5. Interpolating D^m-Splines

In Chapter 5 we consider the most important example of multivariate splines in the variational spline theory. We mean D^m-splines. Many results in D^m-spline theory are due to the works of M.Atteia, J.Duchon, W.Freeden, J.Meigneut, S.L.Sobolev, G.Wahba, etc. A valuable contribution in development of the theory has been made by the authors of this monograph too. It concerns the errors of interpolation for D^m-splines, their finite-element analogs, D^m-splines with boundary conditions. The exact rates of convergence were presented in Chapter 4 on the basis of a general technique, but here we make more precise formulations. All error estimates attained in the Chapter are given in the Sobolev semi-norms. To get them we prove the so called lemma on the Sobolev functions with condensed zeros. Then this lemma is applied in different situations. The sense of the lemma consists in the following: if a function has a dense set of zeros, and its Sobolev norm or semi-norm is bounded, then this function is very small.

Section 5.2 contains the description of well-known B-splines, and an algorithm of interpolation for multivariate functions on scattered meshes based on B-splines is considered. Here, useful for programmers algorithms of assembling the matrices is presented in detail. A more interesting part of this Section concerns the convergence in the anisotropic Sobolev spaces. In Section 5.1 we prove that replacing the minimized semi-norm of isotropic Sobolev space by the semi-norm of anisotropic Sobolev space does not change the error estimates. Here we show that the degrees in error estimates in the interpolation of multivariate functions on scattered meshes do not change if we change a minimized semi-norm by the equivalent one. These facts allow us to construct and use a special semi-norm in B-spline space which is calculated faster than the conventional Sobolev semi-norm. Finishing Section 5.2 we give proofs of error estimates for spline-interpolation on B-splines.

In Section 5.3, we study D^m-spline in the space $\mathbb{R}^n$. Here we formulate the characterization theorems for these splines in the terms of reproducing kernels, and give the error estimates for the spline-interpolating problem. Further we propose a special example of interpolating and smoothing the function given on the finite set of spheres by its average values. The errors of approximation for such an interpolation are also presented.

5.1. D^m-Splines in Bounded Domain

5.1.1. Interpolating D^m-Splines in Isotropic and Anisotropic Sobolev space

Let $\Omega \subset R^n$ be a bounded, simply connected domain with a Lipschitz boundary, and let $m > n/2$ be an integer. We then have a compact imbedding of the space $W_2^m(\Omega)$ into $C(\bar{\Omega})$. The semi-norm

$$\|D^m u\|_{L^2(\Omega)} = \left(\sum_{|\alpha|=m} \frac{m!}{\alpha!} \int_{\Omega} (D^\alpha u)^2 d\Omega \right)^{1/2} \tag{5.1}$$

defines the following norm

$$\|u\|_{W_2^m(\Omega)} = (\|u\|^2_{L^2(\Omega)} + \|D^m u\|^2_{L^2(\Omega)})^{1/2} \tag{5.2}$$

in $W_2^m(\Omega)$. Here $\alpha = (\alpha_1, ..., \alpha_n)$ are multi-indices with non-negative integer components, $|\alpha| = \alpha_1 + ... + \alpha_n$, $\alpha! = \alpha_1! \cdot ... \cdot \alpha_n!$, $D^\alpha u = \partial^m u / \partial^{\alpha_1} x_1 ... \partial^{\alpha_n} x_n$ are partial derivatives of the m-th order. Likewise, other semi-norms,

$$\|D^k u\|_{L^p(\Omega)} = \left(\sum_{|\alpha|=k} \frac{k!}{\alpha!} \int_{\Omega} (D^\alpha u)^p d\Omega \right)^{1/p}, \; p \geq 2,$$

$$\|D^k u\|_{L^p(\Omega)} = \text{vrai} \max_{|\alpha|=k,\, x\in\Omega} |D^\alpha u(x)|, \quad p = \infty$$

can be introduced in the space $W_2^m(\Omega)$, with the parameters k and p satisfying conditions

$$k - \frac{n}{p} \leq m - \frac{n}{2}, \quad p \geq 2, \quad \text{except } (k = m - n/2 \;\&\; p = \infty). \tag{5.3}$$

Note that conditions (5.3) imply the imbedding $W_2^m(\Omega) \subset W_p^k(\Omega)$.

Let $f \in W_2^m(\Omega)$, $K = \{P_1, ..., P_N\}$ be a set of points from $\bar{\Omega}$. The element $\sigma \in W_2^m(\Omega)$ is said to be an *interpolating D^m-spline* (see also Section 1.3.3.) if this element is a solution to problem

$$\sigma = \arg \min_{u \in K^{-1}(f)} \|D^m u\|_{L^2(\Omega)} \tag{5.4}$$

where $K^{-1}(f) = \{u \in W_2^m(\Omega) : \quad u(P_i) = f(P_i),\ i = 1, ..., N\}$. The following necessary and sufficient equalities

$$(D^m \sigma, D^m u)_{L^2(\Omega)} = 0, \quad \forall u \in K^{-1}(0) \tag{5.5}$$

for the spline σ to be an interpolating D^m-spline are known as an orthogonal property.

In Chapter 4, we have shown that if the sets $A_1, A_2, ...$ form a condensed h-net in Ω, then the D^m-splines σ_i, which are the solutions to problems

$$\sigma_i = \arg \min_{u \in A_i^{-1}(f)} \|D^m u\|_{L^2(\Omega)} \tag{5.6}$$

strongly converge to f in the norm of the space $W_2^m(\Omega)$. Further, we shall name this fact as the convergence theorem for D^m-splines.

We define now D^m-spline in the anisotropic Sobolev spaces. Anisotropic Sobolev space $W_2^m(\Omega)$ differs from isotropic space $W_2^m(\Omega)$ only in its norm

$$\|u\|_{W_2^m(\Omega)} = (\|u\|^2_{L^2(\Omega)} + \|D^m u\|^2_{L^2(\Omega)})^{1/2}$$

where the semi-norm is defined as follows

$$\|D^m u\|_{L^2(\Omega)} = \left(\int_\Omega (D^m_{x_1} u)^2 + ... + (D^m_{x_n} u)^2 dx \right)^{1/2}. \tag{5.7}$$

One can easily see that this semi-norm is majorized by (5.1). The properties of the space $W_2^m(\Omega)$ are different in different directions, though, the symmetry in perpendicular directions is saved. For this reason, the Sobolev space $W_2^m(\Omega)$ with such a norm is called the anisotropic Sobolev space. Note that the changed norm is equivalent to the original one.

Similar to the above, we introduce the interpolating D^m-spline

$$\sigma = \arg \min_{u \in K^{-1}(f)} \|D^m u\|_{L^2(\Omega)}$$

in the anisotropic Sobolev space. It is not difficult to prove that the spline σ exists and is unique, if the set K contains an L-solvable set for the space of polynomials $\mathbb{P}_{m-1}$, whose degrees do not exceed $m-1$ in any n variables. Obviously, $P_{m-1} \subset \mathbb{P}_{m-1}$. Certainly, the convergence for the D^m–splines on a condensed h-net in Ω takes place as for the ordinary D^m-splines.

5.1.2. Uniform Equivalence of Norms

Let B be a compact in the Cartesian product Ω^N consisting of L-solvable sets only.

Lemma 5.1. There exist the constants $c_1, c_2 > 0$, such that inequalities

$$c_1 \|u\|_{W_2^m(\Omega)} \le \left(\sum_{i=1}^{R} u^2(b_i) + \|D^m u\|^2_{L^2(\Omega)} \right)^{1/2} \le c_2 \|u\|_{W_2^m(\Omega)} \tag{5.8}$$

hold for any function $u \in W_2^m(\Omega)$ and any L-solvable set $\boldsymbol{b} = (b_1, ..., b_R) \in B$.

Proof. Since the imbedding of $W_2^m(\Omega)$ into $C(\bar{\Omega})$ is compact,

$$\|u\|_C \le K \|u\|_{W_2^m(\Omega)}, \tag{5.9}$$

whence $c_2 = (K^2 N + \|D^m\|^2)^{1/2}$. Any L-solvable set defines a special equivalent norm, hence, for any $\boldsymbol{b} \in B$ there exists a positive constant $c_1(\boldsymbol{b})$, satisfying

$$c_1^2(\boldsymbol{b}) \le \sum u^2(b_i) + \|D^m u\|^2_{L^2(\Omega)}. \tag{5.10}$$

We need to show that there exists a total constant c_1 independent of $\boldsymbol{b}$ such that

$$c_1^2 \le \sum_{i=1}^{R} u^2(b_i) + \|D^m u\|^2_{L^2(\Omega)}, \quad \forall u \in S, \quad \forall \boldsymbol{b} \in B \tag{5.11}$$

where $S = \{u \in W_2^m(\Omega) : \|u\|_{W_2^m(\Omega)} = 1\}$ is the unit sphere. Assume now that no constant satisfies condition (5.11). In other words, there is a sequence of L-solvable sets $\boldsymbol{b}^k = (b_1^k, ..., b_R^k) \in B$ and functions $u_k \in S$ which satisfy inequalities

$$\sum_{i=1}^{R} u_k^2(b_i^k) + \|D^m u_k\|^2_{L^2(\Omega)} \le \frac{1}{k}. \tag{5.12}$$

Since the points b^k belong to the compact B, the sequence of the points $b^1, ...$ can be chosen to converge to a point $\boldsymbol{b} \in B$. Adding inequalities (5.10) with $u = u_k$ to (5.12), we obtain

$$\sum_{i=1}^{R} (u_k^2(b_i) - u_k^2(b_i^k)) \ge c_1^2(\boldsymbol{b}) - \frac{1}{k}. \tag{5.13}$$

If we prove that the left-hand side of inequality (5.13) tends to zero, a contradiction will arise thereby proving the lemma. The convergence to zero is implied by the inequalities

$$\begin{aligned} |\sum_{i=1}^{R} u_k^2(b_i) - u_k^2(b_i^k)\,| &\le 2K \sum_{i=1}^{R} |u_k(b_i) - u_k(b_i^k)| \\ &\le 2KR \max \|\delta_{b_i} - \delta_{b_i^k}\|, \end{aligned}$$

because the latter term converges to zero, owing to the compact embedding of $W_2^m(\Omega)$ into $C(\Omega)$. □

Lemma 5.2. There exist constants $h_0 > 0$ and $\lambda > 0$, such that

$$\|u\|_{W_2^m(\Omega)} \le \lambda \|D^m u\|_{L^2(\Omega)} \tag{5.14}$$

for any function $u \in W_2^m(\Omega)$ having an h-net of zeros in the domain Ω when $h \le h_0$.

Proof. Take in Ω an arbitrary L-solvable set of points $\{t_1, t_2, ..., t_R\}$. If $e_1, e_2, ..., e_R$ is a basis in P_{m-1}, the condition for the set to be an L-solvable set is equivalent to non-singularity of the matrix $[e_i(t_j)]$ $(i, j = 1, ..., R)$. Its determinant depends in a continuous manner on the points $t_1, ..., t_R$; hence, there exists such $h_0 > 0$, that the set

$$B = \prod_{i=1}^{R} (B(t_i, h_0) \cap \Omega)$$

forms a compact of L-solvable sets; $B(t, h_0)$ is a closed ball in R^n of the radius h_0, with the centre at t.

Let a function u have an h-net of zeros in Ω when $h \leq h_0$. Then there are points $b_i \in B(t_i, h_0)$ $(i = 1, ..., R)$, where u equals zero. Using inequalities (5.8), we arrive at (5.14) with the constant $\lambda = 1/c_1$. □

5.1.3. Special Cover of Bounded Domain

We will say that the domain Ω satisfies the *cone condition*, if there exist two parameters $\theta > 0$ and $r > 0$ such that for every point $t \in \Omega$ there exists the vector ξ for which the cone

$$C(t, \xi, \theta, r) = \{t + \lambda\eta : \ \eta \in \mathbb{R}^n, \ |\eta| = 1, \ \angle(\eta, \xi) \leq \theta, \ 0 \leq \lambda \leq r\}$$

is totally contained in Ω. In (Nikolsky 1972) one can find the fact that any domain with the Lipshitz boundary satisfies the cone condition. Taking this fact into account let us formulate the following lemma about a special cover.

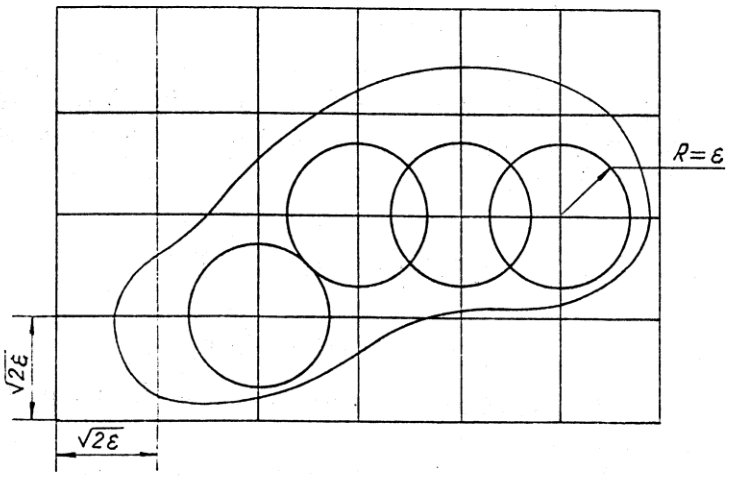

Fig. 5.1. Illustration to Lemma 5.3: the choice of the set T_ε. One can see four points of grid forming T_ε.

Lemma 5.3. There exist positive constants M, M_1, ε_0, such that for any $\varepsilon \leq \varepsilon_0$ there is a finite set of points T_ε satisfying the conditions

(1) the balls $B(t, \varepsilon)$ $(t \in T_\varepsilon)$ are contained in Ω;

(2) the balls $B(t, M\varepsilon)$ cover Ω;

(3) each point of the domain Ω belongs to, at most, M_1 balls of the covering balls $B(t, M\varepsilon)$ $(t \in T_\varepsilon)$.

Proof. Consider the integer grid $\mathbb{Z}^n$ in $\mathbb{R}^n$. Obviously, it forms a $\sqrt{n}/2$-net in $\mathbb{R}^n$. Hence, the grid $(2\varepsilon/\sqrt{n})\mathbb{Z}^n$ is an ε-net in $\mathbb{R}^n$, i.e. for any $q \in \mathbb{R}^n$ there exists a point $z(q) \in (2\varepsilon/\sqrt{n})\mathbb{Z}^n$ such, that $\mathbf{dist}(z(q), q) \leq \varepsilon$.

Let $r > 0$, $r' = r \sin\theta/(1 + \sin\theta)$. It is easy to see that for any point $s \in \mathbb{R}^n$ and any unit vector $\xi \in \mathbb{R}^n$ the ball $B(s + (r - r')\xi, r')$ is contained in $C(s, \xi, \theta, r)$. The choice of r' is illustrated by Fig. 5.2, where the ball is inscribed in the cone.

Let

$$\varepsilon_0 = \frac{R \sin\theta}{2(1 + \sin\theta)}, \quad M = 2 + \frac{2}{\sin\theta}, \quad M_1 = (M\sqrt{n} + 1)^n$$

and find for every $\varepsilon < \varepsilon_0$ a cover, satisfying conditions (1), (2), (3). To do this, consider the finite set of points

$$T_\varepsilon = \{t \in \frac{2\varepsilon}{\sqrt{n}}\mathbb{Z}^n : \quad B(t, \varepsilon) \subset \Omega\}.$$

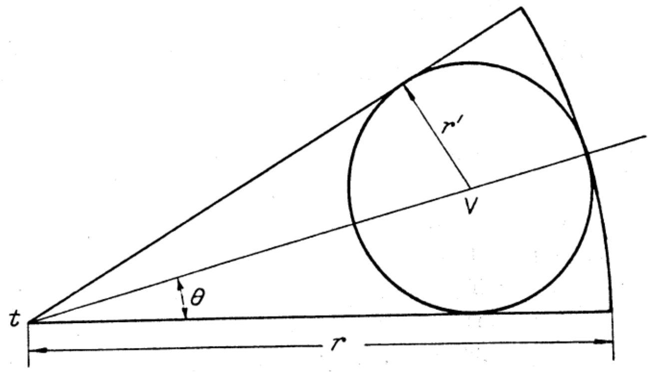

Fig. 5.2. Cone $C(t, \xi, \theta, r)$ contains ball $B(v, r')$, $r/r' = \frac{\sin\theta}{1+\sin\theta}$.

So, condition (1) is fulfilled. Now, prove condition (2). Let s be any point in Ω, then $M\varepsilon < M\varepsilon_0 = R$ and, consequently, there exists a cone $C(s, \xi(s), \theta, M\varepsilon)$ totally contained in Ω. But, we prove that some ball $B(v, 2\varepsilon)$ of the radius 2ε totally lies in the cone.

The ball $B(v, \varepsilon)$ necessarily contains a point t in the ε-net $(2\varepsilon/\sqrt{n})\mathbb{Z}^n$. Since the ball $B(t, \varepsilon)$ is contained in $B(v, 2\varepsilon)$, then $B(t, \varepsilon)$ is totally contained in Ω, i.e. $t \in T_\varepsilon$. From Fig. 5.3 one can see that $B(t, M\varepsilon)$ contains the point s.

So, condition (2) is also fulfilled. The third condition is proved more trivially. It is necessary to count the number of elements of the form $\frac{2\varepsilon}{\sqrt{n}}(z_1, ..., z_n)$, where z_i are integers, in the ball $B(s, M\varepsilon)$. This number is independent of s (approximately), and may by estimated by the number of points $z \in \mathbb{Z}^n$ in the ball $B(0, M\varepsilon \cdot \frac{\sqrt{n}}{2\varepsilon}) = B(0, M\sqrt{n}/2)$. Counting all points from the n-dimensional cube, containing the ball as inscribed, we obtain the number $M_1 = (2M \cdot \sqrt{n}/2 + 1)^n = (M\sqrt{n} + 1)^n$. □

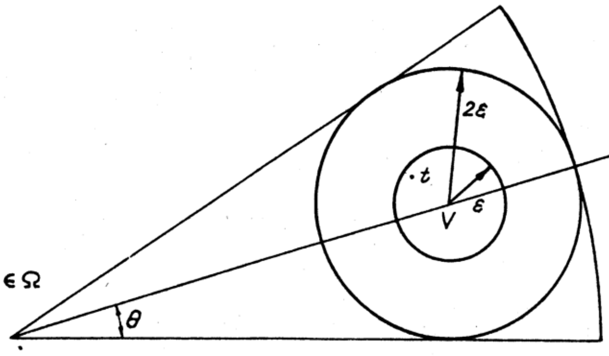

Fig. 5.3. Point s belongs $B(t, M\varepsilon)$, $t \in T_\varepsilon$.

Lemma 5.4. Let $M \geq 1$. Then there exist $c, \rho, h_0 > 0$ such that for any $t \in R^n$ and any function $u \in W_2^m(R^n)$ which has an h-net of zeros in the ball $B(t, \rho h)$ at $h \leq h_0$, the following inequality holds:

$$\|D^k u\|_{L^p(B(t,M\rho h))} \leq c h^{m-k-n/2+n/p} \|D^m u\|_{L^2(B(t,M\rho h))}. \tag{5.13}$$

Proof. Choose an arbitrary L-solvable set $\{b_1, b_2, ..., b_R\}$ from $\mathbb{R}^n$. Clearly, there exists $\delta > 0$, such that the set $B_\delta = \prod_{i=1}^R B(b_i, \delta)$ forms a compact of L-solvable sets. A change of coordinates $x \to x/\delta$ transforms an L-solvable set into an L-solvable set. We designate $p_i = b_i/\delta$; then $B = \prod B(p_i, 1)$ is a compact of L-solvable sets. Let us find such $\rho > 0$ that the ball $B(0, \rho)$ contains all balls $B(p_i, 1)$. The set B then forms a compact of L-solvable sets is $B(0, \rho)^N$, and hence, in $B(0, M\rho)^N$ for any $M \geq 1$. Lemma 5.2 implies that inequality

$$c_1 \|u\|_{W_2^m(B(0,M\rho))} \leq \left(\sum_{i=1}^R u^2(b_i) + \|D^m u\|^2_{L^2(B(0,M\rho))} \right)^{1/2} \tag{5.14}$$

holds for any function $u \in W_2^m(R^n)$ and any $\boldsymbol{b} \in B$.

Let a function u have an 1-net of zeros in the ball $B(0, \rho)$. The set B is so defined that at least one zero, b_i, of the function u can be found in each of the balls $B(p_i, 1)$. Therefore, the sum $\sum u^2(b_i)$ vanishes. In view of the continuity of the embedding $W_2^m(B(0, M\rho)) \subset W_p^k(B(0, M\rho))$, we obtain from inequality (5.14)

$$\|D^k u\|_{L^p(B(0,M\rho))} \leq c \|D^m u\|_{L^2(B(0,M\rho))}. \tag{5.15}$$

A linear change of coordinates, $x' = t + hx$, in (5.15) completes the proof of this lemma. □

5.1.4. Lemma on Sobolev Functions with Condensed Zeros and Convergence Rates for D^m-Splines

Lemma 5.5. There exist constants $c, h_0 > 0$ such that

$$\|D^k u\|_{L^p(\Omega)} \leq c h^{m-k-n/2+n/p} \|D^m u\|_{L^2(\Omega)} \tag{5.16}$$

for any function $u \in W_2^m(\Omega)$ which has an h-net of zeros in the domain Ω for $h \leq h_0$. The constant c depends on the domain Ω and on the parameters k and p, which satisfy (5.3).

Proof. It was shown (Besov et al. 1975) that there exists a bounded operator which continues a function $u \in W_2^m(\Omega)$ from the domain Ω to R^n:

$$\|u^\Omega\|_{W_2^m(R^n)} \leq K \|u\|_{W_2^m(\Omega)}. \tag{5.17}$$

Let ε_0 and ρ be parameters defined in Lemmas 5.3 and 5.4, and the parameters ε and h be chosen using the condition $\varepsilon = \rho h \leq \rho h_0 = \varepsilon_0$. Estimates (5.13) are valid for the function u^Ω for any $t \in T_\varepsilon$ because this function has an h-net of zeros in Ω.

Making use of the Jensen inequality $(\sum x_i^p)^{1/p} \leq (\sum x_i^2)^{1/2}$, inequalities (5.13) and (5.14) from Lemmas 5.4 and 5.2, we obtain a chain of inequalities

$$\begin{aligned}
\|D^k u\|_{L^p(\Omega)} &\leq \left(\sum_{t \in T_\varepsilon} \|D^k u^\Omega\|^p_{L^p(B(t,M\rho h))} \right)^{1/p} \\
&\leq \left(\sum_{t \in T_\varepsilon} \|D^k u^\Omega\|^2_{L^p(B(t,M\rho h))} \right)^{1/2} \\
&\leq c h^{m-k-n/2+n/p} \left(\sum_{t \in T_\varepsilon} \|D^m u^\Omega\|^2_{L^2(B(t,M\rho h))} \right)^{1/2} \\
&\leq c\sqrt{M_1} h^{m-k-n/2+n/p} \|D^m u^\Omega\|_{L^2(R^n)} \\
&\leq cK\sqrt{M_1} h^{m-k-n/2+n/p} \|u\|_{W_2^m(\Omega)} \\
&\leq c\lambda K\sqrt{M_1} h^{m-k-n/2+n/p} \|D^m u\|_{L^2(\Omega)}.
\end{aligned}$$

The proof for $p = \infty$ is quite similar. □

Theorem 5.1. Let A_h be a sequence of condensed h-nets in Ω. Then for any function $f \in W_2^m(\Omega)$ $(m > n/2)$ the sequence of D^m-splines σ_h, which are the solutions to the problems

$$\sigma_h = \arg \min_{u \in A_h^{-1}(f)} \|D^m u\|_{L^2(\Omega)}$$

strongly converge to f in the space $W_2^m(\Omega)$ as $h \to 0$, with the following asymptotic estimates of convergence:

$$\|D^k(\sigma_h - f)\|_{L^p(\Omega)} \leq ch^{m-k-n/2+n/p}\|D^m(\sigma_h - f)\|_{L^2(\Omega)} \tag{5.18}$$

where the constant c is a function of Ω, m, n, p and k (satisfying (5.3)) independent of h and f.

Proof. The convergence theorem for D^m-splines implies the convergence $\sigma_h \to f$ in the norm of the space $W_2^m(\Omega)$. To prove estimate (5.18), it is sufficient to apply Lemma 5.5 to the error $(\sigma_h - f)$ which has an h-net of zeros in Ω. □

Remark 5.1. If h-nets are condensed not in the whole domain Ω but only in its subdomain $\Omega' \subset \Omega$, estimates (5.18) hold in Ω'. In addition, the splines σ_h converge in Ω in the norm of the space $W_2^m(\Omega)$ to the function σ which is the solution to the continuation problem

$$\sigma = \arg \min_{u \in (\Omega')^{-1}(f)} \|D^m u\|_{L^2(\Omega)}.$$

For this reason, it is possible to solve D^m-approximation problems in comprising domains of simpler geometry (for example, in parallelepipeds).

Remark 5.2. D^m-splines in the anisotropic Sobolev spaces converge to the function f with the orders presented in Theorem 5.1. Naturally, since the norms in the isotropic and anisotropic Sobolev space are equivalent:

$$\|u\|_{W_2^m(\Omega)} \leq c\|u\|_{W_2^m(\Omega)},$$

then

$$\|D^m(\sigma_h - f)\|_{L_2(\Omega)} \leq c\|\sigma_h - f\|_{W_2^m(\Omega)}. \tag{5.19}$$

On the basis of the general convergence theory (see Chapter 3) it is easy to prove strong convergence σ_h to f in the space $W_2^m(\Omega)$, like that for D^m-splines. So, combining (5.18), (5.19) we bring about to the required converging orders.

5.1.5. D^m-Splines with Boundary Conditions

Consider in the Sobolev space $W_2^m(\Omega)$ its subspace

$$\overset{\circ}{W}{}_2^m = \{u \in W_2^m(\Omega): \quad D^\alpha u|_\Gamma = 0, \quad |\alpha| \leq m-1\}. \tag{5.20}$$

Let us understand the equality $D^\alpha u|_\Gamma = 0$ in the sense of the space $L^2(\Omega)$. This definition is correct, because the partial derivatives $D^\alpha u$ with the multi-indexes satisfying $|\alpha| \leq m-1$ are at least from $W_2^1(\Omega)$, and their traces are clearly from $L^2(\Gamma)$.

Lemma 5.6. $\overset{\circ}{W}{}_2^m$ is a Hilbert subspace in $W_2^m(\Omega)$.

Proof. It is sufficient to prove, that the set $\overset{\circ}{W}{}_2^m$ is linear and closed. If $u, v \in \overset{\circ}{W}{}_2^m$, then

$$\|D^\alpha(au + bv)\|_{L^2(\Gamma)} \leq |a|\|D^\alpha u\|_{L^2(\Gamma)} + |b|\|D^\alpha v\|_{L^2(\Gamma)} = 0,$$

and linearity is established. Now demonstrate that if u_n is a sequence from $\overset{\circ}{W}{}_2^m$ converging to u, then $u \in \overset{\circ}{W}{}_2^m$, too.

The trace operator on the manifold Γ is bounded, i.e. for any v from $W_2^1(\Omega)$ the following equality

$$\|v\|_{L^2(\Omega)} \leq c\|v\|_{W_2^1(\Omega)} \tag{5.21}$$

is valid. Since the partial derivatives $D^\alpha(u - u_n)$ belong to $W_2^1(\Omega)$, then we have

$$\begin{aligned}\|D^\alpha u\|_{L^2(\Gamma)} &\leq \|D^\alpha(u - u_n)\|_{L^2(\Gamma)} + \|D^\alpha u_n\|_{L^2(\Gamma)} \leq \\ &\leq c\|D^\alpha(u - u_n)\|_{W_2^1(\Omega)} \leq c_\alpha\|u - u_n\|_{W_2^m(\Omega)}.\end{aligned}$$

From here it follows that $\|D^\alpha u\|_{L^2(\Gamma)} = 0$, and $D^\alpha u|_\Gamma = 0$. Thus, $\overset{\circ}{W}{}_2^m(\Omega)$ is closed. □

Let $f \in W_2^m(\Omega)$. Consider in the space $W_2^m(\Omega)$ the affine close subspace $H_f^m = f + \overset{\circ}{W}{}_2^m$. Clearly, H_f^m consists of the functions $g \in W_2^m(\Omega)$, whose α-th derivatives ($|\alpha| < m$) coincide with respective derivatives of the function f on the manifold Γ.

Let A be a subset of Ω. Introduce the set

$$A^{-1}(f) = \{u \in H_f^m : \quad u|_A = f|_A\}.$$

Definition 5.1. *We shall call the function $f^A \in H_f^m$ as an interpolating D^m-spline with boundary conditions, if f^A is a solution to the following problem*

$$f^A = \arg \min_{u \in A^{-1}(f)} \|D^m u\|_{L^2(\Omega)}. \tag{5.22}$$

For the interpolating D^m-spline with boundary conditions the following theorem of existence and uniqueness is valid.

Theorem 5.2. The solution to problem (5.22) always exists and is unique.

Proof. Since the Hilbert space $W_2^m(\Omega)$ is compactly embedded in $C(\bar{\Omega})$, then for any point $a \in \Omega$ the set

$$L_a = \{u \in W_2^m(\Omega) : \quad u(a) = f(a)\}$$

will be an affine hyperplane in $W_2^m(\Omega)$. The set $A^{-1}(f)$ may be written as an intersection of affine closed subspaces in $W_2^m(\Omega)$ in the following way

$$A^{-1}(f) = (\bigcap_{a\in A} L_a) \cap H_f^m.$$

Thus, $A^{-1}(f)$ is the affine closed subspace in $W_2^m(\Omega)$ and it isn't empty, because the function f belongs to $A^{-1}(f)$.

We know that solution to the problem of semi-norm minimization on the closed affine subspace $A^{-1}(f)$ is determined with the accuracy to the intersection of the kernel of the semi-norm and $A^{-1}(0)$. Thus, the uniqueness will be proved if the kernel of the semi-norm, which is the space of polynoms P_{m-1}, and $A^{-1}(0)$ have no common elements except zero. Assume that there exists a nonsingular polynomial p contained in $A^{-1}(0)$. Then, since the polynomial p and all of its derivatives are continuous functions, the conditions $D^\alpha p|_\Gamma = 0$ in $L^2(\Gamma)$ for all $|\alpha| \leq m-1$ are equivalent to $D^\alpha p(a) = 0$ for all points $a \in \Omega$. But such a polynomial may be only zero. Thus, the Theorem is proved. □

Remark 5.3. Similar to orthogonal property (5.14) for the ordinary D^m-splines one can write down the orthogonal property for D^m-splines with boundary conditions

$$(D^m f^A, D^m v)_{L^2(\Omega)} = 0, \quad \forall v \in A^{-1}(0). \tag{5.23}$$

Further we shall need the following equalities, which are consequences of (5.23):

$$(D^m f^A, D^m(f^A - f))_{L^2(\Omega)} = 0 \tag{5.24}$$

and

$$\|D^m(f^A - f)\|^2_{L^2(\Omega)} = -(D^m(f^A - f), D^m f)_{L^2(\Omega)}. \tag{5.25}$$

Theorem 5.3. There exists a parameter $h_0 > 0$ such that for any set A being a h-net in the domain Ω and for any $f \in W_2^{2m}(\Omega)$ the following estimates

$$\|D^k(f^A - f)\|_{L^p(\Omega)} \leq C h^{2m-k-\frac{n}{2}+\frac{n}{p}} \|D^{2m} f\|_{L^2(\Omega)} \tag{5.26}$$

are valid. Here, the constant C is independent of h and f. The parameters k, m, n and p satisfy condition (5.3).

Proof. Inequalities (5.26) will be obtained with the help of Lemma 5.5 about the functions with condensed zeros. Let us choose the constant h_0 defined in the Lemma and assume that A is a h-net with $h \leq h_0$. Utilizing the Lemma we have equalities

$$\|D^k(f^A - f)\|_{L^p(\Omega)} \leq c h^{m-k-\frac{n}{2}+\frac{n}{p}} \|D^m(f^A - f)\|_{L^2(\Omega)}. \tag{5.27}$$

Estimate the right part of (5.27). To this end prove that there exists a constant $M > 0$ independent of f for which the following inequalities

$$\|D^m(f^A - f)\|^2_{L^2(\Omega)} \leq M\|f^A - f\|_{L^2(\Omega)}\|D^{2m}f\|_{L^2(\Omega)} \tag{5.28}$$

are true. Make use of equality (5.25) and definition of D^m-semi-norm (5.2):

$$\begin{aligned}\|D^m(f^A - f)\|^2_{L^2(\Omega)} &= -(D^m(f^A - f), D^m f)_{L^2(\Omega)} \\ &= -\sum_{|\alpha|=m} \frac{m!}{\alpha!} \int_\Omega D^\alpha(f^A - f)D^\alpha f d\Omega.\end{aligned} \tag{5.29}$$

We assert that

$$\int_\Omega D^\alpha(f^A - f)D^\alpha f d\Omega = (-1)^m \int_\Omega (f^A - f)D^{2\alpha} f d\Omega \tag{5.30}$$

for all α with $|\alpha| = m$. To demonstrate that utilize the Green integration formula:

$$\int_\Omega u\frac{\partial v}{\partial x_k} d\Omega = -\int_\Omega v\frac{\partial u}{\partial x_k} d\Omega + \int_\Gamma uv\cos(\mathbf{n}, x_k)d\Gamma. \tag{5.31}$$

which is valid for domains with Lipshitz bounds and functions u, v from $W^1_2(\Omega)$. Here $\mathbf{n}$ is the exterior normal vector.

Throwing over the partial derivatives in the expression

$$\int_\Omega D^\alpha(f^A - f)D^\alpha f d\Omega$$

from the function $f^A - f$ to f in conformity with the Green formula we naturally obtain the formula which is similar to (5.30). One differs from the other by the sum of integrals of the following form

$$\int_\Gamma D^{\alpha-\alpha_1}(f^A - f)D^{\alpha+\alpha_2} f \cos(\mathbf{n}, x_k)d\Gamma,$$

where $0 < |\alpha_1| \leq m, \quad 0 \leq |\alpha_2| \leq m - 1$. Since the function $f^A - f$ belongs to $\overset{0}{W}{}^m_2(\Omega)$, these integrals are equal to zero:

$$\begin{aligned}&\left|\int_\Gamma D^{\alpha-\alpha_1}(f^A - f)D^{\alpha+\alpha_2} f \cos(\mathbf{n}, x_k)d\Gamma\right| \\ &\qquad \leq \|D^{\alpha-\alpha_1}(f^A - f)\|_{L^2(\Gamma)}\|D^{\alpha+\alpha_2} f\|_{L^2(\Gamma)} = 0.\end{aligned}$$

Thus, taking together the proved facts we have

$$\|D^m(f^A - f)\|^2_{L^2(\Omega)} = - \sum_{|\alpha|=m} \frac{m!}{\alpha!} \int_\Omega D^\alpha(f^A - f) D^\alpha f d\Omega$$
$$= (-1)^m \sum_{|\alpha|=m} \frac{m!}{\alpha!} \int_\Omega (f^A - f) D^{2\alpha} f d\Omega$$
$$\leq \|f^A - f\|_{L^2(\Omega)} \sum_{|\alpha|=m} \frac{m!}{\alpha!} \|D^{2\alpha} f\|_{L^2(\Omega)},$$

and the existence of the constant M in inequality (5.28) becomes evident. Putting in (5.27) the constant k being equal to zero and $p = 2$ we obtain

$$\|f^A - f\|_{L^2(\Omega)} \leq ch^m \|D^m(f^A - f)\|_{L^2(\Omega)}. \tag{5.32}$$

Combining inequlities (5.28) and (5.32) we establish

$$\|D^m(f^A - f)\|_{L^2(\Omega)} \leq Mch^m \|D^{2m} f\|_{L^2(\Omega)}, \tag{5.33}$$

from which and from (5.27) follows (5.26) with the constant $C = Mc^2$. □

Remark 5.4. If the function f belongs only to the space $W_2^l(\Omega)$ with $m < l < 2m$, then producing the demonstration as in the latter theorem one can prove

$$\|D^k(f^A - f)\|_{L^p(\Omega)} \leq Ch^{l-k-\frac{n}{2}+\frac{n}{p}} \|D^l f\|_{L^2(\Omega)}. \tag{5.34}$$

Remark 5.5. One can consider D^m-splines with boundary conditions

$$D^\alpha f^A|_\Gamma = D^\alpha f|_\Gamma \tag{5.35}$$

only for $k_1 \leq |\alpha| < m$. In this case the estimates of D^m-splines are also improved and become of the following form

$$\|D^k(f^A - f)\|_{L^p(\Omega)} \leq Ch^{l-k-\frac{n}{2}+\frac{n}{p}-\min(2m-l,k_1)} \|D^l f\|_{L^2(\Omega)}.$$

Remark 5.6. If one considers D^m-spline with boundary conditions (5.35) for $0 < |\alpha| < k_1 < m$ then one obtain no improvement in comparison with the usual estimates of ordinary D^m-splines without boundary conditions.

5.2. Finite-element D^m-splines on B-splines

5.2.1. Theoretical Grounds of Approximation with B-Splines

The utilization of B-splines in interpolation and smoothing algorithms is explained by sparseness of arising SLAEs (systems of linear algebraic equations), and also by good approximating properties of piece-wise polynomials of the defect 1. Finite element spline approximations inherit the convergence rates from the respective analytical splines.

For the analytical spline-functions one can replace the energy functional (or semi-norm) to the equivalent one and this does not provoke the changing in converging rates of spline approximation when interpolating meshes are condensed. This also concerns the spline on subspaces. We consider the anisotropic Sobolev space $W_2^m(\Omega)$ of different smoothness by different variables and the semi-norm

$$|u|_1 = \left(\sum_{l=1}^{n} \int_{\Omega} (D_l^{m_l} u(x))^2 dx \right)^{\frac{1}{2}}. \tag{5.36}$$

On the space of B-splines we introduce the semi-norm equivalent to (5.36) which provide more sparseness in SLAEs in comparison with (5.36). This approach is like the one used in the theory of finite difference schemes and is called the condensation method. The general convergence theory for spline interpolation shows that we must choose an equivalent, but the most effective semi-norm for algorithmic implementation. Equivalence saves the error estimates, i.e. it does not increase error orders.

We also consider the structures of these sparse matrices and the procedures of multiplications in detail.

5.2.2. Semi-Norms in Tensor Product of Finite Dimensional Spaces

Let Ω be the parallelepiped $\prod_{i=1}^{n} [a_i, b_i]$, $S_1, ..., S_n$ be univariate finite dimensional spaces of functions with the basic functions

$$B_i(x_j), \quad j = 1, ..., n, \quad i = 1, ..., y_j.$$

By the tensor product $S = \otimes_{i=1}^{n} S_i$ one names a n-variate space of functions, which is the linear shell of functions

$$B_I(X) = B_{i_1}(x_1) ... B_{i_n}(x_n), \quad I \in \Pi, \tag{5.37}$$

where $\Pi = \{I = (i_1, ..., i_n) \ : \ 1 \le i_1 \le y_1, ..., 1 \le i_n \le y_n\}$. We assume the basic functions in the spaces $S_1, ..., S_n$ form partitions of the unity, i.e.

$$\sum_{i=1}^{y_j} B_i(x_j) = 1, \quad \forall \, x_j \in [a_j, b_j]. \tag{5.38}$$

Obviously, basis (5.37) also forms a partition of the unity. Assume that the bases consist of non-negative functions, and contain the polynomials up to the $(m_i - 1)$-th degree, $i = 1, ..., n$.

Semi-norm (5.36) for the function $\sum_{I \in \Pi} C_I B_I(X)$ in the tensor product of spaces S is represented in the following form

$$|\cdot|_1 = \left(\sum_{l=1}^{n} \int_{\Omega} (D_l^{m_l} \sum_{I \in \Pi} C_I B_I(X))^2 dX \right)^{\frac{1}{2}}.$$

Introduce the following notations

$$\Omega_l = \prod_{\substack{i=1 \\ i \neq l}}^{n} [a_i, b_i], \quad l = 1, ..., n,$$

$$I_l = \begin{cases} (i_1, ..., i_{l-1}, i_{l+1}, ..., i_n) & , l \neq 1,\ l \neq n, \\ (i_2, ..., i_n) & , l = 1, \\ (i_1, ..., i_{n-1}) & , l = n, \end{cases}$$

$$\Pi_l = \{ I_l \ : \ 0 \le i_j \le y_j,\ \forall_j \}$$

$$X_l = \begin{cases} (x_1, ..., x_{l-1}, x_{l+1}, ..., x_n) & , l \neq 1,\ l \neq n, \\ (x_2, ..., x_n) & , l = 1, \\ (x_1, ..., x_{n-1}) & , l = n, \end{cases}$$

$$B_{I_l}(X_l) = \frac{B_I(X)}{B_{i_l}(x_l)}, \quad l = 1, ..., n,$$

which help us to rewrite the semi-norm as follows

$$|\cdot|_1 = \left(\sum_{l=1}^{n} \int_{\Omega} \left(\sum_{I_l \in \Pi_l} B_{I_l}(X_l) \cdot \sum_{i_l=1}^{y_l} C_I B_{i_l}^{(m_l)}(x_l) \right)^2 dX \right)^{\frac{1}{2}}.$$

To estimate the semi-norm use the following lemma.

Lemma 5.7. If $a_j \ge 0$ and $\sum_{j=1}^{n} a_j = 1$, then

$$\left(\sum a_j b_j \right)^2 \le \sum a_j b_j^2. \tag{5.39}$$

Proof. It is sufficient to use the Schwartz inequality $(\sum_{j=1}^{n} x_j y_j)^2 \le \sum_{j=1}^{n} x_j^2 \sum_{j=1}^{n} y_j^2$, with $x_j = \sqrt{a_j}$, $y_j = \sqrt{a_j} b_j$. □

Applying the Lemma we have

$$|\cdot|_1 \le |\cdot|_2 = \left(\sum_{l=1}^{n} \int_{\Omega} \sum_{I_l \in \Pi_l} B_{I_l}(X_l) \cdot \left(\sum_{i_l=1}^{y_l} C_I B_{i_l}^{(m_l)}(x_l)\right)^2 dX\right)^{\frac{1}{2}}.$$

Thus, we obtain another semi-norm $|\cdot|_2$ in the tensor product of spaces, which majorates the initial one. Prove that the second semi-norm is induced by a symmetric scalar semi-product in S. To do this, introduce the functions

$$Z_I^l(X) = \sqrt{B_{I_l}(X_l)} \cdot B_{i_l}(x_l). \tag{5.40}$$

Then we have

$$\sum_{l=1}^{n} \int_{\Omega} \sum_{I_l \in \Pi_l} B_{I_l}(X_l) \cdot \left(\sum_{i_l=1}^{y_l} C_I B_{i_l}^{(m_l)}(x_l)\right)^2 dx =$$

$$\sum_{l=1}^{n} \int_{\Omega} \sum_{I_l \in \Pi_l} B_{I_l}(X_l) \sum_{i_l=1}^{y_l} C_{I_l i_l} B_{i_l}^{(m_l)}(x_l) \cdot \sum_{j_l=1}^{y_l} C_{I_l j_l} B_{j_l}^{(m_l)}(x_l) dX =$$

$$\sum_{l=1}^{n} \int_{\Omega} \sum_{I \in \Pi} C_I D_l^{m_l} Z_I^l(X) \sum_{j_l=1}^{y_l} C_{I_l j_l} D_l^{m_l} Z_I^l(X) dX =$$

$$\sum_{l=1}^{n} \int_{\Omega} \sum_{I \in \Pi} \sum_{J \in \Pi} \delta(I_l, J_l) C_I D_l^{m_l} Z_I^l(X) C_J D_l^{m_l} Z_J^l(X) dX,$$

where δ is the Kronecker symbol. The corresponding scalar semi-products, for the first and second semi-norms are of the following forms

$$\Big(\sum_{I \in \Pi} C_I B_I, \sum_{J \in \Pi} E_J B_J\Big)_1 = \sum_{l=1}^{n} \int_{\Omega} \sum_{I \in \Pi} \sum_{J \in \Pi} C_I D_l^{m_l} B_I(X) \cdot E_J D_l^{m_l} B_J(X) dX.$$

$$\Big(\sum_{I \in \Pi} C_I B_I, \sum_{J \in \Pi} E_J B_J\Big)_2 = \sum_{l=1}^{n} \int_{\Omega} \sum_{I \in \Pi} \sum_{J \in \Pi} \delta(I_l, J_l) \cdot C_I D_l^{m_l} Z_I^l(X) \cdot E_J D_l^{m_l} Z_J^l(X) dX.$$

5.2.3. Polynomial Splines of the Defect 1

For the integer $N > 2$ consider the partition of the real line by N intervals:

$$\Delta : (-\infty, x_1), [x_1, x_2), \ldots, [x_{N-1}, \infty).$$

Definition 5.2. *By polynomial spline of the k-th degree and defect 1 we call the piece-wise polynomial function $f(x) \in C^{k-1}(-\infty, \infty)$, which is a polynomial of the k-th degree on any interval from Δ.*

Otherwise, the function $f(x)$ is composed of polynomial "pieces" and continuous up to $(k-1)$-th derivative in the points $x_1, ..., x_{N-1}$. Polynomial splines form the linear space S^k_Δ, whose dimension is equal to $(N+k)$. It is known, that S^k_Δ has the B-spline basis $\{B_i\}_{i=1}^{N+k}$, satisfying the conditions

$$B_i(x) \neq 0 \quad \text{only if} \quad x \in (x_{i-k-1}, x_i), \tag{5.41}$$

$i = 1, ..., N+k$, $x_i = -\infty$ if $i < 0$, $x_i = +\infty$ if $i > N$. Thus, the B-spline supports consist of, at most, $(k+1)$ intervals of the partition Δ, and we can say that the functions B_j have finite supports.

Any function $f \in S^k_\Delta$ is linear combination:

$$f(x) = \sum_{i=1}^{N+k} C_i B_i(x). \tag{5.42}$$

Enumerate the intervals of the partition Δ in the order $1, ..., N$ and introduce the function

$$i_x : x \to \{1, ..., N\}, \tag{5.43}$$

putting into correspondence to any point x the interval, which contains this point. Taking into account condition (5.41), formula (5.42) is simplified:

$$f(x) = \sum_{j=0}^{k} C_{i_x+j} B_{i_x+j}(x). \tag{5.44}$$

For the integer positive vector $N = (N_1, ..., N_n)$ define the partition $\Delta = \Delta_1 \times ... \times \Delta_n$ of the space $\mathbb{R}^n$ by cells making use of the following partitions of real lines

$$\Delta_i : (-\infty, x_1^{(i)}), [x_1^{(i)}, x_2^{(i)}), ..., [x_{N_i-1}^{(i)}, \infty).$$

Definition 5.3. *The multivariate space of the polynomial spline $S^{\mathbf{k}}_\Delta$ of the degree $\mathbf{k} = (k_1, ..., k_n)$ of the defect 1 is the tensor product of univariate polynomial spline spaces: $S^{\mathbf{k}}_\Delta = \otimes_{i=1}^n S^{k_i}_{\Delta_i}$.*

In particular, this signifies that the dimension of S^k_Δ is equal to $(N_1 + k) \cdot ... \cdot (N_n + k)$ and its basis is formed by the products of univariate B-splines:

$$\begin{aligned} &B_I(X) = B_{i_1}(x_1)...B_{i_n}(x_n), \\ &I \in \Pi = \{i_1, ..., i_n), \ 1 \le i_1 \le N_1 + k_1, ..., 1 \le i_n \le N_n + k_n\}, \\ &X = (x_1, ..., x_n). \end{aligned} \tag{5.45}$$

The function $B_I(X)$ has the finite support consisting of at most $(k_1+1) \cdot ... \cdot (k_n+1)$ cells of the partition.

Any function $f \in S^{\mathbf{k}}_\Delta$ is linear combination

$$f(X) = \sum_{I \in \Pi} C_I B_I(X) \tag{5.46}$$

of multivariate B-splines $B_I(X)$. Analogously to (5.43) introduce the multivariate enumeration of cells and the function

$$I_X : X \to \Phi$$
$$\Phi = \{(i_1, ..., i_n),\ 0 \le i_1 < N_1, ..., 0 \le i_n < N_n\},$$

which puts into correspondence to a point X the multi-index I_X of the cell containing X. Then formula (5.46) may be changed:

$$f(X) = \sum_{J \in \Theta} C_{I_X+J} B_{I_X+J}(X). \tag{5.47}$$

The set $\Theta = \{(s_1, ..., s_n) : 0 \le s_1 \le k_1, ..., 0 \le s_n \le k_n\}$ is called a pattern of shifts.

5.2.4. Assembling of Interpolating Matrix A

Consider two smoothing D^m–spline problems on B-spline spaces ($n = 1, 2$):

$$\sigma_n = \arg \min_{u \in S^k_\Delta} \alpha |u|_n + \sum_{i=1}^{N} w_i (L_i u - r_i)^2$$

with the positive weigths $w_1, ..., w_N$ and linear continuous functionals $L_1, ...,$ L_N. Then, in accordance with the general theory of splines on subspaces (see Chapter 4) the elements of the interpolating matrix A for B-splines case are of the following form

$$a_{iI} = \sqrt{w_i}(L_i B_I)(P_i), \quad i = \overline{1, N}, \quad I \in \Pi. \tag{5.48}$$

The multiplication procedure on an arbitrary vector $(u_I, I \in \Pi)$ is as follows:

$$v_i = \sum_{I \in \Pi} a_{iI} u_I, \quad i = \overline{1, N}. \tag{5.49}$$

Formula (5.49) is computationally inconvenient: first, it does not takes into account the sparseness of A, secondly: the multi-index realization of arrays in programming languages is impossible or very slow. Further, we show how to dispose the matrix in a two-dimensional array in a packed form without zero elements, and organize a fast unpacking with the aims of multiplication of the matrices A, A^* by an arbitrary vector.

Introduce one-dimensional enumerations on the sets of multi-indexes of basic functions Π, multi-indexes of cells Φ, pattern of shifts Θ in accordance with the following formulas

$$\begin{aligned} \pi(I) &= 1 + (i_1 - 1) + (i_2 - 1)(N_1 + k_1) + \dots, \quad I \in \Pi, \\ \varphi(I) &= 1 + (j_1 - 1) + (j_2 - 1)N_1 + \dots, \quad J \in \Phi, \\ \rho(J) &= 1 + j_1 + j_2 k_1 + \dots, \quad J \in \Theta. \end{aligned} \tag{5.50}$$

i.e. in the lexicographic order.

Making use of formula (5.47), multiplication procedure (5.49) is rewritten in the following form:

$$v_i = \sqrt{W_i}\sum_{I\in\Pi} u_I(L_iB_I)(P_i) = \sqrt{W_i}L_i(\sum_{I\in\Pi} u_IB_I(X))(P_i)$$
$$= \sqrt{W_i}L_i(\sum_{J\in\Theta} u_{I_i+J}B_{I_i+J})(P_i) = \sum_{J\in\Theta} u_{I_i+J}\sqrt{W_i}(L_iB_{I_i+J})(P_i),$$

where I_i stands for the multi-index of the cell I_{P_i} containing the point P_i. Thus, in order to calculate (5.49) we must preserve only the following elements of the matrix A:

$$a_{i,I_i+J} = \sqrt{W_i}(L_iB_{I_i+J})(P_i), \quad i=\overline{1,N},\ J\in\Theta \tag{5.51}$$

and produce calculations with the help of the following sum

$$v_i = \sum_{J\in\Theta} a_{i,I_i+J}u_{I_i+J}. \tag{5.52}$$

Unfortunately formulas (5.51),(5.52) contain calculations with multi-indexes. Using notations (5.50), propose another implementation. Replace the elements of the matrix A in two-dimensional array $\bar{A}=(a_{ij})$ as follows

$$\bar{a}_{i,j} = a_{i,I_i+\rho^{-1}(j)}, \quad i=\overline{1,N},\ j=\overline{1,K}, \tag{5.53}$$

where $K=(k_1+1)\cdot\ldots\cdot(k_n+1)$ is the number of different multi-indexes in the pattern array Θ. Enumerate the vector u from (5.49) in the following order

$$\bar{u}_i = u_{\pi^{-1}(i)}, \quad i=\overline{1,H}, \tag{5.54}$$

Here $H=(N_1+k_1)\cdot\ldots\cdot(N_n+k_n)$ is the dimension of the vector u. Now, multiplication procedure (5.52) may be rewritten as follows:

$$\begin{aligned} v_i &= \sum_{j=1}^{k} \bar{a}_{ij}\ u_{I_i+\rho^{-1}(j)} \\ &= \sum_{j=1}^{k} \bar{a}_{ij}\ \bar{u}_{\pi(I_i+\rho^{-1}(j))} = \sum_{j=1}^{k} \bar{a}_{ij}\ \bar{u}_{\pi(I_i)+\pi\rho^{-1}(j)}. \end{aligned} \tag{5.55}$$

Introduce additionally the arrays of attachement M and shifts G:

$$m_i = \pi(I_i), \quad i=\overline{1,N},$$
$$g_j = \pi\rho^{-1}(j), \quad j=\overline{1,K}.$$

After that, the multiplication procedure may be described in following form

$$v_i = \sum_{j=1}^{k} \bar{a}_{ij}\bar{u}_{m_i+g_j} \tag{5.56}$$

which is convenient for computations. We shall not prove analogous formulas for multiplication by A^*, but only give them in ready form

$$\begin{aligned} &\bar{u} = 0, \\ &\bar{u}_{m_i+g_j} = \bar{u}_{m_i+g_j} + \bar{a}_{ij}\bar{v}_i, \quad \forall i = \overline{1,N}, \quad j = \overline{1,k}. \end{aligned} \tag{5.57}$$

First, we annul the vector $\bar{u}$, then, in accordance with (5.57: second line) successively change its components. Formulas (5.56), (5.57) may be effectively realized in accembler codes.

5.2.5. Assembling of Energy Matrix T

Let the domain $\Omega = [a_1, b_1] \times ... \times [a_n, b_n]$ coincide with the rectangular domain $[x_1^{(1)}, x_{N_1-1}^{(1)}] \times ... \times [x_1^{(n)}, x_{N_n-1}^{(n)}]$, i.e. with the union of all internal cells of the partition Δ. The elements of the energy matrix T have of one the following forms (i=1,2):

$$t_{IJ} = (B_I, B_J)_i, \quad I \in \Pi, \quad J \in \Pi. \tag{5.58}$$

The matrices are sparse because of finite supports of the B-splines. Consider the structure of the energy matrix T in dètail for the first semi-norm. Making use of the notations of Section 5.2.2 we have

$$t_{IJ} = \sum_{l=1}^{n} \int_{\Omega_l} B_{I_l}(X_l)B_{J_l}(X_l)dX_l \int_{a_l}^{b_l} B_{i_l}^{(m_l)}(x_l)B_{j_l}^{(m_l)}(x_l)dx_l. \tag{5.59}$$

From the definition of B-splines it follows that $t_{IJ} \neq 0$, if $|I - J| \leq k$ (i.e. $|i_l - j_l| \leq k_l$, $\forall\, l = 1, ..., n$). The latter condition provides for B_I and B_J, whose supports are crossed.

Introduce the pattern $\Theta_1 = \{(i_1, ..., i_n) : -k_1 \leq i_1 \leq k_1, ..., -k_n \leq i_n \leq k_n\}$ and numeration of its elements:

$$\eta_1(I) = i_1 + k_1 + (i_2 + k_2)(2k_1 + 1) + ... \tag{5.60}$$

Then, the multiplication procedure of matrix T on an arbitrary vector may be presented as follows

$$V_I = \sum_{J \in \Theta_1} t_{I,I+J} u_{I+J}, \quad I \in \Pi. \tag{5.61}$$

Similarly Section 5.2.4 the multiplication procedure may be replaced into two-index expression

$$\bar{v}_i = \sum_{j=1}^{K1} \bar{t}_{i,j} \bar{u}_{i+g_1(j)}, \tag{5.62}$$

where

$$K_1 = (2k_1+1) \times ... \times (2k_n+1),$$

$$\bar{t}_{i,j} = t_{\pi^{-1}(i),\pi^{-1}(i)+\eta_1^{-1}(j)}, \quad i=\overline{1,H}, \quad j=\overline{1,K1}, \tag{5.63}$$

$$g_1(j) = \pi(\eta_1^{-1}(j)), \quad j=\overline{1,K1}.$$

Note that in formula (5.61) the index $I+J$ ($I \in \Pi$, $J \in \Theta_1$) can leave the set Π. In this case it is necessary to annul the respective elements $\bar{t}_{ij}$ in formula (5.63), and then the summing in (5.62) must be held when $\bar{t}_{ij}$ is not equal to zero.

Consider now the second energy semi-norm $|\cdot|_2$. For it the element t_{IJ} is calculated in another form:

$$t_{IJ} = \sum_{l=1}^{n} \delta(I_l, J_l) \int_{\Omega_l} B_{I_l}(X_l)dX_l \int_{a_l}^{b_l} B_{i_l}^{(m_l)}(x_l) B_{j_l}^{(m_l)}(x_l)dx_l. \tag{5.64}$$

In this case the matrix T becomes more sparse, there arizes the opportunity to conserve it more efficiently and organize a faster multiplication procedure of the matrix on an arbitrary vector. Naturally, the element t_{IJ} is not equal to zero, iff $I_l = J_l$ & $|I-J| \le k$. Instead of the rectangular pattern Θ_1 there arizes the croix pattern Θ_2, which may conveniently written out in the form of the union

$$\Theta_2 = \bigcup_{k=1}^{n} \Theta^l = \bigcup_{l=1}^{n} \{(0,...,i_l,...,0), \ -k_l \le i_l \le k_l\}.$$

Then, the multiplication procedure of the matrix T by arbitrary vector is presented as follows

$$v_I = \sum_{J\in\Theta_2} t_{I,I+J} u_{I+J}, \quad I \in \Pi. \tag{5.65}$$

Introducing the numeration of different multi-indexes from the pattern Θ_2 in the same manner as previously, we have a simplified formula for multiplication:

$$\bar{v}_I = \sum_{j=1}^{K2} \bar{t}_{i,j} \bar{u}_{i+g_2(j)}, \tag{5.66}$$

$$\bar{t}_{i,j} = t_{\pi^{-1}(i),\pi^{-1}(i)+\eta_2^{-1}(j)}, \quad i=\overline{1,H}, \quad j=\overline{1,K2}, \tag{5.67}$$

$$g_2(j) = \pi(\eta_2^{-1}(j)), \quad j=\overline{1,K2},$$

where $K2 = (2k_1+1) + ... + (2k_n+1) - (n-1)$ is the number of different multi-indexes of the pattern Θ_2. The matrix T may be conserved in a smaller array than before. We do not need the calculation of its elements but only the two following arrays

$$t^l_{I_l} = \int\limits_{\Omega_l} B_{I_l}(X_l)dX_l,$$

$$z_{i_l,j_l} = \int\limits_{a_l}^{b_l} B^{(m_l)}_{i_l}(x_l)B^{(m_l)}_{j_l}(x_l)dx_l, \quad l=\overline{1,n}, \quad I\in\Pi.$$

Then the multiplication procedure is of the following form

$$v_I = \sum_{l=1}^{n}\sum_{j_l\in\Theta^l} t^l_{I_l} z_{i_l,j_l} u_{I+J}.$$

We shall not reduce the formula to two-index form. Note only that the reduced formula is equivalent to the convolution on one-dimensional lines (on columns and horizontal lines in two-dimensional case $n=2$). Unfortunately, the realization formula is twice as expensive.

5.2.6. Convergence in Anisotropic Space

The aim of this Section is to prove the equivalence of the semi-norms $|\cdot|_1$ and $|\cdot|_2$ on the space of B-splines. This equivalence is sufficient grounds for the application of the second semi-norm because the error estimates are the same, but the algorithm (see previous Section) is cheaper.

For the sake of simplicity consider the domain $\Omega = [0,1]^n$. Let $S^k_\Delta(\Omega)$ be the space of B-splines with equidistant meshes in each direction, given by the vector $\boldsymbol{\tau} = (\tau_1,...,\tau_n)$, where $\tau_1 = 1/N_1,...,\tau_n = 1/N_n$ are reciprocal to the integers. Prove the following preliminary lemma.

Lemma 5.8. If the semi-norms $\|\cdot\|_a$ and $\|\cdot\|_b$ in the finite-dimensional space X have the same kernel P, and the first semi-norm is majorated by the second one: $\|u\|_a \le C_2\|u\|_b$, then the semi-norms are equivalent.

Proof. Decompose X in the direct sum of subspaces $X = P\oplus P^\perp$. Since the semi-norms become the norms in $P^\perp$, and $P^\perp$ is finite-dimensional, then the norms are equivalent on $P^\perp$. The same, obviously, concerns the semi-norms because

$$\|u\|_a = \|u_1+u_2\|_a = \|u_2\|_a, \quad \|u\|_b = \|u_1+u_2\|_b = \|u_2\|_b,$$

where $u_1\in P$, $u_2\in P^\perp$ are the elements, uniquely determined by the decomposition $X = P\oplus P^\perp$. □

The kernel of the first semi-norm $|\cdot|_1$ is the tensor product $P_{m_1-1}\otimes ... \otimes P_{m_n-1}$ of polynomial spaces. It consists of the polynomials of n variables, whose degree in the variable x_l does not exceed (m_l-1), $l=1,...,n$. Further we show that the kernel of the semi-norm $|\cdot|_2$ is the same space, and the equivalence could be followed from the Lemma. Show that each term of sum

$$(u,u)_1 = \sum_{l=1}^{n} \int_{\Omega} (D_l^{m_l} u)^2 dX, \tag{5.68}$$

is equivalent to the respective term of sum

$$(u,u)_2 = \sum_{l=1}^{n} \int_{\Omega} \sum_{I_l \in \Pi_l} B_{I_l}(X_l) \left(\sum_{i_l=1}^{N_l+k_l} C_I B_{i_l}^{(m_l)}(x_l) \right)^2 dX. \tag{5.69}$$

Before we proved the following estimates

$$\int_{\Omega} (D_l^{m_l} u)^2 dX \le \int_{\Omega} \sum_{I_l \in \Pi_l} B_{I_l}(X_l) \left(\sum_{i_l=1}^{N_l+k_l} C_I B_{i_l}^{(m_l)}(x_l) \right)^2 dX,$$

hence, in order to prove the equivalence it is sufficient to show, that the kernel of the first semi-norm (in the latter inequality) is the kernel of the second one. The kernel of $\int_{\Omega} (D_l^{m_l} u) dX$ is the tensor product $T_l = S_1 \otimes ... \otimes S_{l-1} \otimes P_{m_l-1} \otimes S_{l+1} \otimes ... \otimes S_n$, i.e. the tensor product of $(n-1)$ univariate B-spline's spaces and the space of polynomials of $(m_l - 1)$ degree in the remaining direction. Represent the function from this space by the basis B_I, $I \in \Pi$. Let

$$\sum_{i_l=1}^{N_l+k_l} C_{i_l}^j B_{i_l}(x_l), \quad j = \overline{1, m_l}$$

be the basis of P_{m_l-1}, $B_{I_l}(X_l)$, $I_l \in \Pi_l$ be the basis of $S_1 \otimes ... \otimes S_{l-1} \otimes S_{l+1} \otimes ... \otimes S_n$. Then the basis of the space T_l is the set of functions

$$B_{I_l}(X_l) \sum_{i_l=1}^{N_l+k_l} C_{i_l}^j B_{i_l}(x_l), \quad I_l \in \Pi_l,\ j = \overline{1, m_l}.$$

Evidently, any of these functions and, consequently, an arbitrary linear combination of the functions, annul the semi-norm

$$\int_{\Omega} \sum_{I_l \in \Pi_l} B_{I_l}(X_l) \left(\sum_{i_l=1}^{N_l+k_l} C_I B_{i_l}^{(m_l)}(x_l) \right)^2 dX.$$

Thus, we have proved the equivalence of semi-norms (5.68), (5.69). However, this proof does not guarantee the boundness of the constants of the equivalence. For this reason we go further.

Theorem 5.4. The semi-norm $|\cdot|_1$ is uniformely equivalent to the semi-norm $|\cdot|_2$ when $N_1, ..., N_n$ accept various natural values.

Proof. The basic splines in the space $S_\Delta^k(\Omega)$ satisfy equality

$$B_I(X) = B(\frac{X}{\tau} - I), \tag{5.70}$$

where $X \in \Omega \cap \text{supp}(B_I)$. They are obtained with the help of the linear transformation of the argument of the model B-spline [1].

It would be recalled that the first energy semi-norm in the tensor product of $B-$spline spaces

$$(u,u)_1 = \sum_{l=1}^{n} \int_{\Omega} \sum_{I\in\Pi} \sum_{J\in\Pi} C_I D_l^{m_l} B_I(x) C_J D_l^{m_l} B_J(x) dX.$$

The domain Ω is decomposed on the cells with the edges of the length $\tau = (\tau_1, ..., \tau_n)$, which may be enumerated with the help of the multi-indexes

$$P \in \Phi = \{(P_1, ..., P_n)\},\ 0 \le P_1 < N_1, ..., 0 \le P_n < N_n\}.$$

Denote the cell corresponding to the multi-index P as Ω_P. From the condition of equidistant partition Δ in each direction it follows that $\Omega_P = \tau(\Omega + P)$. Replace the latter integral by the sum of integrals by cells:

$$(u,u)_1 = \sum_{l=1}^{n} \sum_{P\in\Phi} \int_{\Omega_P} \sum_{I\in P} \sum_{J\in P} C_I D_l^{m_l} B_I(X) C_J D_l^{m_l} B_J(X) dX$$

and apply formula (5.47) of the spline representation

$$(u,u)_1 = \sum_{l=1}^{n} \sum_{P\in\Phi} \int_{\Omega_P} \sum_{I\in\Theta} \sum_{J\in\Theta} C_{I+P} D_l^{m_l} B_{I+P}(X) C_{J+P} D_l^{m_l} B_{J+P}(X) dX.$$

In the latter expression change the variables $X = \tau(Y + P)$. Then, using equality (5.70) and taking into account that Ω_P is mapped into the model domain Ω, we have

$$(u,u)_1 = \sum_{l=1}^{n} \frac{|\tau|}{\tau_l^{2m_l}} \sum_{P\in\Phi} \int_{\Omega} \sum_{I\in\Theta} \sum_{J\in\Theta} [C_{I+P} D_l^{m_l} B(Y - I) \times C_{J+P} D_l^{m_l} B(Y - J)]\, dY, \tag{5.71}$$

where $|\tau| = \tau_1 \cdot ... \cdot \tau_n$. Analogously one can obtain the following representation for the second semi-norm:

$$(u,u)_2 = \sum_{l=1}^{n} \frac{|\tau|}{\tau_l^{2m_l}} \sum_{P\in\Phi} \int_{\Omega} \sum_{I\in\Theta} \sum_{J\in\Theta} [\delta(I_l, J_l) C_{I+P} D_l^{m_l} Z^l(Y - I) \times C_{J+P} D_l^{m_l} Z^l(Y - J)]\, dY. \tag{5.72}$$

Analyzing expressions (5.71), (5.72) one can see that the energy semi-norms of a polynomial spline consist of the energies on the cells for the first semi-norm as well as for the second one. Thus, since we first prove equivalence for the model domain, then the total equivalence is obtained by summation. The Theorem is proved.

□

[1] The model B-spline has its nodes on the integer mesh.

5.2.7. Convergence Rates in Isotropic Space

Let $\Omega = [0,1]^n$, $\{A_h\}$ be a set of condensed h-nets in Ω. Denote by E^τ finite element spaces, which approximate the space $W_2^m(\Omega)$ when $\tau \to 0$. In this Section we are interested the convergence problem for D^m-splines on subspaces

$$\sigma_{h,\tau} = \arg \min_{u \in E_h^\tau(f)} \|D^m u\|_{L^2(\Omega)}, \tag{5.73}$$

where $E_h^\tau(f)$ is the subset of interpolants for the function f in the space E^τ, connected with the scattered mesh A_h, i.e.

$$E_h^\tau(f) = \{u_{h,\tau} \in E^\tau : \quad u_{h,\tau}(P) = f(P),\ \forall P \in A_h\}.$$

In particular, we consider B-spline spaces S_Δ^k as the spaces E^τ and prove the theorem giving the estimate for the function $\tau = \tau(h)$, which ensures the convergence $\sigma_{h,\tau(h)}$ to f in norms $W_p^k(\Omega)$.

Theorem 5.5. If the following estimates

$$\inf_{u_{h,\tau} \in E_h^\tau(f)} \|D^m(u_{h,\tau} - f)\|_{L^2(\Omega)} \le c\|D^m f\|_{L^2(\Omega)} \tag{5.74}$$

are valid with the constant c independent of $h, \tau(h), f$, then the interpolating D^m-splines $\sigma_{h,\tau}$ converge to f with the following orders

$$\|D^k(\sigma_{h,\tau} - f)\|_{L^p(\Omega)} \le c h^{m-k-\frac{n}{2}+\frac{n}{p}} \|D^m f\|_{L^2(\Omega)}. \tag{5.75}$$

Proof. To prove (5.75) we may use inequalities (5.16) for the function $\sigma_{h,\tau} - f$ and the following inequality

$$\|D^m(\sigma_{h,\tau} - f)\|_{L^2(\Omega)} \le c\|D^m f\|_{L^2(\Omega)}.$$

It is proved in the following way:

$$\begin{aligned}
\|D^m(\sigma_{h,\tau} - f)\|_{L^2(\Omega)} &\le \|D^m \sigma_{h,\tau}\|_{L^2(\Omega)} + \|D^m f\|_{L^2(\Omega)} \\
&\le \|D^m \bar{u}_{h,\tau}\|_{L^2(\Omega)} + \|D^m f\|_{L^2(\Omega)} \\
&\le \|D^m(\bar{u}_{h,\tau} - f)\|_{L^2(\Omega)} + 2\|D^m f\|_{L^2(\Omega)} \\
&\le (c+2)\|D^m f\|_{L^2(\Omega)}.
\end{aligned}$$

Here $\bar{u}_{h,\tau}$ is the function, which gives the minimum in the left-hand side of (5.74). □

Let us assume E^τ to be the B-spline space S_Δ^k with the parameters $\tau_1 = \tau_2 = \ldots = \tau$ and $k_1 = k_2 = \ldots = k \ge m$. Connect with these parameters the uniform τ-mesh, which coincides for odd k with B-spline nodes from S_Δ^k, and is shifted in the centres of B-spline cells for even k. Let us assume in addition that the following hypothesis is true.

Hypothesis. Denote by $E_\tau^\tau(f)$ the subset of interpolants for the function f in the space E^τ, connected with the uniform τ-mesh. Then, the following estimates are valid:

$$\inf_{u_\tau \in E_\tau^\tau(f)} \|D^m(u_\tau - f)\|_{L^2(\Omega)} \leq c\|D^m f\|_{L^2(\Omega)} \tag{5.76}$$

Theorem 5.6. The interpolating splines $\sigma_{h,\tau}$ converge to f with estimates (5.75) if for any function $g \in W_2^m(\Omega)$ there exists an interpolant

$$u_{h,\tau} = \sum_{I \in \Pi_\tau} \alpha_I B_I \tag{5.77}$$

satisfying $u_{h,\tau}(P) = g(P)$, $\forall P \in A_h$, whose coefficients satisfy the following condition

$$\sum_{I \in \Pi_\tau} |\alpha_I| \leq c_0 \sum_{P \in A_h} |g(P)|. \tag{5.78}$$

Here Π_τ is the pattern for multivariate indexes of B-splines, c is the constant independent of $h, \tau(h), g$.

Proof. Let $\sigma_\tau \in E_\tau^\tau(f)$ be the function giving The minimum in the left-hand side of (5.76), i.e.

$$\|D^m(\sigma_\tau - f)\|_{L^2(\Omega)} \leq c\|D^m f\|_{L^2(\Omega)}. \tag{5.79}$$

Choose the coefficients α_I such that function

$$u_{h,\tau} = \sigma_\tau + \sum_{I \in \Pi_\tau} \alpha_I B_I \tag{5.80}$$

would be interpolant of the function f on the h-mesh A_h. In accordance with the conditions of the Theorem this choice is possible, moreover, due to (5.78) we have

$$\sum_{I \in \Pi_\tau} |\alpha_I| \leq c_0 \sum_{P \in A_h} |f(P) - \sigma_\tau(P)|. \tag{5.81}$$

From (5.80) it follows

$$\begin{aligned}\|D^m(u_{h,\tau} - f)\|_{L^2(\Omega)} &\leq \|D^m(\sigma_\tau - f)\|_{L^2(\Omega)} + \|\sum_{I \in \Pi_\tau} \alpha_I D^m B_I\|_{L^2(\Omega)} \\ &\leq \|D^m(\sigma_\tau - f)\|_{L^2(\Omega)} + \max_{I \in \Pi_\tau} \|D^m B_I\|_{L^2(\Omega)} \sum_{I \in \Pi_\tau} |\alpha_I|.\end{aligned}$$

Making use of (5.70), (5.81) we bring about the latter inequality to the following

$$\begin{aligned}\|D^m(u_{h,\tau} - f)\|_{L^2(\Omega)} &\leq \|D^m(\sigma_\tau - f)\|_{L^2(\Omega)} \\ &+ \frac{c_1}{\tau^{m-\frac{n}{2}}} \sum_{P \in A_h} |f(P) - \sigma_\tau(P)|,\end{aligned} \tag{5.82}$$

where $c_1 = c_0 \|D^m B\|_{L^2}$, B is the model B-spline (see Section 5.2.6).

Now, let us apply inequality (5.16) of Lemma 5.4 to the uniform τ-mesh for $p = \infty$, $k = 0$, $M = 1$. This inequality assumes the following form

$$\|u\|_{C(B(t,\rho\tau))} \le c\tau^{m-\frac{n}{2}} \|D^m u\|_{L^2(B(t,\rho\tau))}. \tag{5.83}$$

Here t is any point, u is any function with the zeros at the points of the uniform τ-mesh, c, ρ are constants. From (5.83) it follows

$$\begin{aligned}\frac{c_1}{\tau^{m-\frac{n}{2}}} \sum_{P\in A_h} |f(P) - \sigma_\tau(P)| &\le c_2 \sum_{P\in A_h} \|D^m(f-\sigma_\tau)\|_{L^2(B(P,\rho\tau))} \\ &\le c_2 M_2 \|D^m(f-\sigma_\tau)\|_{L^2(\Omega)}\end{aligned} \tag{5.84}$$

The latter inequality from the fact that each point $Q \in \Omega$ is covered by the finite number of balls $B(p, \rho\tau)$, $P \in A_h$, which does not exceed the fixed constant M_2. Really, since interpolant (5.77) exists, then each cell of B-splines contains at most k^n points of the scattered mesh A_h. Thus, each point is covered at most by $(2\rho+1)^n k^n$ balls.

Combining (5.82) and (5.84) we have

$$\|D^m(u_{h,\tau} - f)\|_{L^2(\Omega)} \le c\|D^m f\|_{L^2(\Omega)}$$

and, hence, inequality (5.74) which is sufficient for D^m-splines to be converging in accordance with Theorem 5.5. □

Theorem 5.7. If $\tau < \frac{h}{k+1}$, then D^m-splines $\sigma_{h,\tau}$ converge to f with estimates (5.75).

Proof. In accordance with Theorem 5.6 it is sufficient to prove existence of interpolant (5.77) and correctness of (5.78). Let $B_{I(P)}$ be the B-spline which has the maximal value at the point P. This is the B-spline whose support centre is the nearest to the point P. Then it is easy to prove that

1) $B_{I(P)}(Q) = 0, \quad P, Q \in A_h, \ P \ne Q,$

2) $B_{I(P)}(P) \ge d(k) > 0,$

where $d(k)$ is independent of τ. Under these two conditions the function

$$u_{h,\tau}(x) = \sum_{P\in A_h} \frac{g(P)}{B_{I(P)}(P)} B_{I(P)}(X)$$

interpolates any function $g(X)$ in the points $P \in A_h$. Besides the coefficients of this expansion satisfy the inequality

$$\sum_{I\in \Pi_\tau} |\alpha_I| = \sum_{P\in A_h} \left| \frac{g(P)}{B_{I(P)}(P)} \right| \le d(K) \sum_{P\in A_h} |g(P)|,$$

i.e. condition (5.78) is fulfilled. □

5.3. D^m-Splines in $\mathbb{R}^n$

Sections 5.3.1, 5.3.2 contain the authors' interpretention of the results by (Duchon 1976, 1977, 1978, et al.) concerning D^m-splines, in $\mathbb{R}^n$. In the next Section we thoroughly describe a less known variant of interpolating smoothing D^m-splines by mean square integrals.

5.3.1. Reproducing Kernel in $D^{-m}L^2$

Denote by $D^{-m}L^2(\mathbb{R}^n)$ (or simply $D^{-m}L^2$) a space of distributions, whose derivatives of order m lie in $L^2(\mathbb{R}^n)$, and equip it with the scalar semi-product

$$(u, v)_m = (D^m u, D^m v)_{L^2(\mathbb{R}^n)} = \sum_{|\alpha|=m} \frac{m!}{\alpha!} \int_{\mathbb{R}^n} D^\alpha u D^\alpha v dX$$

and semi-norm

$$|u|_m = \|D^m u\|_{L^2(\mathbb{R}^n)} = \left(\sum_{|\alpha|=m} \frac{m!}{\alpha!} \int_{\mathbb{R}^n} (D^\alpha u)^2 dX \right)^{\frac{1}{2}} . \tag{5.85}$$

For $m > n/2$ the space $D^{-m}L^2$ consists of the continuous functions $u(X)$, $X \in \mathbb{R}^n$, moreover, for $k < m - n/2$ the functions of the space $D^{-m}L^2$ belongs to the class $C^k(\mathbb{R}^n)$.

The factor space $D^{-m}L^2/P_{m-1}$ with the norm (5.85) is a Hilbert space. The Hilbert structure may be introduced also in $D^{-m}L^2$. To do this consider a uniconnected domain $\Omega \subset \mathbb{R}^n$ with a Lipshitz boundary. Then, the expression

$$\|u\|_m = (\|u\|^2_{L^2(\Omega)} + |u|_m)^{1/2}$$

is the norm defining the Hilbert structure. Thus, we see that $D^{-m}L^2$ is a semi-Hilbert space and we can apply the theory of reproducing kernels (see Chapter 2) to get characterization for D^m-spline in $\mathbb{R}^n$. Fortunately, the reproducing kernel in the space $D^{-m}L^2(\mathbb{R}^n)$ is known in the exact form.

Remember the form of the Green function of the polyharmonic operator Δ^m in $\mathbb{R}^n$ (Sobolev 1974):

$$G_{m,n}(P) = \begin{cases} c_{m,n}\|P\|^{2m-n}, & \text{n odd,} \\ c_{m,n}\|P\|^{2m-n} \ln \|P\|, & \text{n even.} \end{cases}$$

We call the linear functional l on the space $D^{-m}L^2(\mathbb{R}^n)$ continuous if it has a compact support and if its restriction to the Sobolev space $W_2^m(\mathbb{R}^n)$ is continuous.

Lemma 5.9. The Green function is the reproducing kernel in the space $D^{-m}L^2(\mathbb{R}^n)$, i.e. each continuous linear functional which vanishes on the polynomial space P_{m-1} can be represented in the form

$$l(f) = (l(G_{m,n}(P - X)), f(P))_m, \quad \forall f \in D^{-m}L^2(\mathbb{R}^n).$$

Remark 5.7. The Green function $G_{m,n}$ is not in $D^{-m}L^2(\mathbb{R}^n)$, and formally the expression $l(G_{m,n}(P - X))$ does not make sense. Nevertheless, as long as the functional has a compact support, the functions $G_{m,n}$ can be extended to the boundaries of the support in such a fashion that the extension belongs to $D^{-m}L^2$.

5.3.2. Interpolating Smoothing Spline

Assume that $A \subset \mathbb{R}^n$, $f \in D^{-m}L^2$. Denote by $A^{-1}(f)$ a set of functions from $D^{-m}L^2$, coinciding with f on the set A.

Lemma 5.10. Let A contains an L-solvable set for the polynomial space P_{m-1}. Then,

(1) the solution of the problem

$$\sigma = \arg \min_{u \in A^{-1}(f)} |u|_m, \tag{5.86}$$

which is called interpolating D^m-spline in $\mathbb{R}^n$, exists and is unique;

(2) the following orthogonal property is valid:

$$(\sigma, u)_m = 0, \quad v \in A^{-1}(0). \tag{5.87}$$

Theorem 5.8. Let A_h be a sequence of condensed h-nets in Ω. Then for any function $f \in D^{-m}L^2(\mathbb{R}^n)$ $(m > n/2)$ the sequence of interpolating D^m-splines σ_h, which are the solutions to the problems

$$\sigma_h = \arg \min_{A_h^{-1}(f)} \|D^m u\|_{L^2(\mathbb{R}^n)}$$

converge to f when $h \to 0$, with the following asymptotic estimates of convergence:

$$\|D^k(\sigma_h - f)\|_{L^p(\Omega)} \le O(h^{m-k-\frac{n}{2}+\frac{n}{p}})$$

where the constants m, n, p, k satisfy (5.3).

One can see that the estimates do not differ from the ones for interpolating D^m-splines in bounded domain.

The definition of the interpolating D^m-spline can be generalized for the finite set A. Let $k_1, ..., k_{s1}, ..., k_s$, $s1 \le s$ be linearly independent continuous linear functionals on $D^{-m}L^2$. Introduce the real numbers $r_1, ..., r_s$ and the set

$$K_{s1}(\boldsymbol{r}) = \{u \in D^{-m}L^2 : \; k_i(u) = r_i, \quad i = s1+1, ..., s\}$$

and positive numbers ρ_i, $i = 1, ..., s1$.

Definition 5.9. *The function σ_ρ is said to be an interpolating smoothing D^m-spline in $\mathbb{R}^n$ if it is the solution to the problem*

$$\sigma_\rho = \arg \min_{u \in K_{s1}(r)} |u|_m + \sum_{i=1}^{s1} \rho_i (k_i(u) - r_i)^2. \tag{5.88}$$

where $r_1, ..., r_s$ are real numbers.

Making use of the technique, developed in Chapter 2, it is easy to show that the spline σ_ρ satisfies the representation

$$\sigma_\rho(P) = \sum_{i=1}^{s} \lambda_i k_i (G_{m,n}(P - X)) + \sum_{i=1}^{M} \nu_i e_i(P) \tag{5.89}$$

where $G_{m,n}$ is the Green function, $e_1, e_2, ..., e_M$ is the basis in the space P_{m-1}. The coefficients λ_i and ν_i are defined by the system of linear equations

$$\begin{bmatrix} I_\rho + A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{\lambda} \\ \boldsymbol{\nu} \end{bmatrix} = \begin{bmatrix} \boldsymbol{r} \\ \boldsymbol{0} \end{bmatrix} \tag{5.90}$$

The elements of the matrices A and B are defined by the relations $a_{ij} = k_i(k_j(G_{m,n}(P - X)))$ and $b_{ij} = k_j(e_i(X))$. The matrix I_ρ is diagonal with elements $c_{ii} = 1/\rho_i$, $i = 1, ..., s1$, $c_{ii} = 0$, $i = s1 + 1, ..., s$.

The condition for the existence and uniqueness of spline (5.88) and, hence, the condition for the non-singularity of the matrix of equation (5.90) is of the following form:

$$K_0(\boldsymbol{0}) \cap P_{m-1} = \{0\}. \tag{5.91}$$

In the particular case, where $k_i(u) = u(P_i)$, $i = 1, ..., s$, for the spline to be unique it is necessary that the points $P_1, ..., P_s$ should contain the L-solvable set for the space P_{m-1}.

5.3.3. Approximation by Sphere Integral Means

Introduce the linear functional k_S of the integral mean for 3-variate function $u(X)$ by the sphere S of radius h with the center at the point P:

$$k_S(u) = \frac{1}{4\pi h^2} \int_{\|X-P\|=h} u(X) dS_X. \tag{5.92}$$

Here, $4\pi h^2$ is the area of the sphere S, dS_X is an elementary area of S.

1. Definition and Convergence. Let Ω be an uniconnected domain in $\mathbb{R}^3$ with a Lipshitz boundary, $W_2^m(\Omega)$ be the Sobolev space. Take in the domain Ω the spheres $S_1, ..., S_N$ of the radii $h_1, ..., h_N$, whose centers are at the points $P_1, ..., P_N$.

Definition 5.10. *Let $m > \frac{3}{2}$ be an integer. Call the function $\sigma(X)$ the interpolating D^m-spline by integral means by the spheres $S_1, ..., S_N$ of the function $f \in W_2^m(\Omega)$, if it is the solution to the following problem*

$$\sigma = \arg \min_{u \in A^{-1}(f).} |u|_m \tag{5.93}$$

where

$$A^{-1}(f) = \{u \in D^{-m}L^2(\mathbb{R}^3): \quad k_{S_i}(u) = k_{S_i}(f), \quad i = 1, ..., n\}.$$

A propos of the definition of the spaces $D^{-m}L^2(\mathbb{R}^n)$ and the norms $|\cdot|_m$ we refer the reader to Section 5.3.1. Taking into account the form of the Green function in accordance with Chapter 2 we obtain the following form of the spline σ:

$$\sigma(X) = \sum_{i=1}^{N} \lambda_i k_{S_i}(\|X - S\|^{2m-3}) + \sum_{|\alpha| \le m-1} C_\alpha X^\alpha, \tag{5.94}$$

where the functionals k_{S_i} effect the function with respect to the variable S;

$$X^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3}, \quad |\alpha| = \alpha_1 + \alpha_2 + \alpha_3 \le m - 1,$$

are monomials of the degree not exceeding $m - 1$. The coefficients $\boldsymbol{\lambda} = (\lambda_1, ..., \lambda_N)^T$, $\quad C = (C_\alpha, |\alpha| \le m - 1)$ are determined from the system

$$\begin{bmatrix} K & B \\ B^T & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{\lambda} \\ C \end{bmatrix} = \begin{bmatrix} \boldsymbol{k}(f) \\ \mathbf{0} \end{bmatrix} \tag{5.95}$$

with a symmetric matrix. Here the matrices K and B have the elements $k_{ij} = k_{S_i} k_{S_j}(\|P - S\|^{2m-3})$ and $b_{i\alpha} = k_{S_i}(X^\alpha)$, respectively. B^T is the transpose matrix with respect to B, the column vector $\boldsymbol{k}(f)$ has the elements $k_{S_i}(f)$.

We formulate estimates for introduced interpolating spline in the following theorem.

Theorem 5.10. Let $\kappa > 0$. There exists a positive constant $\tilde{h}_0 > 0$ such that for any $h \le \tilde{h}_0$ and for any number of spheres $S_1, ..., S_N$, whose centers form h-net in Ω and radii do not exceed κh, the following equalities

$$\|D^k(\sigma - f)\|_{L^p(\Omega)} \le \tilde{c} h^{m-k-\frac{n}{2}+\frac{n}{p}} \|D^m(\sigma - f)\|_{L^2(\Omega)} \tag{5.96}$$

are valid.

Proof. In accordance with the Lemma 5.5 on Sobolev functions with condensed zeros we can choose the parameter h_0 which provides the estimates of the form

$$\|D^k u\|_{L^p(\Omega)} \le c h^{m-k-\frac{n}{2}+\frac{n}{p}} \|D^m u\|_{L^2(\Omega)} \tag{5.97}$$

for the functions $u \in W_2^m(\Omega)$ with h-net of zeros (with $h \le h_0$). Now prove that the parameters $\tilde{h}_0 = h_0/(1+\kappa)$ and $\tilde{c} = c(1+\kappa)^{m-k-\frac{n}{2}+\frac{n}{p}}$ satisfy the theorem. Since

$$\int_{S_i} (\sigma - f) dS_X = 0$$

then there exist points $Q_i \in S_i$ such that $\sigma(Q_i) = f(Q_i)$. The points $Q_1, ..., Q_N$ form $h(1+\kappa)$-net. Naturally, by the condition of the theorem the centers $P_1, ..., P_N$ of the spheres form h-net, but each point Q_i lies not farther than κh from the respective center P_i. Finally, replace in inequality (5.97) the parameter h by $h(1+\kappa)$ and the function u by $(\sigma - f)$, which has the $h(1+\kappa)$-net of zeros. Thus, we obtain (5.96). □

1. Problem Let $\kappa > 0$, $h_1, h_2, ...$ be parameters converging to zero. Let $T_i = \{S_1^i, ..., S_{N(i)}^i\}$ be the set of condensed spheres, such that the centers of the spheres from T_i form h_i-net, and their radii are less than κh_i. If σ^i is the interpolating D^m-spline by integral means of the function $f \in W_2^m(\Omega)$ by the spheres from the T_i, then the following convergence takes place

$$\|D^m(\sigma^i - f)\|_{L^2(\Omega)} \to 0 \tag{5.98}$$

with $i \to \infty$.

Remark. Though the problem is not solved yet, from Theorem 5.10 follows the following convergence

$$\|D^k(\sigma^i - f)\|_{L^p(\Omega)} \le \tilde{c} h^{m-k-\frac{n}{2}+\frac{n}{p}} \|D^m f\|_{L^2(\Omega)},$$

because in accordance with the orthogonal property

$$\|D^m(\sigma^i - f)\|^2_{L^2(\Omega)} + \|D^m \sigma^i\|^2_{L^2(\Omega)} = \|D^m f\|^2_{L^2(\Omega)}.$$

The rest of the Section 5.3.3 will be devoted to obtaining the analytic representation for the functions $k_{S_i}(\|X - S\|^{2m-1})$ and the elements of the matrices $K = \{k_{ij}\}$ and $B = \{b_{i\alpha}\}$. This will allow one to construct a numerical algorithm for D^m-spline computation.

2. Integration of Radial Functions on Sphere. Let $f(u)$ be a continuous locally integrable function of one variable. We call by a radial function of the points P and Q from $\mathbb{R}^3$ any function of the form $f(\|P - Q\|)$, which depends only on the distance from P to Q. Our aim is to calculate integrals

$$F(P, R) = \frac{1}{4\pi h^2} \int_{\|Q-R\|=h} f(\|P - Q\|) dS_Q. \tag{5.99}$$

For this we need the Poisson formula (Fichtengoltz 1969):

$$\int\limits_{\|X\|=1} f((X,P))dS_X = 2\pi \int\limits_{-1}^{1} f(u\|P\|)du. \tag{5.100}$$

Here the scalar product $(X,P) = x_1p_1 + x_2p_2 + x_3p_3$ has the fixed point $P \in \mathbb{R}^3$ and the point $X \in \mathbb{R}^3$ running the unit sphere with the center in the origin. Replacing the coordinates one can readily obtain the following consequence of (5.100):

$$\int\limits_{\|X\|=h} f((X,P))dS_X = 2\pi h \int\limits_{-h}^{h} f(u\|P\|)du. \tag{5.101}$$

Make the replacing of variables: $X = Q - R$. Then, using representation of the norm with the help of the scalar product we have

$$\begin{aligned} 4\pi h^2 F(P,R) &= \int\limits_{\|X\|=h} f(\|(P-R)-X\|)dS_X \\ &= \int\limits_{\|X\|=h} f(\sqrt{\|P-R\|^2 - 2(P-R,X) + h^2})dS_X \\ &= 2\pi h \int\limits_{-h}^{h} f(\sqrt{\|P-R\|^2 - 2\|P-R\|u + h^2})du. \end{aligned}$$

In the latter integral produce the substitution

$$v = \sqrt{\|P-R\|^2 - 2\|P-R\|u + h^2}, \quad 2vdv = -2\|P-R\|du.$$

Then, we obtain the final formula for integration of radial sphere functions in the following form

$$\begin{aligned} F(P,R) &= \frac{1}{4\pi h^2} \int\limits_{\|Q-R\|=h} f(\|P-Q\|)dS_Q \\ &= \frac{1}{2\|P-R\|h} \int\limits_{|\|P-R\|-h|}^{|\|P-R\|+h|} vf(v)dv. \end{aligned} \tag{5.102}$$

From of this formula it is clear that the integral of the radial function has again a radial form. Formula (5.102) is valid only if $P \neq R$. When the points coincide, then

$$F(P,P) = f(h). \tag{5.103}$$

3. Analytic Representation of Spline.

Theorem 5.11. The solution to problem (5.93) has the following analytic expansion

$$\sigma_m(X) = \sum_{i=1}^{N} \lambda_i K_i(\|X - P_i\|) + \sum_{|\alpha|\le m-1} c_\alpha X^\alpha, \tag{5.104}$$

where

$$K_i(t) = \frac{1}{(2m-1)t}\left[\frac{|t+h_i|^{2m-1} - |t-h_i|^{2m-1}}{2h_i}\right], \quad t > 0,$$
$$K_i(0) = h_i^{2m-3}.$$

Proof. Utilize expansion (5.94) and formulas (5.102),(5.103). Supposing $t = \|X - P_i\|$ make the following transformations

$$K_i(t) = K_i(\|X - P_i\|) = k_{S_i}(\|X - S\|^{2m-3})$$
$$= \frac{1}{2th_i}\int_{|t-h_i|}^{|t+h_i|} v \cdot v^{2m-3}dv = \frac{1}{(2m-1)t}\frac{v^{2m-1}}{2h_i}\Bigg|_{|t-h_i|}^{|t+h_i|}.$$

The formula for $K_i(0)$ readily follows from (5.103). □

4. Determination of Matrix K. The elements of the matrix K of system (5.95) are of the following form

$$k_{ij} = k_{S_i}(k_{S_i}\|X - S\|^{2m-3})). \tag{5.105}$$

For $i = j$ from (5.104) it follows $k_{ii} = k_{S_i}(K_i(\|X - P_i\|)) = K_i(h_i)$, or

$$k_{ii} = K_i(h_i) = 2\frac{(2h_i)^{2m-3}}{2m-1}. \tag{5.106}$$

For $i \neq j$ we have to find an analytic expression for radial function $K_j(\|X - P_j\|) = k_{S_j}(\|X - S\|^{2m-3})$. Making use of formula (5.102) we obtain

$$k_{ij} = \frac{1}{2t_{ij}h_i}\int_{|t_{ij}-h_i|}^{|t_{ij}+h_i|} tK_j(t)dt, \tag{5.107}$$

where $t_{ij} = \|P_i - P_j\|$. For the sake of simplicity investigate the case, when the interiors of the spheres S_i and S_j are not intersected, i.e. $t_{ij} \ge h_i + h_j$. Then, we can remove the modulus in (5.107) and, taking into account the determination of the functions $K_j(t)$, obtain the expression

$$k_{ij} = \frac{1}{4(2m-1)t_{ij}h_ih_j} \int\limits_{t_{ij}-h_i}^{t_{ij}+h_i} (t+h_j)^{2m-1} - (t-h_j)^{2m-1} dt.$$

After integration the final formula is the following

$$\begin{aligned} k_{ij} =& \frac{(t_{ij}+h_i+h_j)^{2m} - (t_{ij}-h_i+h_j)^{2m}}{8m(2m-1)t_{ij}h_ih_j} \\ &- \frac{(t_{ij}+h_i-h_j)^{2m} - (t_{ij}-h_i-h_j)^{2m}}{8m(2m-1)t_{ij}h_ih_j}. \end{aligned}$$

5. Integration of Monomials on Spheres in $\mathbb{R}^3$. To find the elements of the matrix B of system (5.95) we need to calculate the following integrals

$$k_S(X^\alpha) = \frac{1}{4\pi h^2} \int\limits_S x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} dS_X \tag{5.108}$$

with $|\alpha| \leq m-1$. To do this, let us consider integral

$$\int\limits_S [(A,X)]^{m-1} dS_X = \int\limits_{\|X-P\|=h} (a_1x_1 + a_2x_2 + a_3x_3)^{m-1} dS_X. \tag{5.109}$$

Change the $(m-1)$-th power of the scalar product into a sum of the terms

$$\int\limits_S [(A,X)]^{m-1} dS_X = 4\pi h^2 \sum_{|\alpha|=m-1} \frac{(m-1)!}{\alpha!} A^\alpha k_S(X^\alpha). \tag{5.110}$$

On the other hand making use of (5.101) integral (5.109) can be calculated exactly:

$$\begin{aligned} \int\limits_{\|X-P\|=h} [(A,X)]^{m-1} dS_X &= \int\limits_{\|Y\|=h} [(A,P)+(A,Y)]^{m-1} dS_Y \\ &= 2\pi h \int\limits_{-h}^{h} [(A,P) + \|A\|u]^{m-1} du \\ &= \frac{2\pi h}{m} \left[\frac{((A,P)+\|A\|h)^m - ((A,P)-\|A\|h)^m}{\|A\|} \right] \\ &= \frac{2\pi h}{m} \left[\frac{\sum_{i=0}^{m} \binom{i}{m} (A,P)^{m-i} \|A\|^i h^i (1-(-1)^i)}{\|A\|} \right] \\ &= \frac{4\pi h^2}{m} \left[\sum_{i=1,3,\ldots}^{m} \binom{i}{m} (A,P)^{m-i} \|A\|^{i-1} h^{i-1} \right]. \end{aligned}$$

The latter expression is summed by odd indexes i. Thus, we obtain the equality

$$\sum_{|\alpha|=m-1} \frac{m!}{\alpha!} A^\alpha K_S(X^\alpha) = \sum_{i=1,3,\ldots}^{m} \binom{i}{m} (A,P)^{m-i} \|A\|^{i-1} h^{i-1}. \tag{5.111}$$

Clearly, the latter sum is expressed as a linear combination of the monomials A^α of the $(m-1)$-th degree, i.e. in the form $\sum\limits_{|\alpha|=m-1} c_\alpha A^\alpha$. From here we have

$$k_S(X^\alpha) = \frac{\alpha!}{m!} c_\alpha, \quad |\alpha| = m-1. \tag{5.112}$$

We use this equality to calculate the functionals $k_S(X^\alpha)$ up to the third degree, i.e. for the cases $m = 1, 2, 3, 4$:

1) $\sum c_\alpha A^\alpha = 1.$

2) $\sum c_\alpha A^\alpha = 2(a_1 p_1 + a_2 p_2 + a_3 p_3).$

3) $\sum c_\alpha A^\alpha = 3(a_1 p_1 + a_2 p_2 + a_3 p_3)^2 + h^2(a_1^2 + a_2^2 + a_3^2).$

4) $\sum c_\alpha A^\alpha = 4(a_1 p_1 + a_2 p_2 + a_3 p_3) + 4h^2(a_1 p_1 + a_2 p_2 + a_3 p_3) \times (a_1^2 + a_2^2 + a_3^2).$

Making use of the given expansions and (5.112) one can obtain

$$\begin{aligned} &k_S(1) = 1, \quad k_S(x_i) = p_i, \quad k_S(x_i^2) = p_i^2 + \frac{h^2}{3}, \\ &k_S(x_i x_j) = p_i p_j, \quad k_S(x_i^3) = p_i(p_i^2 + h^2), \\ &k_S(x_i^2 x_j) = p_j(p_i^2 + \frac{h^2}{3}), \quad k_S(x_1 x_2 x_3) = p_1 p_2 p_3. \end{aligned}$$