a} \hat{\pi} \) or \( \pi \xrightarrow{\{a\}} \hat{\pi} \), if \( T_C = \{a\} \) and \( l_C(v) = a \). A process \( \hat{\pi} \) is an extension of \( \pi \) by multiset of actions, or step, if \( \prec_C = \emptyset \). In such a case we write \( \pi \xrightarrow{\{a\}} \hat{\pi} \) or \( \pi \xrightarrow{\delta} \hat{\pi} \), if \( T = V_C \) and \( l_C(T_C) = A \); \( A \in \mathcal{M}(\mathfrak{N}) \).

### 2.4. Mappings

Given nets \( N = (P_N, T_N, F_N, l_N, M_N) \) and \( N' = (P_{N'}, T_{N'}, F_{N'}, l_{N'}, M_{N'}) \). We call \( \beta \) a mapping of \( N \) into \( N' \), written \( \beta : N \rightarrow N' \), if \( \beta : P_N \cup T_N \rightarrow P_{N'} \cup T_{N'} \), \( \beta(P_N) \subseteq P_{N'} \) and \( \beta(T_N) \subseteq T_{N'} \). We write \( \beta(N) = N' \), when \( \beta(P_N) = P_{N'} \) and \( \beta(T_N) = T_{N'} \).

A mapping \( \beta : N \rightarrow N' \) is an isomorphism between \( N \) and \( N' \), written \( \beta : N \cong N' \), if:

1) \( \beta \) is a bijection and \( \beta(N) = N' \);
2) \( \forall u \in T_N \quad l_N(u) = l_{N'}(\beta(u)) \);
3) \( \forall u \in T_N \quad \beta(\star(u)) = \beta(\star(u)) \) and \( \beta(u) = \beta(u) \star \).

Nets \( N \) and \( N' \) are isomorphic, written \( N \cong N' \), if there exists an isomorphism \( \beta : N \cong N' \).

Given two labelled causal nets

\[
C = (P_C, T_C, F_C, l_C) \quad \text{and} \quad C' = (P_{C'}, T_{C'}, F_{C'}, l_{C'}). 
\]

A mapping \( \beta : T_C \rightarrow T_{C'} \) is a label preserving bijection between \( T_C \) and \( T_{C'} \), written \( \beta : T_C \cong T_{C'} \), if:

1) \( \beta \) is a bijection and \( \beta(T_C) = T_{C'} \);
2) \( \forall u \in T_C \quad l_C(v) = l_{C'}(\beta(u)) \).

We write \( T_C \cong T_{C'} \), if there exists a label-preserving bijection \( \beta : T_C \cong T_{C'} \).

A mapping \( \beta : T_C \rightarrow T_{C'} \) is a homomorphism between \( T_C \) and \( T_{C'} \), written \( \beta : T_C \subseteq T_{C'} \), if:

1) \( \beta : T_C \cong T_{C'} \);
2) \( \forall u, w \in T_C \quad u \prec_C w \Rightarrow \beta(u) \prec_{C'} \beta(w) \).

We write \( T_C \subseteq T_{C'} \), if there exists a homomorphism \( \beta : T_C \subseteq T_{C'} \).

A mapping \( \beta : T_C \rightarrow T_{C'} \) is an isomorphism between \( T_C \) and \( T_{C'} \), written \( \beta : T_C \cong T_{C'} \), if \( \beta : T_C \subseteq T_{C'} \) and \( \beta^{-1} : T_{C'} \subseteq T_C \). We write \( T_C \cong T_{C'} \), if there exists an isomorphism \( \beta : T_C \cong T_{C'} \).
3. Trace equivalences

A sequential trace of a net $N$ is a sequence $a_1 \cdots a_n \in \mathcal{A}^*$ such that $\pi_N \xrightarrow{a_1} \pi_1 \xrightarrow{a_2} \cdots \xrightarrow{a_n} \pi_n$, where $\pi_i \in \Pi(N)$ ($1 \leq i \leq n$) and $\pi_N$ is an initial process of $N$. SeqTraces($N$) denotes a set of all sequential traces of $N$. Two nets $N$ and $N'$ are interleaving trace equivalent, written $N \equiv_i N'$, if SeqTraces($N$) = SeqTraces($N'$).

A step trace of a net $N$ is a sequence $A_1 \cdots A_n \in (\mathcal{M}(\mathcal{A}))^*$ such that $\pi_N \xrightarrow{A_1} \pi_1 \xrightarrow{A_2} \cdots \xrightarrow{A_n} \pi_n$, where $\pi_i \in \Pi(N)$ ($0 \leq i \leq n$), and $\pi_N$ is an initial process of $N$. StepTraces($N$) denotes a set of all step traces of $N$. Two nets $N$ and $N'$ are step trace equivalent, written $N \equiv_s N'$, if StepTraces($N$) = StepTraces($N'$).

A pomset trace of a net $N$ is a pomset $\rho$, an isomorphism class of $T_C$ for $\pi = (C, f) \in \Pi(N)$, where $C = (P_C, T_C, F_C, l_C)$. We write $\rho \subseteq \rho'$, if $T_C \subseteq T_{C'}$, $P_C \subseteq P_{C'}$ and $F_C \subseteq F_{C'}$. In such a case we say that pomset $\rho$ is less sequential or more parallel than $\rho'$. Let us denote a set of all pomset traces of $N$ by Pomsets($N$). Two nets $N$ and $N'$ are partial word trace equivalent, written $N \equiv_{pw} N'$, if Pomsets($N$) $\subseteq$ Pomsets($N'$) and Pomsets($N'$) $\subseteq$ Pomsets($N$), i.e., for any $\rho' \in$ Pomsets($N'$) there exists $\rho \in$ Pomsets($N$) such that $\rho \subseteq \rho'$ and vice versa. Two nets $N$ and $N'$ are pomset trace equivalent, written $N \equiv_{pom} N'$, if Pomsets($N$) = Pomsets($N'$).

A process trace of a net $N$ is an isomorphism class of $C$ for $\pi = (C, f) \in \Pi(N)$. ProcessNets($N$) denotes a set of all process traces of $N$. Two nets $N$ and $N'$ are process trace equivalent, written $N \equiv_{pr} N'$, if ProcessNets($N$) = ProcessNets($N'$).

4. Bisimulation equivalences

In this section we consider the definitions of different bisimulations. A notation $\mathcal{R} : N \equiv_\alpha N'$ means that $\mathcal{R}$ is a bisimulation of $\alpha$ type between nets $N$ and $N'$. Nets $N$ and $N'$ are called $\alpha$-bisimulation equivalent, written $N \equiv_\alpha N'$, if $\mathcal{R} : N \equiv_\alpha N'$ for some $\alpha$-bisimulation $\mathcal{R}$.

4.1. Simple bisimulations

Let $\mathcal{R} \subseteq \Pi(N) \times \Pi(N')$. In the following definition $\tilde{\pi} = (\tilde{C}, \tilde{f})$, $\tilde{\pi}' = (\tilde{C}', \tilde{f}')$.

$\mathcal{R}$ is a $\alpha$-bisimulation between $N$ and $N'$, $\alpha \in \{\text{interleaving, step, partial word, pomset, process}\}$, written $\mathcal{R} : N \equiv_\alpha N'$, $\alpha \in \{i, s, pw, pom, pr\}$, if:

1. $(\pi_N, \pi_{N'}) \in \mathcal{R}$;
2. $(\pi, \pi') \in \mathcal{R}$, $\pi \xrightarrow{a} \tilde{\pi}$,
   (a) $|\tilde{P}_C| = 1$, if $\alpha = i$;
   (b) $|\tilde{P}_C| = 0$, if $\alpha = s$;
then $\exists \tilde{\pi}' : \pi' \xrightarrow{a} \tilde{\pi}'$, $(\tilde{\pi}, \tilde{\pi}') \in \mathcal{R}$ and
   (a) $T_{C'} \subseteq T_{C'}$, if $\alpha = pw$;
   (b) $T_{C} \simeq T_{C'}$, if $\alpha \in \{i, s, pom\}$;
   (c) $C \simeq C'$, if $\alpha = pr$;
3. As previous item but $N$ and $N'$ are transposed.
4.2. ST-bisimulations

A ST-process of a net $N$ is a pair $(\pi_E, \pi_P)$ such that $\pi_E, \pi_P \in \Pi(N)$, $\pi_P \preceq \pi_E$ and $\forall v, w \in T_C \exists v \prec C w \Rightarrow v \in T_C$. In such a case $\pi_E$ is a process which began to work, i.e., all actions of $\pi_E$ began working. A process $\pi_P$ corresponds to the terminated part of $\pi_E$, and $\pi_P$ corresponds to the still working part. Clearly, $\prec C w = \emptyset$. ST$ - \Pi(N)$ denotes a set of all ST-processes of $N$. $(\pi_N, \pi_N)$ will be an initial ST-process of $N$. Let $(\pi_E, \pi_P), (\tilde{\pi}_E, \tilde{\pi}_P) \in ST - \Pi(N)$. We write $(\pi_E, \pi_P) \vdash (\tilde{\pi}_E, \tilde{\pi}_P)$, if $\pi_E \rightarrow \tilde{\pi}_E$ and $\pi_P \rightarrow \tilde{\pi}_P$.

Let $\mathcal{R} \subseteq ST - \Pi(N) \times ST - \Pi(N') \times B$, where $B = \{ \beta \mid \beta : T_C \rightarrow T_{C'}, \pi = (C, f) \in \Pi(N), \pi' = (C', f') \in \Pi(N') \}$. In the following definitions $\pi_E = (C_E, f_E)$, $\pi_P = (C_P, f_P)$, $\pi_E^\prime = (C_E, f_E')$, $\pi_P^\prime = (C_P, f_P')$, $\pi = (C, f)$, $\pi' = (C', f')$.

$\mathcal{R}$ is an $\alpha$-ST-bisimulation between $N$ and $N'$, $\alpha \in \{\text{interleaving, partial word, pomset, process}\}$, written $\mathcal{R} : N \equiv_{\alpha ST} N'$, $\alpha \in \{i, pw, pom, pr\}$, if:

1. $((\pi_N, \pi_N'), (\pi_N', \pi_N')) \in \mathcal{R}$;
2. $((\pi_E, \pi_P), (\pi_E', \pi_P'), \beta) \in \mathcal{R} \Rightarrow \beta : T_{C_E} \approx T_{C_E'}$ and $\beta(T_{C_P}) = T_{C_P'}$;
3. $((\pi_E, \pi_P), (\pi_E', \pi_P'), \beta) \in \mathcal{R}, (\pi_E, \pi_P) \rightarrow (\tilde{\pi}_E, \tilde{\pi}_P)$, $\exists \tilde{\beta}, (\tilde{\pi}_E^\prime, \tilde{\pi}_P') :$ $(\pi_E', \pi_P') \rightarrow (\tilde{\pi}_E^\prime, \tilde{\pi}_P'),$ $\tilde{\beta}[T_{C_E} = \beta$, $((\tilde{\pi}_E, \tilde{\pi}_P), (\tilde{\pi}_E^\prime, \tilde{\pi}_P'))$, then:
   a. $\tilde{\beta}[T_{C_E}]^{-1} : T_{C_E^\prime} \subseteq T_{C'}$, if $\alpha = pw$;
   b. $\tilde{\beta}[T_{C_E}] : T_{C} \approx T_{C'}$, if $\alpha \in \{pom, pr\}$;
   c. $C \approx C'$, if $\alpha = pr$;

4. As previous item but $N$ and $N'$ are transposed.

4.3. History preserving bisimulations

Let $\mathcal{R} \subseteq \Pi(N) \times \Pi(N') \times B$, where $B = \{ \beta \mid \beta : T_C \rightarrow T_{C'}, \pi = (C, f) \in \Pi(N), \pi' = (C', f') \in \Pi(N') \}$. In the following definitions $\pi = (C, f)$, $\tilde{\pi} = (\tilde{C}, \tilde{f})$, $\pi' = (C', f')$, $\tilde{\pi}' = (\tilde{C}', \tilde{f}')$.

$\mathcal{R}$ is a $\alpha$-history preserving bisimulation between $N$ and $N'$, $\alpha \in \{\text{partial word, pomset, process}\}$, written $N \equiv_{\alpha h} N'$, $\alpha \in \{pw, pom, pr\}$, if:

1. $(\pi_N, \pi_N', \emptyset) \in \mathcal{R}$;
2. $(\pi, \pi', \beta) \in \mathcal{R} \Rightarrow \beta : T_C \approx T_{C'}$;
3. $(\pi, \pi', \beta) \in \mathcal{R}, \pi \rightarrow \tilde{\pi} \Rightarrow \exists \tilde{\beta}, \tilde{\pi}' : \pi' \rightarrow \tilde{\pi}', \tilde{\beta}[T_C = \beta$, $(\tilde{\pi}, \tilde{\pi}', \tilde{\beta})] \in \mathcal{R}$ and
   a. $\tilde{\beta}[T_{C_E}]^{-1} : T_{C_E^\prime} \subseteq T_{C'}$, if $\alpha = pw$;
   b. $\tilde{\beta}[T_{C_E}] : T_{C} \approx T_{C'}$, if $\alpha \in \{pom, pr\}$;
   c. $\tilde{C} \approx \tilde{C}'$, if $\alpha = pr$;

4. As previous item but $N$ and $N'$ are transposed.
5. A comparison of the equivalences

In this section a theorem establishing a correlation of all introduced equivalences is proved.

Theorem 1. Let $\sim \in \{\equiv, \equiv_s\}$ and $\alpha, \beta \in \{i, s, pw, pom, pr, iST, pwST, pomST, prST, pwh, pomh, prh\}$. For nets $N$ and $N'$, $N \sim_\alpha N' \Rightarrow N \sim_\beta N'$ if there exists a directed path $\sim_\alpha \rightarrow \cdots \rightarrow \sim_\beta$ in a graph in Figure 1.

![Figure 1: Correlation of the equivalences](image)

Proof.

$\Leftarrow$ By definitions of the equivalences.

$\Rightarrow$ It is sufficient to consider the following examples.

- In Figure 2.1: $N \equiv_i N'$ but $N \not\equiv_s N'$ since there exists a step trace \{a, b\} in $N'$ which is not in $N'$.
- In Figure 2.2: $N \equiv_{pr} N'$ but $N \not\equiv_i N'$ since only in $N$ an action $a$ can happen such that it is impossible to run $b$ after it.
- In Figure 2.3: $N \equiv_{pwh} N'$ but $N \not\equiv_{pom} N'$ since $b$ can depend on $a$ in $N$.
- In Figure 2.4: $N \equiv_{pomh} N'$ but $N \not\equiv_{pr} N'$ since $N$ is a causal net which is not isomorphic to causal net $N'$.
- In Figure 2.5: $N \equiv_{iST} N'$ but $N \not\equiv_{pw} N'$ since a net $N$ is corresponded by a pomset such that there is not even less sequential pomset in $N'$.
- In Figure 3.1: $N \equiv_{pr} N'$ but $N \not\equiv_{iST} N'$ since an action $a$ is able to begin working in $N'$ so that no $b$ can start later.
- In Figure 3.2: $N \equiv_{prST} N'$ but $N \not\equiv_{pwh} N'$ since only in $N'$ actions $a$ and $b$ can happen so that the next action, $c$, must depend on $a$. \hfill \Box

6. Equivalences on different net subclasses

In the literature several subclasses of nets were proposed by introduction some restrictions on the initial definition of nets, and merging of equivalences was obtained on these types of nets. See for example [2, 7]. We will consider the introduced equivalences on sequential nets, on T-nets and on nets with strict labelling.
Figure 2. Examples of nets
A net $N = \langle P_N, T_N, F_N, I_N, M_N \rangle$ is sequential if $\forall \pi = (C, f) \in \Pi(N)$, $\forall v, w \in T_C (v \prec_C w) \lor (w \prec_C v)$, i.e., $\prec_C$ is a strict (total) ordering on causal net transitions of any process $\pi = (C, f)$ of the net $N$.

**Proposition 1.** For sequential nets $N$ and $N'$,

1. $[2]$ $N \preceq_i N' \iff N \preceq_{\text{pomh}} N'$;
2. $N \equiv_i N' \iff N \equiv_{\text{pom}} N'$.

**Theorem 2.** Let $\sim \in \{\equiv, \preceq\}$, $\alpha, \beta \in \{i, \text{pr}, \text{prST}, \text{prh}\}$. For sequential nets $N$ and $N'$ $N \sim_{\alpha} N' \Rightarrow N \sim_{\beta} N'$ if there exists a directed path $\sim_{\alpha} \rightarrow \cdots \rightarrow \sim_{\beta}$ in graph in Figure 4.

**Figure 4.** Equivalences on sequential nets

**Proof.** $\Leftarrow$ By Theorem 1.

$\Rightarrow$ It is sufficient to consider the following examples on sequential nets.

- In Figure 2.4: $N \preceq_i N'$ but $N \not\preceq_{\text{pr}} N'$.
- In Figure 2.2: $N \equiv_{\text{pr}} N'$ but $N \not\equiv_i N'$.
- In Figure 5.1: $N \preceq_{\text{prh}} N'$ but $N \not\preceq_{\text{prST}} N'$ since only in $N'$ we can begin running a process with action $a$ so that it may be extended by action $b$ in the only way (i.e., so that extended process be only one).
In Figure 5.2: $N \tTR_{pr-ST} N'$ but $N \not\equiv_{pr} N'$ since only in $N'$ it is possible to run a process with sequential occurring actions $a$ and $b$ so that the next action, $c$, may extend this process only in one way (i.e., causal net with action $c$, extending a causal net corresponding to sequence $ab$, connects with its subnet containing $a$, in the only way).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure5.png}
\caption{Examples of sequential nets}
\end{figure}

A T-net is a net $N = (P_N, T_N, F_N, l_N, M_N)$ such that $\forall p \in P_N \ |\!^* p\!| \leq 1$ and $|p^*| \leq 1$.

**Proposition 2.** For autoconcurrency free T-nets $N$ and $N'$, $N \equiv_i N' \iff N \equiv_{jST} N'$.

No pomset equivalence is a consequence of partial word one, and no process equivalence is a consequence of pomset one on T-nets without autoconcurrency. It is demonstrated correspondently by Figure 6.2 where $N \equiv_{push} N'$ but $N \not\equiv_{pom} N'$ since only in $N'$ an action $b$ can depend on $a$ and by Figure 2.4 where $N \equiv_{pomh} N'$ and $N \not\equiv_{pr} N'$. Let us note that for safe autoconcurrency free T-nets we can use the results of [10] and establish the coincidence of interleaving and pomset trace equivalences.

A net $N = (P_N, T_N, F_N, l_N, M_N)$ is a strictly labelled, if its labelling function is $l_N$ bijective, i.e., $\forall t, u \in T_N \ t \neq u \Rightarrow l_N(t) \neq l_N(u)$. 

Proposition 3. For strictly labelled nets $N$ and $N'$, $N \equiv_\alpha N' \Leftrightarrow N \equiv_\alpha N'$, $\alpha \in \{i, s, pw, pom, pr\}$.

For strictly labelled nets we can not draw any arrow in a graph in Figure 1 from interleaving to step, from partial word to pomset and from pomset to process equivalences. In addition, in all semantics from interleaving to pomset the history preserving bisimulation equivalences are strictly stronger than ST-bisimulation ones. It is proved by the following examples.

- In Figure 6.1: $N \equiv_i N'$ but $N \not\equiv_s N'$, since only in $N$ actions $a$ and $b$ can work concurrently.
- In Figure 6.2: $N \equiv_{pw} N'$ but $N \not\equiv_{pom} N'$.
- In Figure 2.4: $N \equiv_{pom} N'$ but $N \not\equiv_{pr} N'$.
- In Figure 6.3: $N \equiv_{pom,ST} N'$ but $N \not\equiv_{push} N'$, since in $N'$ the sequence $ab$ can happen so that the next action, $c$, must depend on $a$.

![Figure 6. Examples of strictly labelled nets](image)

7. Conclusion

A group of the Petri net equivalences is introduced in the paper. A correlation of these equivalences on nets with finite processes without $\lambda$-actions is found. In addition, it is considered which equivalences coincide on different subclasses of nets. The development of the subject consists in further exploration of the introduced equivalences on T-nets and strictly labelled nets.

The next direction of the development of this theme may be an examination of the proposed equivalences on the wider net class, exactly, on nets with $\lambda$-actions. Probably some equivalences will not be connected on such nets. In [11] the example
of event structures with \(A\)-actions was considered. It is demonstrated the independence of ST-bisimulation equivalences and h-bisimulation equivalence on such event structures.

Finally, it would be interesting to find out how ST- and history preserving equivalences are connected with place bisimulation equivalences introduced in [1].

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References


