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The specific basis of trigonometric functions in the problem of approximate solution of integral equations with the kernel of the kind $K(x-t)$

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1. In this paper we will deal with the approximate solution of Fredholm's and Volterra's equations with the kernel of the kind $K(x-t)$. We shall use the known algorithm for the search of the approximate solution in the form of a linear combination of preassigned basic functions

$$\varphi(x) \approx \sum_{k=0}^{n} c_k \varphi_k(x),$$

with the help of Galerkin's method.

The principal matter of the paper is the choice of the specific basis $\{\varphi_k(x)\}$ which:

1) possesses high approximate properties, i.e., makes possible to find the approximate solution with a good accuracy, but with a small number of basic functions;
2) makes possible (by using the inner properties of the functions $\varphi_k(x)$) to easily transform the double integral by Galerkin's algorithm to a simple (of multiplicity 1) integral;
3) reduces the problem to a system of equations with a reducible matrix, i.e., reduces it to parallelizing an algorithm to two independent subsystems of equations if the kernel is $K(|x-t|)$.

In the Appendix we illustrate the use of the specific basis of functions by solving the integral Peierls equation.

2. Let correspond the interval $[a,b]$ to an integral equation. We take for the basis $\{\varphi_k(x), a \le x \le b\}$ the eigenfunctions of the problem

$$-\varphi''(x) = \lambda\varphi(x), \quad \varphi'\Big(a - \frac{H}{2}\Big) = \varphi'\Big(b + \frac{H}{2}\Big) = 0 \quad (H = b - a).$$

(It is very important that problem (2) is considered in the extended interval $[a - \frac{H}{2}, b + \frac{H}{2}]$, but its eigenfunctions will be used only within the interval $[a,b]$.)

The solutions to problem (2) are the functions

$$\varphi_k(x) = \cos\frac{\pi k}{2H}\Big(\frac{H}{2} + b - x\Big), \quad k = 0, 1, 2, \ldots,$$

that are even with respect to the point $c = 0.5(a+b)$ for even k and are odd for odd k.

An important property of functions (3) follows from the below stated fragment of the book [1].

Let L be the Sturm-Liouville operator

$$L \equiv \frac{1}{r(x)}\Big[-\frac{d}{dx}p(x)\frac{d}{dx} + q(x)\Big], \quad A < x < B,$$

where $p(x) \geq p_0 > 0$, $r(x) \geq r_0 > 0$, $q(x) \geq 0$ are sufficiently smooth functions. Let us consider the spectral problem

$$\begin{aligned} &L\varphi(x) = \lambda\varphi(x), \quad A < x < B, \\ &\eta_0\varphi(A) + \theta_0\varphi'(A) = 0, \\ &\eta_1\varphi(B) + \theta_1\varphi'(B) = 0, \end{aligned} \tag{4}$$

where the constants η_0, θ_0, η_1, θ_1 satisfy the conditions

$$\eta_i^2 + \theta_i^2 \neq 0, \quad (-1)^i\eta_i\theta_i \leq 0, \quad i = 0, 1.$$

Let $\varphi_k(x)$, $k = 0, 1, 2, \ldots$ be the normalized ($\|\varphi_k\|^2 = \int_A^B r(x)\varphi_k^2(x)dx = 1$) eigenfunctions of problem (4). In this assumption holds the following

Theorem 1. *Any function $\varphi_k(x) \in H^N(a, b)$, $N \geq 0$, $A < a < b < B$, admits infinitely many representations in the form of series in the eigenfunctions of problem (4)*

$$\varphi(x) = \sum_{k=0}^{\infty} c_k\varphi_k(x), \quad a \leq x \leq b,$$

which are convergent in the norm of the space $H^N(a, b)$. The coefficients of series (5) for $k \geq 1$ are representable in the form

$$c_k = \sigma_k k^{-N}, \quad k = 1, 2, \ldots,$$

where

$$\sum_{k=1}^{\infty} \sigma_k^2 < \infty.^*$$

*It is assumed that the smoothness of the functions r, p, q ensures the inclusions $\varphi_k \in H^N(a,b)$.

If the function $\varphi(x)$ is sufficiently smooth (i.e., $N \gg 1$), then the law of the decrease for coefficients (6) means the asymptotic rapid convergence of the whole infinite set of ambiguous expansions (5).

Since problem (2) is a partial case of problem (4), functions (3) (by virture of Theorem 1) corresponds to the above-mentioned Item 1), if a solution of the integral equation is sufficiently smooth.

3. At first we will consider the equation

$$\varphi(x) - \lambda \int_a^b K(|x-t|)\varphi(t)\,dt = f(x),$$

where λ is not an eigenvalue of the kernel $K(|x-t|)$.

We will seek the coefficients c_k in (1) with the help of Galerkin's method, i.e, by solving the next system of linear algebraic equations:

$$\sum_{k=0}^{n} c_k\Big(\int_a^b \varphi_k(x)\varphi_m(x)\,dx - \lambda I_{km}\Big) = \int_a^b f(x)\varphi_m(x)\,dx, \qquad (7)$$
$$m = 0, 1, \dots, n,$$

where

$$I_{km} = \int_a^b \int_a^b K(|x-t|)\varphi_k(t)\varphi_m(x)\,dtdx. \qquad (8)$$

Remark 1. The possibility to find with a good accuracy the solution of the integral equation on the basis of approximate equality (1) assumes as well the good accuracy in the following representation of the right-hand side:

$$f(x) \approx \sum_{k=0}^{n} g_k\varphi_k(x),$$

where the coefficients g_k may be found from system (7) for $\lambda = 0$.

Now we will transform the integral I_{km} with the help of the following change of variables:

$$x = c + \frac{2H}{\pi}(\xi+\tau), \quad t = c + \frac{2H}{\pi}(\xi-\tau),$$

where $c = 0.5(a+b)$, $H = b-a$. Then

$$I_{km} = 8\Big(\frac{H}{\pi}\Big)^2 \int_0^{\frac{\pi}{4}} K\Big(\frac{4H}{\pi}\tau\Big)\,d\tau \int_0^{\frac{\pi}{4}-\tau} (\Pi_{km}(\xi,\tau) + \Pi_{mk}(\xi,\tau))\,d\xi,$$

where

$$\Pi_{km}(\xi,\tau) = \varphi_k\Big(c-\frac{2H}{\pi}(\xi-\tau)\Big)\varphi_m\Big(c-\frac{2H}{\pi}(\xi+\tau)\Big)+ \\ \varphi_k\Big(c+\frac{2H}{\pi}(\xi+\tau)\Big)\varphi_m\Big(c+\frac{2H}{\pi}(\xi-\tau)\Big). \quad (9)$$

Using the particular basic functions (3) we find*

$$\Pi_{km}(\xi,\tau) = 2\cos k\Big(\frac{\pi}{2}-\tau\Big)\cos m\Big(\frac{\pi}{2}+\tau\Big)\cos k\xi\cdot\cos m\xi + \\ 2\sin k\Big(\frac{\pi}{2}-\tau\Big)\sin m\Big(\frac{\pi}{2}+\tau\Big)\sin k\xi\cdot\sin m\xi. \quad (10)$$

If we use the identities

$$\cos k\Big(\frac{\pi}{2}-\tau\Big)\equiv(-1)^k\cos k\Big(\frac{\pi}{2}+\tau\Big),\quad \sin k\Big(\frac{\pi}{2}-\tau\Big)\equiv(-1)^{k+1}\sin k\Big(\frac{\pi}{2}+\tau\Big),$$

then we have

$$\Pi_{km}(\xi,\tau)+\Pi_{mk}(\xi,\tau) \\ = 2[(-1)^k+(-1)^m]\Big\{\cos k\Big(\frac{\pi}{2}+\tau\Big)\cos m\Big(\frac{\pi}{2}+\tau\Big)\cos k\xi\cdot\cos m\xi - \\ \sin k\Big(\frac{\pi}{2}+\tau\Big)\sin m\Big(\frac{\pi}{2}+\tau\Big)\sin k\xi\cdot\sin m\xi\Big\}. \quad (11)$$

Hence $I_{km}=0$, if k and m have the different parity. It is easy to see that for such k and m the expression in the parentheses in equalities (7) equals zero.

If the finite system of basic functions (3) is ordered as

$$\varphi_0(x),\varphi_2(x),\ldots;\quad \varphi_1(x),\varphi_3(x),\ldots,$$

then the matrix of system (7) will be of the following stucture:

$$A=\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \quad (12)$$

where A_1 and A_2 are the square matrices. Thus, the algebraic system (7) decomposes into two independent subsystems of the half dimension.

According to formulas (11) the values I_{km} may be easily reduced to a simple integral, namely

$$I_{00}=2\Big(\frac{4H}{\pi}\Big)^2\int_0^{\frac{\pi}{4}}K\Big(\frac{4H}{\pi}\tau\Big)\Big(\frac{\pi}{4}-\tau\Big)\,d\tau; \quad (13)$$

*Here and below the important role play the inner properties of functions (3).

$$I_{kk} = (-1)^k \Big(\frac{4H}{\pi}\Big)^2 \times$$
$$\int_0^{\frac{\pi}{4}} K\Big(\frac{4H}{\pi}\tau\Big)\Big\{(-1)^k\Big(\frac{\pi}{4}-\tau\Big)\cos 2k\tau + \frac{1}{2k}\sin k\Big(\frac{\pi}{2}-2\tau\Big)\Big\}\,d\tau, \qquad (14)$$

$k \neq 0$;

$$I_{km} = \frac{(-1)^k+(-1)^m}{2}\Big(\frac{4H}{\pi}\Big)^2 \times$$
$$\int_0^{\frac{\pi}{4}} K\Big(\frac{4H}{\pi}\tau\Big)\Big\{\frac{\sin[(k-m)(\frac{\pi}{4}-\tau)]\cos[(k+m)(\frac{\pi}{2}+\tau)]}{k-m} +$$
$$\frac{\sin[(k+m)(\frac{\pi}{4}-\tau)]\cos[(k-m)(\frac{\pi}{2}+\tau)]}{k+m}\Big\}\,d\tau, \qquad (15)$$

$k \neq m$.

4. In the case of the equation

$$\varphi(x) - \lambda\int_a^b K(x-t)\varphi(t)\,dt = f(x),$$

(where λ is not an eigenvalue of the kernel $K(x-t)$) it is sufficient to perform the following substutions:

(a) in formulas (13) and (14) it is necessary to replace

$$K\Big(\frac{4H}{\pi}\tau\Big) \quad \text{on} \quad \frac{1}{2}\Big[K\Big(-\frac{4H}{\pi}\tau\Big)+K\Big(\frac{4H}{\pi}\tau\Big)\Big],$$

(b) in formula (15) it is necessary to replace

$$[(-1)^k+(-1)^m]K\Big(\frac{4H}{\pi}\tau\Big) \quad \text{on} \quad (-1)^k K\Big(-\frac{4H}{\pi}\tau\Big)+(-1)^m K\Big(\frac{4H}{\pi}\tau\Big).$$

It is necessary to note, that the matrix of the system in this case has not the form (12).

5. As a next example we consider Volterra's equation of the second kind

$$\varphi(x) - \lambda\int_a^x K(x-t)\varphi(t)\,dt = f(x), \quad a \le x \le b.$$

After the realization of the whole above-described procedure of transformation of the double integral

$$I_{km} = \int_a^b dx\int_a^x K(x-t)\varphi_k(t)\varphi_m(x)\,dt,$$

which appears in Galerkin's method, we obtain the results, that are compared to (13)–(15). Namely, in the present variant it is necesary to take the right-hand side of formulas (13) and (14) with coefficient 1/2, while in formula (15) it is necesary to replace $(-1)^k+(-1)^m$ for $(-1)^m$. In this case the matrix of the system is also irreducible to the form (12).

6. Now it is appropriate to clarify statements 1) and 2) from Item 1. Since polynomials are the simpliest functions, it is interesting to compare the basis (3) to an arbitrary polynomial basis

$$\{P_k(x)\}_k^\infty, \quad P_k(x) \text{ is the polynomial of degree } k.$$

(It is necessary to note, that all bases of the kind (16) are equivalent). We state that the basis (3) has clear advantages as compared to the polynominal basis.

In the first place it is impossible to get such a refined separation of the variables τ and ξ like in formula (10) and in the subsequent equality (11). As the final result the polynominal variant reduces to the expressions I_{km}, which are incomparable with respect to the simplicity of formulas (13)–(15).

In the second place, for the rows

$$\varphi(x) = \sum_{k=0}^{\infty} c_k P_k(x), \quad a \le x \le b, \tag{17}$$

it is possible to attain the convergence to a smooth function $\varphi \in H^N(a,b)$, $N \ge 2$ in the general case only in the norm $\|\cdot\|_{H^M(a,b)}$, where $M = [\frac{N}{2} - 1]$, but the convergence of rows in the basic functions (3) is attained (by virtue of Theorem 1) in the norm $\|\cdot\|_{H^N(a,b)}$. The convergence of the polynomial row (17) in the uniform metric is also slower than of the rows in basis (3). The latter facts are proved in [1], where all investigations with rows (17) are realized with the use of the basis of the Legendre's polynomials.

It is low-probability that there exists the possibility of constructing the bases $\{\varphi_k(x)\}_{k=0}^\infty$ for other functions, which are different from trigonometric and polynomial functions and which are more efficient than (3) (more efficient whithin statements 1) and 2) from Item 1).

Appendix

The above-stated algorithm is realized to solve a one-dimensional problem of the search for a one-velocity neutron flux density in a homogeneous plate in the presence of a distibuted isotropic source inside of the plate and external isotropic sources outside of its each side.

Mathematically this problem may be writen in the form of the following integral Peierls equation [2]:

$$\varphi(x) = \frac{1}{2\sigma}\int_0^H [\sigma_s\varphi(t) + 2q(t)]E_1(|x-t|)\,dt + Q_1E_2(x) + Q_2E_2(H-x), \tag{18}$$

where $\varphi(x)$ is the neutron flux density, $q(x)$ is density of the inner source, $0 < \sigma_s \le \sigma$ are the macroscopic scattering cross-section and the total cross-section of neutrons, Q_1 and Q_2 are the external isotropic sources and $E_n(x)$ is the integral exponent function:

$$E_n(x) = \int_0^1 \frac{\exp(-\nu x)}{\nu^n}\, d\nu, \quad x > 0.$$

(Note that equation (18) is written in mean free paths $l = \frac{1}{\sigma}$.)

If one introduces the function $\Phi(x) = \sigma_s \varphi(x) + 2q(x)$, then equation (18) takes the form

$$\Phi(x) - \lambda \int_0^H \Phi(t) E_1(|x - t|)\, dt = f(x),$$

where $\lambda = \frac{\sigma_s}{2\sigma}$, $f(x) = 2q(x) + \sigma_s[Q_1 E_2(x) + Q_2 E_2(H - x)]$.

Since $E_1(x) \sim (-\ln x)$ for $x \to 0$ $(x > 0)$, the integrals in formulas (13)–(15) are improper. This case represents a certain inconvenience for the numerical integration. In order to get rid of the improper integration, we will transform the integral I_{km}. To this end we take the integral

$$\int_0^H E_1(|x - t|)\varphi_k(t)\, dt = \int_0^x E_1(x - t)\varphi_k(t)\, dt + \int_x^H E_1(t - x)\varphi_k(t)\, dt, \quad (19)$$

and using the identities

$$\frac{dE_n(x - t)}{dt} \equiv E_{n-1}(x - t), \quad \frac{dE_n(t - x)}{dt} \equiv -E_{n-1}(t - x),$$

we twice realize integration by parts of every summand in the right-hand side of (19). As a result integral (8) (with the basis functions (3)) takes the form

$$\begin{aligned} I_{km} = {} & 2\int_0^H \varphi_k(x)\varphi_m(x)\, dx - \\ & \cos\frac{\pi k}{4} \int_0^H \varphi_m(x)[E_2(H - x) + (-1)^k E_2(x)]\, dx - \\ & \frac{\pi k}{2H} \sin\frac{\pi k}{4} \int_0^H \varphi_m(x)[E_3(H - x) + (-1)^k E_3(x)]\, dx - \\ & \left(\frac{\pi k}{2H}\right)^2 \int_0^H \int_0^H E_3(|x - t|)\varphi_m(x)\varphi_k(t)\, dt dx. \end{aligned} \quad (20)$$

The functions $E_n(x)$ for $n \geq 2$ have not singularity near zero, and the double integral in the right-hand side of equality (20) should be calculated by formulas (13)–(15) for $K(\frac{4H}{\pi}\tau) \equiv E_3(\frac{4H}{\pi}\tau)$. It should be noted in passing, that every summand in (20) is equal to zero if k and m have a different parity.

The results of a series of calculations by using twelve basis functions (3) are compared with the values that are obtained from the transport integro-differential equation. This equation is solved by a difference method on the basis of the splitting algorithm [2]. These two approximate approaches turn out to be in the good conformity with each other, i.e., they have sufficient accuracy for practical purposes. In the presence of external sources and in the absence of an interior source the problem is considered at H of the order of two mean free parts (see Remark 1).

Remark 2. The derivative $\frac{d\varphi}{dx}$ may have a logarithmic singularity near the boundaries of the plate [3]. In this case $\varphi \in H^1(0, H)$ and therefore according to Theorem 1 series (5) must converge sufficiently slowly. However such a deterioration of the smoothness of $\varphi(x)$ has the local (near-boundary) nature, so that on the whole this circumstance has a small effect for our method.

References

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