1. Splines in the Hilbert Spaces

The aim of this chapter is to introduce the main definitions in the abstract variational spline theory and describe the basic properties of interpolating, smoothing and mixed abstract splines.

1.1. Interpolating, Smoothing and Mixed Splines

1.1.1. Main Definitions

Let X, Y and Z be real separable Hilbert spaces and $T : X \to Y$, $A : X \to Z$ be some linear bounded operators. Consider an element $z \in Z$.

Definition 1. *Solution of the variational problem*

$$\sigma = \arg \min_{x \in A^{-1}(z)} \|Tx\|_Y^2 \tag{1.1}$$

where $A^{-1}(z) = \{x \in X : Ax = z\}$, we call the interpolating spline corresponding to the initial data $z \in Z$, to the measurement operator A and the energy operator T.

In this situation we assume $A^{-1}(z) \neq \emptyset$.
Let $\alpha > 0$ be any parameter.

Definition 2. *Solution of variational problem*

$$\sigma_\alpha = \arg \min_{x \in X} \{\alpha \|Tx\|_Y^2 + \|Ax - z\|_Z^2\} \tag{1.2}$$

we call the smoothing spline, corresponding to the described objects and the smoothing parameter α.

For the smoothing spline we do not assume $A^{-1}(z) \neq \emptyset$.
The first question arises: what requirements provide the existence and uniqueness of interpolating and smoothing splines?

1.1.2. Interpolation

Denote by $N(T)$ and $N(A)$ the null spaces (kernels) of the operator T and A correspondingly, and by $R(T), R(A)$ their ranges.

Theorem 1.1. Let $A^{-1}(z) \neq \emptyset$. If the subspace $TN(A)$ is closed in Y, then the interpolating spline σ does exist. Furthermore, if $N(T) \cap N(A) = \{\Theta_X\}$ (Θ_X is the null vector in X) then the spline σ is unique.

Proof. Let us fix any element $x_* \in A^{-1}(z)$. Then

$$A^{-1}(z) = x_* + N(A).$$

Hence the manifold

$$TA^{-1}(z) = Tx_* + TN(A)$$

is closed in the space Y. But variational problem (1.1) can be reduced to minimization of the distance between zero vector Θ_Y of the space Y and the manifold $TA^{-1}(z)$.

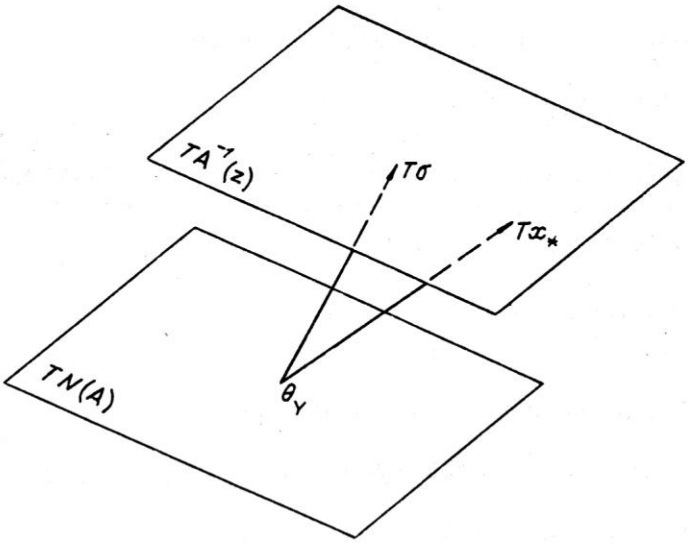

Fig. 1.1.

If $TA^{-1}(z)$ is closed, then this minimal distance is realized at the unique point $f = T\sigma \in TA^{-1}(z)$. If the condition $N(T) \cap N(A) = \{\Theta_X\}$ takes place, then spline σ is unique because the equalities

$$T\sigma_1 = T\sigma_2 = f, \quad A\sigma_1 = A\sigma_2 = z$$

bring about $\sigma_1 - \sigma_2 \in N(T) \cap N(A)$, in other words $\sigma_1 - \sigma_2 = \Theta_X, \quad \sigma_1 = \sigma_2$. □

Theorem 1.2. If $N(T) \cap N(A) = \{\Theta_X\}$, the range $R(T)$ of the operator T is closed in Y and the null-space $N(T)$ is finite-dimensional, then the subspace $TN(A)$ is closed in Y.

Proof. Consider the sequence $y_k \in TN(A)$ which converges in the Y-norm to y_*, the range $R(T)$ is closed and $y_* \in R(T)$. There exists a sequence $x_k \in N(A)$ such that $y_k = Tx_k$. We will prove now that the sequence x_k is bounded in X. We represent x_k in the form $x_k = x_{k,1} + x_{k,2}$, where $x_{k,1} \in N(T)$, $x_{k,2} \in N(T)^\perp$. It is clear that $Tx_{k,2} = y_k$. The restriction $\tilde{T}$ of the operator T to the subspace $N(T)^\perp$ is a bijective mapping between $N(T)^\perp$ and $R(T)$ and by the Banach inversion theorem (see Appendix 1, Theorem 8) the operator $\tilde{T}^{-1}$ is bounded. Then

$$\|x_{k,2}\|_X = \|\tilde{T}^{-1} y_k\|_X \leq \|\tilde{T}^{-1}\| \cdot \|y_k\|_Y.$$

It is clear that $\|y_k\|_Y$ is bounded and $\|x_{k,2}\|_X$ is bounded, too. Remember now that $A(x_{k,1} + x_{k,2}) = \Theta_Z$, $Ax_{k,1} = -Ax_{k,2}$. The restriction of $\tilde{A}$ of the operator A to the finite-dimensional space $N(T)$ is one-to-one operator from $N(T)$ to $AN(T)$ (see $N(T) \cap N(A) = \{\Theta_X\}$), and $\tilde{A}^{-1}$ is bounded. Hence,

$$\|x_{k,1}\| = \|\tilde{A}^{-1} A x_{k,2}\|_X \leq \|\tilde{A}^{-1}\| \cdot \|A\| \cdot \|x_{k,2}\|_X.$$

Finally, the sequence $\{x_k\}$ is bounded. We extract now from $\{x_k\}$ some weakly tending subsequence $x_{k'} \xrightarrow{W} x_*$. Then $Ax_{k'} \xrightarrow{W} Ax_*$, $Tx_{k'} \xrightarrow{W} Tx_*$. However we know that $Ax_{k'} = \Theta_Z$, $Tx_{k'} \xrightarrow{S} y_*$. Finally $Ax_* = \Theta_Z$, $Tx_* = y_*$, in other words, $y_* \in TN(A)$. □

Remark. It is possible to change the requirement "$N(T)$ is finite-dimensional" to the condition "$AN(T)$ is closed in Z".

From the geometrical interpretation (see Fig.1.1) it is obvious that the minimal distance from the point Θ_Y up to the manifold $TA^{-1}(z)$ is realized in the vector $T\sigma$, which is orthogonal to the linear subspace $TN(A)$. From this consideration we can write the orthogonal property of the interpolating spline in the following useful forms

$$\forall x \in A^{-1}(z) \quad \|T(x-\sigma)\|_Y^2 = \|Tx\|_Y^2 - \|T\sigma\|_Y^2, \tag{1.3}$$

$$\forall x \in A^{-1}(z) \quad \|T\sigma\|_Y^2 = (Tx, T\sigma)_Y, \tag{1.4}$$

$$\forall x \in N(A) \quad (T\sigma, Tx)_Y = 0. \tag{1.5}$$

The forms (1.3), (1.4) are dependent on the data vector $z \in Z$, but the latter form (1.5) is independent it. The latter form shows us that every vector σ from the spline space belongs to the space $[T^*TN(A)]^\perp$.

1.1.3. Smoothing

We consider now the problem of the existence and uniqueness of the smoothing splines. Let us define a space $F = Y \times Z$ of the pairs $f = [y, z]$, $y \in Y$, $z \in Z$. Let $f_1 = [y_1, z_1]$, $f_2 = [y_2, z_2]$ be two elements of F. We introduce the scalar product by the formula

$$(f_1, f_2)_F = ([y_1, z_1], [y_2, z_2])_F \stackrel{\text{df}}{=} \alpha(y_1, y_2)_Y + (z_1, z_2)_Z. \tag{1.6}$$

It is obvious that F becomes some new Hilbert space. We define the linear bounded operator $L : X \to F$ by the formula

$$Lx = [Tx, Ax]$$

and introduce the vector $a = [\Theta_Y, z]$. Then the variational functional

$$\Phi_\alpha(x) = \alpha\|Tx\|_Y^2 + \|Ax - z\|_Z^2$$

can be written in the equivalent form

$$\Phi_\alpha(x) = \|Lx - a\|_F^2.$$

Really,

$$\begin{aligned}(Lx - a, Lx - a)_F &= ([Tx, Ax - z], [Tx, Ax - z])_F \\ &= \alpha\|Tx\|_Y^2 + \|Ax - z\|_Z^2 = \Phi_\alpha(x).\end{aligned}$$

Theorem 1.3. If the range $R(L)$ of the operator $L : X \to F$ is closed in F and $N(T) \cap N(A) = \{\Theta_X\}$, then the smoothing spline σ_α which provides the solution of variational problem (1.2) does exist and is unique.

Proof. It is clear that $N(L) = \{\Theta_X\}$. In fact, the equality $Lx = \Theta_F$ means $Tx = \Theta_Y$, $Ax = \Theta_Z$; thus $x \in N(T) \cap N(A)$ and $x = \Theta_X$.

To solve the problem

$$\|L\sigma_\alpha - a\|_F^2 = \min_{x \in X} \|Lx - a\|_F^2,$$

it is necessary to find the minimal distance between the fixed point $a \in F$ and the linear closed subspace $R(L)$ from F (see Fig.1.2). This minimal distance is always realized at the unique point $f = L\sigma_\alpha$. The inversion of this nonsingular operator L is always possible, and $\sigma_\alpha = L^{-1}f$ exists and is unique. □

It is evident that $R(L)$ is closed in F if the ranges $R(T)$ and $R(A)$ are closed in Y and Z respectively and $N(A) + N(T)$ is closed.

From the geometric representation (see Fig.1.2) it is obvious that

$$\forall x \in X \quad (Lx, L\sigma_\alpha - a)_F = 0$$

or in the complete form

$$\forall x \in X \quad \alpha(T\sigma_\alpha, Tx)_Y + (A\sigma_\alpha - z, Ax - z)_Z = -(A\sigma_\alpha - z, z)_Z. \tag{1.7}$$

This identity is the orthogonal property of the smoothing spline. Note that the right-hand side of (1.7) is independent of x.

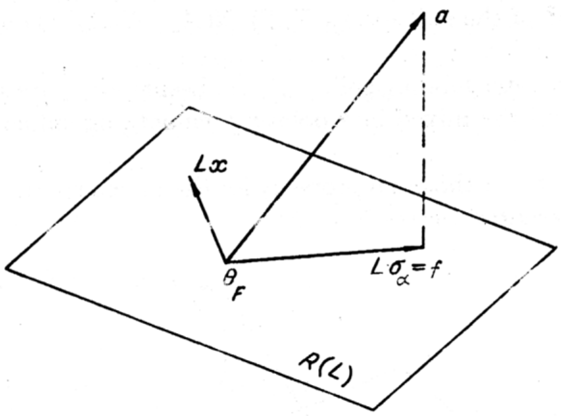

Fig. 1.2.

1.1.4. Mixed Splines

It is possible to consider the mixed case of interpolating-smoothing splines. In this situation the measurement operator $A : X \to Z$ is splitted up two operators $A_1 : X \to Z_1$, $A_2 : X \to Z_2$, Z_1, Z_2 are any Hilbert spaces and $Z = Z_1 \times Z_2$. If $z = [z_1, z_2]$, the interpolating-smoothing spline σ_α is the solution of the variational problem

$$\sigma_\alpha = \arg \min_{x \in A_1^{-1}(z_1)} \{\alpha \|Tx\|_Y^2 + \|A_2 x - z_2\|_{Z_2}^2\}, \tag{1.8}$$

where $A_1^{-1}(z_1) = \{x \in X : A_1 x = z_1\}$. In this case the initial data z_1 will be interpolated and the data z_2 will be smoothed. Now if we introduce the Hilbert space $F_2 = Y \times Z_2$ with the scalar product .

$$([y_1, z_2'], [y_2, z_2''])_{F_2} = \alpha(y_1, y_2)_Y + (z_2', z_2'')_{Z_2},$$

the composite operator $L_2 : X \to F_2$ by the formula

$$L_2 x = [Tx, A_2 x]$$

and the fixed vector $a_2 = [\Theta_Y, z_2]$, then variational problem (1.8) can be reduced to

$$\sigma_\alpha = \arg \min_{x \in A_1^{-1}(z_1)} \|L_2 x - a_2\|_{F_2}^2.$$

With respect to the general consideration (see Theorem 1.1) it means that the existence and uniqueness of σ_α are provided if $L_2 N(A_1)$ is closed in F_2 and $N(A_1) \cap N(L_2) = \{\Theta_X\}$, but $N(L_2) = N(T) \cap N(A_2)$. To simplify the formulation of the existence and uniqueness theorem we do not assume the most general but very suitable in practice conditions:

1. Null space of T is finite-dimensional.
2. Ranges $R(T), R(A_1), R(A_2)$ are closed in Y, Z_1, Z_2, respectively.
3. Intersection of the null spaces $N(T), N(A_1), N(A_2)$ is only zero vector of the space X.

Then the pure interpolating spline (A_2 is absent), the pure smoothing spline (A_1 is absent) and the mixed interpolating- smoothing spline do always exist and are unique.

We write now the orthogonal property for the mixed spline. From the trivial geometrical representation (see Fig.1.3)

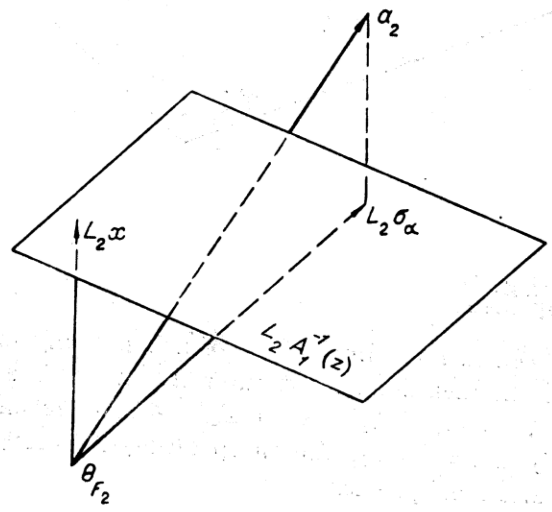

Fig. 1.3.

we have

$$\forall x \in A_1^{-1}(z_1)\ (L_2\sigma_\alpha - L_2 x, L_2\sigma_\alpha - a_2)_{F_2} = 0,$$

or in detail

$$\forall x \in A_1^{-1}(z_1) \quad ([T(\sigma_\alpha - x), A_2(\sigma_\alpha - x)], [T\sigma_\alpha, A_2\sigma_\alpha - z_2])_{F_2}$$
$$= \alpha\|T\sigma_\alpha\|_Y^2 - \alpha(T\sigma_\alpha, Tx)_Y + (A_2\sigma_\alpha - z_2, A_2\sigma_\alpha - A_2 x)_{F_2} = 0.$$

Finally, the following orthogonal identity is valid

$$\alpha(T\sigma_\alpha, Tx)_Y + (A_2\sigma_\alpha - z_2, A_2 x - z_2)_{Z_2} = \alpha\|T\sigma_\alpha\|_Y^2 + \|A_2\sigma_\alpha - z_2\|_{Z_2}^2 \quad (1.9)$$

for every $x \in X$ under the constraints $A_1 x = z_1$. The right-hand side of this identity is independent of x.

Remark. We have already considered the problem of the uniqueness for the main type of splines. In every case the condition $N(T) \cap N(A) = \{\Theta_x\}$ is necessary for the uniqueness of a spline. If the subspace $N(T) \cap N(A)$ is not trivial, then the suitable spline is not unique and can be determined with the accuracy to the arbitrary element from this subspace. But sometimes we do not need to find the spline $\sigma \in X$ but only $B\sigma$, where B is a linear operator from X to some space U. If the null space of the operator B includes $N(T) \cap N(A)$, then the element $B\sigma$ is the same for every solution of the spline-problem. So, if

$$N(B) \supset N(T) \cap N(A),$$

then the element $B\sigma$ is unique while the spline σ is not unique.

1.1.5. Functional Equations on Splines

We obtain now the general functional equations on the mixed interpolating-smoothing spline, and in particular cases of the pure interpolating and smoothing splines we obtain the corresponding equations too.

We assume that the conditions providing the existence and uniqueness of the mixed spline σ_α take place. Then to find the spline σ_α it is necessary to minimize the quadratic functional

$$\Phi_\alpha(x) = \alpha \|Tx\|_Y^2 + \|A_2 x - z_2\|_{Z_2}^2$$

under the linear constraints $A_1 x = z_1$. The corresponding Lagrange function has the form

$$L(x, \lambda) = \frac{1}{2}\Phi_\alpha(x) + (\lambda, A_1 x - z_1)_{Z_1},$$

where $\lambda \in Z_1$ is any Lagrangian parameter. We rewrite $L(x,\lambda)$ in the following form

$$\begin{aligned} L(x, \lambda) = {} & \frac{\alpha}{2}(T^*Tx, x)_X + \frac{1}{2}(A_2^* A_2 x, x)_X - (A_2^* z_2, x)_X \\ & + \frac{1}{2}\|z_2\|_{Z_2}^2 + (A_1^*\lambda, x)_X - (\lambda, z_1)_{Z_2}. \end{aligned}$$

Using the Frechet differentiations with respect to the arguments x and λ we obtain the following conditions for the point $\sigma_\alpha, \lambda_\alpha$ of the minimum

$$\frac{\partial L}{\partial x} = \alpha T^* T \sigma_\alpha + A_2^* A_2 \sigma_\alpha - A_2^* z_2 + A_1^* \lambda_\alpha = \Theta_X$$

$$\frac{\partial L}{\partial \lambda} = A_1 \sigma - z_1 = \Theta_{Z_1}.$$

Then in the matrix form we have functional equations

$$\begin{bmatrix} \alpha T^*T + A_2^* A_2 & A_1^* \\ A_1 & \Theta_{Z_1 \to X} \end{bmatrix} \begin{bmatrix} \sigma_\alpha \\ \lambda_\alpha \end{bmatrix} = \begin{bmatrix} A_2^* z_2 \\ z_1 \end{bmatrix} \tag{1.10}$$

where $\Theta_{z_1 \to x}$ is zero operator from Z_1 to X. For the pure smoothing spline (the constraint $A_1\sigma_\alpha = z_1$ is absent) we obtain equation

$$(\alpha T^*T + A^*A)\sigma_\alpha = A^*z, \tag{1.11}$$

since $A_2 = A,\ z_2 = z$. For the pure interpolating case we have

$$\begin{bmatrix} \alpha T^*T & A^* \\ A & \Theta_{z \to x} \end{bmatrix} \begin{bmatrix} \sigma \\ \lambda_\alpha \end{bmatrix} = \begin{bmatrix} \Theta_x \\ z \end{bmatrix} \tag{1.12}$$

with an arbitrary constant $\alpha > 0$ (for example, $\alpha = 1$); in this situation the operator A_2 is absent, $A_1 = A,\ z_1 = z$. In fact we assume that the uniqueness for the pure interpolating and pure smoothing splines takes place.

1.1.6. Pseudo-Interpolating Splines

Sometimes in practice the interpolating spline as solution of the problem

$$\sigma = \arg \min_{x \in A^{-1}(z)} \|Tx\|_Y^2$$

does not exist because the set $A^{-1}(z)$ is empty or, in other words, the interpolation condition $A\sigma = z$ is contradictory. In this situation it is natural to change the exact interpolation condition $A\sigma = z$ to the least square condition

$$\|A\sigma - z\|_Z^2 = \min_{x \in X} \|Ax - z\|_Z^2. \tag{1.13}$$

If the range $R(A)$ of the operator A is closed in Z then the solution of this least square problem does always exist, but is not unique if the null space of A is not trivial. It is easy to show (with the help of the Frechet differentiation) that condition (1.13) is equivalent to the equation $A^*A\sigma = A^*z$ which is always solvable.

Definition 3. *The pseudo-interpolating spline is the solution of problem*

$$\sigma = \arg \min_{x \in (A^*A)^{-1}(z)} \|Tx\|_Y^2, \tag{1.14}$$

where $(A^*A)^{-1}(z)$ is a set of solutions for the equation $A^*Ax = A^*z$.

It is easy to modify a suitable functional equation for the pseudo interpolating spline. Really, change the operator A for the operator A^*A in (1.12) and obtain

$$\begin{bmatrix} T^*T & A^*A \\ A^*A & \Theta_{X \to X} \end{bmatrix} \begin{bmatrix} \sigma \\ \lambda \end{bmatrix} = \begin{bmatrix} \Theta_X \\ A^*z \end{bmatrix} \tag{1.15}$$

where $\lambda \in X$ is the Lagrangian parameter. It is a trivial fact that $N(A^*A) = N(A)$ and a special theorem on the existence and uniqueness for the pseudo-interpolating spline is not required.

1.1.7. "Any Smoothing Spline is Interpolating One"

Let us consider a closed subspace $R(A)$ in Z. For $z \in R(A)$ the interpolation condition $Ax = z$ is non-contradictory. Thus, for every $z \in R(A)$ under the standard condition the interpolating spline

$$\sigma = \arg \min_{x \in A^{-1}(z)} \|Tx\|_Y^2$$

does exist and is unique. Then the spline space $S(A,T)$ arises.

Theorem 1.4. For every $z \in Z$ the smoothing spline which gives the solution to the problem

$$\sigma_\alpha = \arg \min_{x \in X} \alpha \|Tx\|_Y^2 + \|Ax - z\|_Z^2$$

belongs to the space $S(A,T)$.

Proof. Let σ_α be the solution of the smoothing problem. Consider the element $z_\alpha = A\sigma_\alpha$ and the corresponding interpolating spline $\hat{\sigma}$, which we find from the problem

$$\hat{\sigma} = \arg \min_{x \in A^{-1}(z_\alpha)} \|Tx\|_Y^2.$$

Then $\|T\hat{\sigma}\|_Y^2 \leq \|T\sigma_\alpha\|_Y^2$. On the other hand

$$\alpha \|T\sigma_\alpha\|_Y^2 + \|A\sigma_\alpha - z\| \leq \alpha \|T\hat{\sigma}\|_Y^2 + \|A\hat{\sigma} - z\|_Z^2.$$

As $A\sigma_\alpha = A\hat{\sigma}$, therefore $\|T\sigma_\alpha\|_Y^2 \leq \|T\hat{\sigma}\|_Y^2$. Finally $\|T\sigma_\alpha\|_Y^2 = \|T\hat{\sigma}\|_Y^2$; using the uniqueness of the interpolating spline we have $\sigma_\alpha = \hat{\sigma} \in S(A,T)$. □

This proof can be repeated for the mixed splines without serious transformations.

1.2. Splines and Equivalent Norms in Hilbert Spaces

1.2.1. Main Theorem

Let X, Y and Z be some Hilbert spaces, $T : X \to Y$, $A : X \to Z$ be some linear bounded operators. We assume the standard requirements on these operators, ($\dim N(T)$ is finite, the ranges $R(T)$ and $R(A)$ are closed and the intersection of the null spaces $N(T) \cap N(A)$ is only zero vector). In this situation we call (T, A) the *spline-pair* because the existence and uniqueness of the non-contradictory interpolating spline problem is provided.

Theorem 1.5. If (T, A) is the spline-pair, then the special norm $\|x\|_{(T,A)}$ in the space X introduced by the formula

$$\|x\|_{(T,A)} = (\|Tx\|_Y^2 + \|Ax\|_Z^2)^{1/2} \tag{1.16}$$

is equivalent to the initial norm $\|x\|$, i.e. independent of x constants $C_2 \geq C_1 > 0$ do exist such that

$$C_1\|x\|_X \leq \|x\|_{(T,A)} \leq C_2\|x\|_X. \tag{1.17}$$

Proof. The expression introduced by (1.16) is actually the norm in X, since the expression $\|x\|_{(T,A)}$ is positively homogeneous, the triangle inequality takes place and the condition $\|x\|_{(T,A)} = 0$ brings about to $x = \Theta_X$ because $N(T) \cap N(A) = \{\Theta_X\}$. The constant C_2 trivially exists,

$$\|x\|_{(T,A)} \leq (\|T\|^2_{X\to Y} + \|A\|^2_{X\to Z})^{1/2}\|x\|_X = C_2\|x\|_X.$$

Now we consider the operator $L : X \to Y \times Z$ introduced by the formula $Lx = [Tx, Ax]$, the inner product in $Y \times Z$ will be determined in the natural manner:

$$([y_1, z_1], [y_2, z_2])_{Y\times Z} = (y_1, y_2)_Y + (z_1, z_2)_Z.$$

Then L is the bijective mapping from X onto the closed range $R(L)$. By the Banach inversion theorem the bounded inverse operator L^{-1} does exist and

$$\begin{aligned} C_1 &= \sup_{\|x\|_{(T,A)}=1} \|x\|_X = \sup_{\|Lx\|_{Y\times Z}=1} \|x\|_X \\ &= \sup_{\|f\|_{Y\times Z}=1} \|L^{-1}f\|_X = \|L^{-1}\|_{Y\times Z\to X} > 0. \end{aligned}$$

Theorem is proved. □

1.2.2. Examples of Equivalent Norms in Sobolev Spaces

Example 1. Let $[a, b]$ be any finite interval of the real line and $X = W_2^m[a, b]$ be the Sobolev space of the functions with the squared integrable derivatives up to the order $m \geq 1$. We introduce the norm in this space in the usual manner

$$\|x\|^2_{W_2^m} = \int_a^b x^2(t)dt + \sum_{i=1}^m \int_a^b [x^{(i)}(t)]^2 dt. \tag{1.18}$$

Let $m = 1$ and $T = d/dt$ is linear bounded operator from W_2^1 to $Y = L_2, Z = R^1$ and A be a linear functional

$$Ax = \int_a^b \omega(t)x(t)dt,$$

where $\omega(t) \geq \omega_0 > 0$ for $a \leq t \leq b$. The ranges of the operators T and A are trivially closed. The null space of T is only constants and $N(T) \cap N(A)$ is only a zero constant. Therefore by Theorem 1.5 the norm defined by expression

$$\|x\|^2_{(T,A)} = \left(\int_a^b \omega(t)x(t)dt\right)^2 + \int_a^b [x^{(m)}(t)]^2 dt \tag{1.19}$$

is equivalent to initial norm (1.18).

Example 2. Let us preserve the notations of Example 1, i.e. $X = W_2^m$, $Y = L_2$, $T = d^m/dt^m$. Let $Z = E_n$ be the usual Euclidean n-dimensional space and the operator A be defined by formula

$$Ax(t) = [x(t_1), x(t_2), ..., x(t_n)] \tag{1.20}$$

where $a \leq t_1 < t_2 < ... < t_n \leq b$ is any mesh in $[a,b]$. It is trivial that $R(A) = E_n$ is closed. The null space of T is the space π_{m-1} of the polynomials of the degree less than m. The null space of A is the set of the functions from W_2^m which vanish at mesh points. Then the intersection of these null spaces is only a zero function, if the number of a mesh points is greater than $m-1$. So, if $n \geq m$ then the a norm defined by the formula

$$\|x\|^2_{(d^m/dt^m,A)} = \sum_{i=1}^{n} x^2(t_i) + \int_a^b [x^{(m)}(t)]^2 dt \tag{1.21}$$

is equivalent to the initial one.

Example 3. Preserve again the notations. Let $Z = E_m$ and the operator A be defined by

$$Ax(t) = [x(t_*), x'(t_*), ..., x^{(m-1)}(t_*)], \tag{1.22}$$

where t_* is any fixed point in $[a,b]$. The intersection of $N(A)$ and π_{m-1} is the only zero function, because only zero polynomial of the degree $m-1$ has the root of the multiplicity m at the fixed point. Thus, the norm defined by

$$\|x\|^2_{(d^m/x^m,A)} = \sum_{k=0}^{m-1} [x^{(k)}(t_*)]^2 + \int_a^b [x^{(m)}(t)]^2 dt \tag{1.23}$$

is equivalent to norm (1.18).

Example 4. Let $X = W_2^2[a,b]$, $Y = L_2[a,b]$, $T = d^2/dt^2 + \omega^2 I$, A being a mesh operator (1.20). The null space of the operator T is functions of the type

$$f(t) = C_1 \sin \omega t + C_2 \cos \omega t$$

with arbitrary constants C_1, C_2. By the well-known transformation it is possible to obtain

$$f(t) = A_0 \sin(\omega t + \psi_0),$$

where $A_0^2 = \sqrt{C_1^2 + C_2^2}$, $\psi_0 = \arcsin(C_2/\sqrt{C_1^2 + C_2^2})$. The distance between the neighbouring roots is equal to π/ω. Therefore the number of roots is not greater than $s = 1 +$ entier $((b-a)\omega/\pi)$. Finally if $n \geq s$ then the norm introduced by

$$\|x\|^2_{(T,A)} = \sum_{k=1}^{n} x^2(t_k) + \int_a^b [x''(t) + \omega^2 x(t)]^2 dt \tag{1.24}$$

is equivalent to norm (1.18).

Example 5. Let Ω be any bounded domain in R^n. Consider the Sobolev space $W_2^m(\Omega)$ with the norm

$$\|x\|^2_{W_2^m} = \sum_{|\alpha| \leq m} \int_\Omega (D^\alpha x)^2 d\Omega. \tag{1.25}$$

This space consists of the functions with the square summarized generalized derivatives up to the m-th order. Let us introduce the operator $T = D^m$ of the generalized gradient of the m-th order in the following form

$$D^m x(t_1, ..., t_n) = \left[(\frac{m!}{\alpha!})^{1/2} D^\alpha x, \ |\alpha| = m \right]. \tag{1.26}$$

If the domain Ω is a star one with respect to any ball (i.e. we are able to observe the boundary of Ω from every point of any ball lying in Ω) or if Ω is the union of the finite number of the star subdomains, then the range of the operator D^m is closed in the Cartesian product $Y = \otimes_{i=1}^{R} L_2(\Omega)$, where $R = (n+m-1)!/(m-1)!/n!$ is the number of various multi-indices $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ with the condition $|\alpha| = \alpha_1 + ... + \alpha_n = m$ (see Appendix 1, Sobolev 1950).

The null space of the operator D^m is the space π_{m-1} of the polynomials with n variables $t_1, t_2, ..., t_n$ of the degree $m-1$. The dimension of π_{m-1} is exactly R.

Assume that $m > n/2$ (the embedding of the space $W_2^m(\Omega)$ to the space $C(\Omega)$ of continuous functions is provided) and let $P_1, P_2, ..., P_N$ be arbitrary situated points in Ω (in other words, the scattered mesh). We introduce the mesh operator A from $W_2^m(\Omega)$ to the Euclidean space E_N of the dimension N by

$$Ax(t) = [x(P_1), x(P_2), ..., x(P_N)]. \tag{1.27}$$

The intersection $N(D^m) \cap N(A)$ is a set of the polynomials $Q_{m-1} \in \pi_{m-1}$ which vanish at the points $P_1, P_2, ..., P_N$. If we express the polynomial Q_{m-1} in the form

$$Q_{m-1}(t) = \sum_{|\alpha| \leq m-1} C_\alpha t^\alpha, t^\alpha = t_1^{\alpha_1} t_2^{\alpha_2} ... t_n^{\alpha_n}$$

then it means that

$$\sum_{|\alpha|\leq m-1} C_\alpha P_i^\alpha = 0, i = 1, 2, ..., N,$$

and with respect to the variables C_α a homogeneous linear algebraic system arises. The matrix of this system is a rectangular of the order $N \times R$. To provide the only zero solution of this system it is necessary to assume that the rank of this matrix is not less than R. In other words, there exist R points $P_{i_1}, P_{i_2}, ..., P_{i_R}$ such that the square matrix $R \times R$ composed of the elements $P_{i_k}^\alpha$, $|a| \leq m-1$, $k = 1, 2, ..., R$ is nonsingular. The set of points with this property is called the *Lagrangian set* or, shortly, *L-set.* Finally, if the scattered mesh $P_1, P_2, ..., P_N$ contains any L-set then the norm introduced by

$$\|x\|^2_{(D^m,A)} = \sum_{k=1}^{N} x^2(P_k) + \sum_{|\alpha|\leq m} \frac{m!}{\alpha!} \int_\Omega (D^\alpha x)^2 d\Omega, \tag{1.28}$$

is equivalent to the initial norm.

In the simple case $n = m = 2$, L-set always contains three points which do not lie on the straight line.

1.3. Examples of Splines

1.3.1. One-Dimensional Splines by Point Evaluations

Let us consider the finite interval $[a, b]$ of the real line with the mesh

$$a \leq t_1 < t_2 < ... < t_N \leq b$$

and introduce subsets $J_0, J_1, ..., J_{m-1}$ of the set $J = \{1, 2, ..., N\}$. We want to interpolate the function by its values at the mesh points with the numbers from J_0, by the values of its first derivatives for the numbers from J_1 and so on and, at last, by its $(m-1)$-th derivatives at the mesh points from J_{m-1}. In other words, we have the interpolation conditions of the form

$$\begin{aligned} &\sigma(t_k) = z_k^0, \quad k \in J_0, \\ &\sigma\prime(t_k) = z_k^1, \quad k \in J_1, \\ &. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\ &\sigma^{(m-1)}(t_k) = z_k^{(m-1)}, \quad k \in J_{m-1}. \end{aligned} \tag{1.30}$$

In the simple case $J_1, J_2, ..., J_{m-1}$ are empty sets, $J_0 = J$, and we have the classical interpolation problem. To formulate this problem in the spline form we introduce the Sobolev space $X = W_2^m[a, b]$, $m \geq 1$, and find the spline-function $\sigma(t) \in W_2^m[a, b]$ minimizing the functional

$$\int_a^b [\sigma^{(m)}(t)]^2 dt \tag{1.31}$$

under linear constraints (1.30). Thus, $X = W_2^m[a,b]$, $Y = L_2[a,b]$, $T = d^m/dt^m$, the measurement operator A is defined by (1.30), the data vector z is defined by the right-hand sides in (1.30).

The range of d^m/dt^m is the whole space $L_2[a,b]$ and the range of A is finite-dimensional. Both ranges are trivially closed. It is clear that interpolation conditions (1.30) are always non-contradictory, and the suitable spline $\sigma(t)$ does always exist.

The null space of d^m/dt^m is the polynomial space π_{m-1} of the degree $(m-1)$. The null space of A is a set of functions from W_2^m which vanish at t_k, $k \in J_0$, these 1-st derivatives vanish at t_k, $k \in J_1$, and so on. There are many possibilities to provide $N(T) \cap N(A) = \{\Theta_X\}$. For example, if the number of integers in J_0 is greater than $(m-1)$, then only zero polynomial of the degree $(m-1)$ vanishes at the mesh points t_k, $k \in J_0$ (the main algebra theorem). In this case structures of $J_1, J_2, ..., J_{m-1}$ do not matter, and the interpolating spline is always unique.

Remark. It is possible to analyze the uniqueness of splines with a more complicated operator T (for example, T is the ordinary differential operator with constant coefficients). In this case the null space of T consists of any quasi-polynomials (exponents, trigonometric functions, polynomials and, probably, their multiplicative combinations). In the general case we have no results on the number of their roots, but in the particular cases this analysis is possible (see example 4 in Section 1.2).

1.3.2. One-Dimensional Splines by Local Integrals

Consider again the finite interval $[a,b]$ and assume that the initial information on the function is its local mean integrals,

$$\frac{1}{t_k^+ - t_k^-} \int_{t_k^-}^{t_k^+} \sigma(t)dt = z_k, \quad k = 1,2,...,N \tag{1.32}$$

where $t_k^- < t_k^+$, and they both lie in $[a,b]$. Moreover, the open interval (t_k^-, t_k^+) does not intersect with others. Formally, we have $X = W_2^m[a,b]$, $m \geq 1$, $Y = L_2[a,b]$, $T = d^m/dt^m$, A is the measurement operator, defined by (1.32). The requirements providing the existence of the minimal point of the functional $\int_a^b [\sigma^{(m)}(t)]^2 dt$ under constraints (1.32) are trivially valid.

The null space of A consists of the functions from W_2^m with zero integrals (1.32). It means that any of these functions changes the sign in every interval

(t_k^-, t_k^+) and it has the root in any interval. Thus, if $N > m - 1$, then $\pi_{m-1} \cap N(A) = \{\Theta_x\}$ and the interpolating spline is unique.

1.3.3. Multi-Dimensional D^m-Splines by Point Evaluations

Let $\Omega \subset R^n$ be some bounded domain star with respect to any ball (or a union of the finite number of this type subdomains), $X = W_2^m(\Omega)$ be the Sobolev space with the standard embedding condition $m > n/2$ to the space $C(\Omega)$, $T = D^m$ be the operator of the generalized gradient of the order m. We introduce the scattered mesh $P_1, P_2, ..., P_N$ in Ω and the suitable measurement operator A is the trace operator to the scattered mesh,

$$Ax(t) = [x(P_1), x(P_2), ..., x(P_N)].$$

Then the interpolating D^m-spline $\sigma(t) \in W_2^m(\Omega)$ by point evaluations is the result of minimization of the quadratic functional

$$\|D^m\sigma\|^2_{L_2(\Omega)} \stackrel{\text{df}}{=} \sum_{|\alpha|=m} \frac{m!}{\alpha!} \int_\Omega (D^\alpha\sigma)^2 d\Omega \tag{1.33}$$

under the interpolation constraints

$$\sigma(P_k) = z_k, \quad k = 1, 2, ..., N. \tag{1.34}$$

As we know (see example 5 from Section 1.2) the ranges $R(D^m)$ and $R(A)$ are closed and D^m-spline $\sigma(t)$ does always exist if the scattered mesh does not contain equal points.

The null space $N(D^m) = \pi_{m-1}$ is the space of polynomials of the degree $m-1$, and D^m-spline $\sigma(t)$ is unique if the scattered mesh $P_1, ..., P_N$ contains any L-set.

1.3.4. D^m-Splines by Local Integrals

We preserve the notations of Section 1.3.3 and assume that instead of interpolation conditions (1.34) we have the conditions in the following form:

$$\int_{B_k} \sigma(t) d\Omega = z_k, \quad k = 1, 2, ..., N, \tag{1.35}$$

where B_k are any subdomains in Ω (for example, balls or cubes) such that the intersection of every B_k with other subdomains is empty or a set of zero measure. D^m-spline by local integrals is the result of minimization of functional (1.33) under constraints (1.35). The suitable operator A defined by

$$Ax(t) = \left[\int_{B_1} x(t) d\Omega, ..., \int_{B_N} x(t) d\Omega\right] \tag{1.36}$$

has the null space of the functions with zero integrals over B_k, $k = 1, 2, ..., N$. Therefore, any of these functions has the root in B_k, $k = 1, 2, ..., N$. Thus $N(D^m) \cap N(A)$ consists of only zero vector if there are $B_{k_1}, B_{k_2}, ..., B_{k_m}$ do exist such that the product $B_{k_1} \times B_{k_2} \times ... \times B_{k_m}$ contains L-sets only. In this situation $\sigma(t)$ is unique.

1.3.5. Finite-Dimensional D^m-Splines

Let $X = W_2^m(\Omega)$, $\Omega \subset R^n$ be the Sobolev space, $m > n/2$, and E_τ be the finite-dimensional subspace in $W_2^m(\Omega)$. We consider the scattered mesh $P_1, P_2, ..., P_N$ in Ω and the following interpolation problem: find $\sigma^\tau \in E_\tau$ providing interpolation,

$$\sigma^\tau(P_k) = z_k, \quad k = 1, 2, ..., N \tag{1.37}$$

and minimizing the functional

$$\|D^m \sigma^\tau\|_{L_2}^2 = \sum_{|\alpha|=m} \frac{m!}{\alpha!} \int_\Omega (D^\alpha \sigma^\tau)^2 d\Omega. \tag{1.38}$$

Interpolation conditions (1.37) being non-contradictory in the whole $W_2^m(\Omega)$ may be contradictory in the finite-dimensional space E_τ. Actually, let E_τ be some finite-element space connected with some division of the domain Ω into the finite number of simplexes. If the large number of scattered points is concentrated in one finite element it is impossible to provide the interpolation in this element because the number of free parameters is not sufficiently large. For this reason we change the pure interpolation conditions (1.37) with pseudo-interpolation condition

$$\sum_{k=1}^{N} [\sigma^\tau(P_k) - z_k]^2 = \min_{u^\tau \in E_\tau} \sum_{k=1}^{n} [u^\tau(P_k) - z_k]^2. \tag{1.39}$$

Then using the convensional consideration of the existence and uniqueness for the pseudo-interpolating spline $\sigma^\tau \in E_\tau$ we obtain that existence always takes place and σ^τ is unique if the scattered mesh $P_1, P_2, ..., P_N$ contains any L-set.

1.4. Structure of Spline Projectors

1.4.1. Maximal Spline-Pairs

Let us return to the general consideration of the interpolating and smoothing spline processes. Let X, Y and Z be the Hilbert spaces, $T : X \to Y$, $A : X \to Z$ be linear bounded operators. Remember (see Section 1.2) that (T, A) forms the so-called spline-pair if the ranges of these operators are closed in Y and Z respectively, $N(T) \cap N(A) = \{\Theta_x\}$ and $q = \dim N(T)$ is finite.

Definition 1. *We say that the spline-pair (T, A) is submitted by the other spline-pair $(T, \tilde{A})$ if $N(\tilde{A}) \supset N(A)$, $N(\tilde{A}) \neq N(A)$.*

Definition 2. *We call $(T, \tilde{A})$ is the maximal spline-pair with respect to the initial spline-pair (T, A), if (T, A) is submitted by $(T, \tilde{A})$ but there is no other spline-pair which submits $(T, \tilde{A})$.*

It is easy to see that the maximal spline-pair with respect to any fixed spline-pair is not unique. Let illustrate us Definitions 1, 2 and non-uniqueness on the following simple example.

Let $[a, b]$ be the finite interval with the mesh

$$\Delta = \{a \leq x_1 < x_2 < ... < x_N \leq b\}$$

and $X = W_2^m(a, b)$, $Y = L_2(a, b)$, $T = d^m/dx^m$, $m \geq 1$, $Z = E_N$ be the usual Euclidean vector space, and

$$Au(x) = [u(x_1), u(x_2), ..., u(x_N)].$$

As we already know, (T, A) forms the spline-pair if $N \geq m$. If we consider any submesh $\tilde{\Delta} \subset \Delta$ and the corresponding mesh operator $\tilde{A}$, it is easy to see that $N(\tilde{A}) \supset N(A)$. If the number of mesh points in the submesh $\tilde{\Delta}$ is greater or equal to m, the spline-pair (T, A) is submitted by $(T, \tilde{A})$. The maximal spline-pair $(T, \tilde{A})$ is any spline-pair corresponding to the mesh operator $\tilde{A}$ on the mesh $\tilde{\Delta}$ with exactly m nodes from Δ. It is clear that this operator is not unique if $N > m$.

We return again to the general consideration. The following theorem is valid

Theorem 1.6. Let $(T, \tilde{A})$ be the maximal spline-pair with respect to the initial spline-pair (T, A). Then

$$X = N(\tilde{A}) + N(T). \tag{1.40}$$

Proof. $N(\tilde{A}) + N(T)$ is a closed subspace in X. Consider some basis n_1, $n_2, ..., n_q$ in $N(T)$. Since $N(\tilde{A}) \cap N(T) = \{\Theta_X\}$, the equality $\tilde{A}(\sum_{i=1}^{q} \lambda_i n_i) = \Theta_{\tilde{z}}$ leads to $\sum_{i=1}^{q} \lambda_i^2 = 0$. It means that the elements $\tilde{A}n_1, \tilde{A}n_2, ..., \tilde{A}n_q$ are linear independent.

Assume that $N(\tilde{A}) + N(T)$ is not the whole X. It means that the element $x_* \in X$ does exist and is orthogonal to the subspace $N(\tilde{A}) + N(T)$. Let us construct an operator $\hat{A}$ which is equal to $\tilde{A}$ in $N(\tilde{A}) + N(T)$ and is expanded by zero to the orthogonal complement. It is obvious that $N(\hat{A})$ strictly contains $N(\tilde{A})$. But $\hat{A}n_i = \tilde{A}n_i$, $i = 1, 2, ..., q$ and these elements are linear independent as before. It means that the equality $\hat{A}(\sum_{i=1}^{q} \lambda_i n_i) = \Theta_{\tilde{Z}}$ leads to $\sum_{i=1}^{q} \lambda_i^2 = 0$, and $N(\hat{A}) \cap N(T) = \{\Theta_X\}$. In other words, the maximal spline-pair $(T, \tilde{A})$ is strictly submitted by spline-pair $(T, \hat{A})$. This situation is impossible and the theorem is proved. □

Remark. It is obvious that $R(\tilde{A}) = \tilde{A}N(T)$. Therefore using the linear independence of the elements $\tilde{A}n_1, \tilde{A}n_2, ..., \tilde{A}n_q$ we obtain

$$\dim R(\tilde{A}) = \dim N(T) = q. \qquad (1.41)$$

In other words, the action of the "maximal" operator $\tilde{A}$ can be always described by q linear independent functionals over X.

1.4.2. Interpolating Spline-Projector

Let $(T, \tilde{A})$ be the maximal spline-pair with respect to the initial spline-pair (T, A). Let us introduce in the space X a special scalar product by the formula

$$(u, v)_* = (\tilde{A}u, \tilde{A}v)_{\tilde{Z}} + (Tu, Tv)_Y \qquad (1.42)$$

and the corresponding norm by

$$\|u\|_* = (u, u)_*^{1/2}, \qquad (1.43)$$

which is equivalent to the initial norm $\|u\|_X$. Consider the null space $N(T)$ and its orthogonal complement $N(T)_*^\perp$ with respect to a special scalar product (1.42).

Theorem 1.7. $N(T)_*^\perp = N(\tilde{A})$.

Proof. Let $n_1, n_2, ..., n_q$ be some basis in $N(T)$ and $u \in N(T)_*^\perp$. It means that

$$(u, n_i)_* = (\tilde{A}u, \tilde{A}n_i)_{\tilde{Z}} = 0, \quad i = 1, 2, ..., q.$$

Using $R(\tilde{A}) = \tilde{A}N(T)$ we have

$$\tilde{A}u = \sum_{j=1}^{q} \lambda_j \tilde{A}n_j,$$

where λ_j are any constants. Hence the linear algebraic system arises with respect to the coefficients λ_j:

$$\sum_{j=1}^{q} \lambda_j (\tilde{A}n_j, \tilde{A}n_i)_{\tilde{Z}} = 0, \quad i = 1, 2, ..., q.$$

The elements $\tilde{A}n_i$, $i = 1, 2, ..., n$ are linear independent and the system has only zero solution. It means that $u \in N(\tilde{A})$. Conversely if $u \in N(\tilde{A})$, then we have

$$(u, n_i)_* = (Tu, Tn_i)_Y = 0$$

and $u \in N(T)_*^\perp$. □

Let us fix any element $\varphi_* \in X$ and approximate it by the interpolating spline $\sigma \in X$ which is the solution of problem

$$\begin{aligned} &A\sigma = A\varphi_* = z, \\ &\|T\sigma\|_Y^2 = \min. \end{aligned} \tag{1.44}$$

It is natural that (T, A) is the spline-pair, and $(T, \tilde{A})$ is any maximal spline-pair with respect to (T, A). We introduce the interpolating spline-operator $S : X \to X$ which maps every element $\varphi_* \in X$ to the spline $\sigma = S\varphi_*$ by the solution of problem (1.44). It is obvious that S is the linear projector (see functional equations (1.12)).

Let us represent the element $\varphi_* \in X$ as the sum

$$\varphi_* = \varphi_*^1 + \varphi_*^2,$$

where $\varphi_*^1 \in N(T)$, $\varphi_*^2 \in N(T)_*^\perp$. It is easy to see that $S\varphi_*^1 = \varphi_*^1$ (the elements from the null space $N(T)$ are exactly reproduced in the spline-interpolation process because the energy $\|T\sigma\|_Y^2$ of these elements is only zero!). Thus, the non-trivial interpolation process takes place only in the subspace $N(T)_*^\perp$, and problems of the convergence and error estimates can be considered only for the elements $\varphi_*^\perp \in N(T)_*^\perp$.

In the closed subspace $N(T)_*^\perp$ the expressions

$$(u, \nu)_* = (Tu, T\nu)_Y, \quad \|u\|_* = \|Tu\|_Y$$

become the scalar product and the norm. For simplicity we preserve the notation A for the restriction of the operator A to $N(T)_*^\perp$. Let $\varphi_*^\perp \in N(T)_*^\perp$. We transform initial interpolation spline problem (1.44) for the subspace: find $\sigma^\perp = S\varphi_*^\perp$ from conditions

$$\begin{aligned} &A\sigma^\perp = A\varphi_*^\perp, \\ &\|\sigma^\perp\|_*^2 = \min. \end{aligned} \tag{1.45}$$

Thus, we have the problem for the normal spline $\sigma^\perp$ in the Hilbert space $X_* = N(T)_*^\perp$ with the special scalar product. Using the general functional equations for the interpolating splines (see Section 1.1.5) we have

$$\begin{bmatrix} I & A^* \\ A & \Theta_{Z\to X_*} \end{bmatrix} \begin{bmatrix} \sigma^\perp \\ \Lambda \end{bmatrix} = \begin{bmatrix} \Theta_X \\ A\varphi_*^\perp \end{bmatrix} \tag{1.46}$$

where I is identical operator in X_*, A^* is the adjoint operator to A with respect to the special scalar product, $\Lambda \in Z$ is the Lagrangian coefficient.

Since the range of the operator $A : X \to Z$ is closed in Z we assume that $R(A)$ is the whole Z. We prove now that $N(AA^*)$ is only zero. Really, if for any $z \in Z$ $AA^*z = \Theta_Z$, multiplying both sides by z we have $(AA^*z, z)_Z = \|A^*z\|^2 = 0$, i.e. $z \in N(A^*)$. However $N(A^*) = R(A)^\perp = Z^\perp = \{\Theta_Z\}$ and $(AA^*)^{-1}$ does exist.

Using equation (1.46) we obtain

$$\sigma^{\perp} + A^* \Lambda = \Theta_X, \quad A\sigma^{\perp} = A\varphi_*^{\perp},$$

or acting with A into the first equation we have

$$AA^*\Lambda = -A\sigma^{\perp} = -A\varphi_*^{\perp}.$$

If we express Λ we obtain

$$\sigma^{\perp} = A^*(AA^*)^{-1}A\varphi_*^{\perp}.$$

Finally, the interpolation operator S in the subspace $X_* = N(T)_*^{\perp}$ can be expressed by formula

$$S = A^*(AA^*)^{-1}A. \tag{1.47}$$

It is evident now that the interpolating spline operator $S : X_* \to X_*$ is really the projector ($S^2 = S$) and the self-adjoint operator in the special scalar product ($S^* = S$) and therefore the orthogonal projector. Actually,

$$\forall u \in X_* \quad (Su, u - Su)_* = (Su, u)_* - (S^*Su, u)_* = 0.$$

1.4.3. Smoothing Spline-Operator

Let (T, A) be any spline-pair, $\varphi_* \in X$. We consider the smoothing spline problem: find $\sigma_\alpha \in X$ from the condition

$$\alpha\|T\sigma_\alpha\|_Y^2 + \|A\sigma_\alpha - A\varphi_*\|_Z^2 = \min_{x \in X} \alpha\|Tx\|_Y^2 + \|Ax - A\varphi_*\|_Z^2. \tag{1.48}$$

The corresponding smoothing operator we denote by S_α. S_α maps the element φ_* onto the smoothing spline $\sigma_\alpha = S_\alpha\varphi_*$. It is easy to see that

$$\forall n \in N(T) \quad S_\alpha n = n \tag{1.49}$$

independently of $\alpha > 0$. Actually, the corresponding functional equation for the smoothing spline (see (1.11)) is

$$(\alpha T^*T + A^*A)\sigma_\alpha = A^*A\varphi_* \tag{1.50}$$

and has the unique solution for any $\alpha > 0$. If $\varphi_* = n \in N(T)$, then this solution is only n independent of α. Thus, the element n from the null space of the energy operator can not be smoothed. Finally, the non-trivial smoothing process like that in the interpolation case takes place only in the subspace $N(T)_*^{\perp}$. In the subspace $X_* = N(T)_*^{\perp}$ corresponding to the special scalar product $(u, \nu)_*$ problem (1.48) can be reduced to

$$\alpha\|\sigma_\alpha^{\perp}\|_*^2 + \|A\sigma_\alpha^{\perp} - A\varphi_*^{\perp}\|_Z^2 = \min_{x \in X_*} \alpha\|x\|_*^2 + \|Ax - A\varphi_*^{\perp}\|_Z^2.$$

This is equivalent to the functional equation

$$(\alpha I_{X_*\to X_*} + A^*A)\sigma_\alpha^\perp = A^*A\sigma_*^\perp \tag{1.51}$$

where $I_{X_*\to X_*}$ is the unit operator from X_* to X_*. Act by the operator A on (1.51) and obtain

$$\alpha A\sigma_\alpha^\perp + AA^*A\sigma_\alpha^\perp = AA^*A\varphi_*^\perp,$$

or

$$[\alpha(AA^*)^{-1} + I_{Z\to Z}]A\sigma_\alpha^\perp = A\varphi_*^\perp,$$

or in the other form:

$$A\sigma_\alpha^\perp = [I_{Z\to Z} + \alpha(AA^*)^{-1}]^{-1}A\varphi_*^\perp.$$

If we interpolate the data $A\sigma_\alpha^\perp$ we obtain $\sigma_\alpha^\perp$. It means that

$$\sigma_\alpha^\perp = A^*(AA^*)^{-1}A\sigma_\alpha^\perp = A^*(AA^*)^{-1}[I_{Z\to Z} + \alpha(AA^*)^{-1}]^{-1}A\varphi_*.$$

Finally,

$$S_\alpha = A^*(\alpha I_{Z\to Z} + AA^*)^{-1}A. \tag{1.52}$$