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Comparative analysis of finite element approximation for the Navier–Stokes equations with basis functions of different orders

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Abstract. Finite element methods (the Uzawa algorithm and a mixed finite element method) for the solution of the Navier–Stokes equations on triangular grids are considered. For approximation of velocity and pressure, interpolating functions from different finite element spaces are chosen. The properties of the algorithms are tested on an analytical test example.

1. Introduction

The solution to the Navier–Stokes equations for an incompressible fluid flow is one of the main problems in the field of computational fluid dynamics. The Navier–Stokes equations form a set of coupled equations for both velocity and pressure (the gradient pressure). One of the main problems related to the numerical solution to these equations is imposition of the incompressibility constraint and, consequently, the calculation of pressure. The method of calculation of pressure is not obvious. Pressure is not a thermodynamic variable, as there is no equation of state for calculation of pressure. The mathematical importance of the pressure in an incompressible flow lies in the theory of saddle — the point problems where it acts as a Lagrangian multiplier that constrains the velocity to remain divergence-free.

For modeling of an incompressible flow, oscillations of solutions are arisen when these features are not taken into account in numerical schemes. Using the finite element approximation, there arise major difficulties in solving the incompressible Navier–Stokes equations: 1) how to choose interpolating functions for approximation of velocity and pressure to ensure the existence of the solution pair $(\boldsymbol{u}, p)$, where $\boldsymbol{u}$ is velocity, p is pressure; 2) how to deal with nonlinearity in the momentum equations; 3) how to interrelate “velocity–pressure”. A variety of incompressible viscous flows can be analyzed via three formulations for the Navier–Stokes equations which are: 1) the original “velocity–pressure formulation” [2, 6–10, 16]; 2) vector potential—vorticity vector, formulation [4, 6, 7]; 3) the velocity and the vorticity vector are taken as dependent variables, formulation [13, 14, 17]. Each formulation has its own advantages and drawbacks. A more detailed review of these formulations is given in the book [8].

The velocity–pressure formulation for the Navier–Stokes equations is considered in the present study. There are numerous methods for approximation of the Navier–Stokes equations in the velocity–pressure formulation, which are finite element method (FEM) [2–4, 10, 12, 16], finite volume method (FV) [6, 8], spectral method [8, 9], and methods combining advantages of the FEM and the FV [2, 12].

Essential advantages of the FEM for an incompressible fluid flow are conservatism and absolute stability [4, 5, 11]. The main difficulties of using the FEM are dealt with selection of approximating and weight functions. Interpolating functions of equal orders for the velocity and the pressure bring about the singularity associated with application of a discrete continuity equation [8]. For the finite element approximation of the Navier–Stokes equations it is necessary that the degrees of approximation for velocity and pressure be satisfied by the Ladyzhenskaya–Babuska–Brezzi (LBB) conditions [2, 3, 7]. This condition means that the degree of approximation for the pressure should be one or two degrees lower than that for the velocity. Interpolating functions satisfying the LBB for the approximation of the Stokes equations (the Navier–Stokes equations) are called div-stability. In the case of unsatisfactory div-stability of the finite element approximation, the resulting solution is less accurate, and the convergence conditions are not valid.

One of the methods of the fulfilment of the incompressibility constraint is the penalty method, which is gaining in importance in the finite element computations [8, 12]. Here the continuity equation is perturbed with a small term proportional to pressure. This leads to decoupled problems for both velocity and pressure. The adequacy of this mathematical performance for the Stokes equations is considered by Temam [7]. The penalty method is more efficient as compared to the mixed method. The velocity and the pressure are computed in the hypervector, and this is a characteristic feature of the mixed method.

Coulliette and Koch [12] reported that neither the penalty nor the mixed finite element method is able to sufficiently satisfy the condition $\operatorname{div} \boldsymbol{u} = 0$. Essentially better results are obtained by the Uzawa algorithm [9, 10], which is used in the theory of saddle-point problems. For the improvement of convergence properties the Uzawa algorithm is combined with the penalty procedure [12].

A semi-implicit method for pressure-linked equations (SIMPLE) are proposed by Patankar [6] to couple the velocity and the pressure fields. The FV discretization of equations on staggered grids is used in this class of algorithms. Application of SIMPLE in the original formulation for a wide class of problems implies that the insertion of pressure correction adjusts the efficiently of the velocity field, but a rapid convergence for pressure is not obtained. The modified algorithm SIMPLER (SIMPLE Revised) is developed for the improvement of convergence. Application of the SIMPLE algorithm allows us to obtain non-degenerate approximations using equal-order basis functions for velocity and pressure. For the convergence of this algorithm several iterations are necessary although the number of operations for a SIMPLER iteration need more than those for a SIMPLE iteration.

Patankar [6] reported that the superposition of pressure points with velocity points leads to the pressure fields oscillations in the finite difference context. For the calculation of the gradient pressure, using a center-difference scheme, the pressure is taken on a coarser grid, and therefore a "saw tooth" pressure field is interpreted as homogeneous. Application of a staggered grid enables us to constitute a link between values of velocity and pressure. As a rule, the oscillations increase for a high Reynolds number, since the dissipative terms joining the values of velocity and pressure at neighboring vertices are small. To solve this problem, Harlow and Welch [15] proposed to calculate the pressure and the velocity components, respectively, at vertices and edges of a finite difference grid.

The "pressure problem" is actual on an unstructured grid. The major achievements in this field are associated with a collocated stencil. These methods are associated with averaging of a coefficient on the edge in the momentum equations. This technique needs calculation and storage of the edge and the nodal values of velocity that make difficulties for application of a collocated stencil. To overcome such difficulties, the staggered discretization was proposed for an unstructured grid. A staggered stencil of variables is equivalent to using of different orders of interpolating functions for velocity and pressure in the finite element context to satisfy the LBB condition. The div-stability interpolating functions do not lead to oscillations of a pressure field. However, in many case the choice of interpolating functions for pressure displays the chess effect. The classical approximation of velocity–pressure using 12 degrees of freedom for velocity and three degrees for pressure can give oscillations of a pressure field.

In paper [16], four types of mixed interpolation elements are considered and compared. These are, namely: six-node triangular elements, eight-node serendipity elements, nine-node Lagrangian elements and four-node quadrilateral elements. The results obtained indicate that for same number of pressure unknowns, serendipity elements can give considerably less accurate pressure fields than most of other types of elements. The Lagrangian elements give the most accurate pressure and velocity distributions. The numerical performance of triangular elements is intermediate in accuracy and is dependent on the triangular pattern used. The four-node element can generate spurious pressure modes depending on the boundary condition specifications.

2. Governing equations

The stationary, incompressible Navier–Stokes equations in the velocity–pressure formulation are given by the momentum equation

$$-\nu\nabla^2\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p = \boldsymbol{f} \tag{1}$$

and the continuity equation

$$\nabla\cdot\boldsymbol{u} = 0 \tag{2}$$

in $\Omega \in \mathbb{R}^2$, $\boldsymbol{x} = (x, y) \in \Omega$. Here $\boldsymbol{u} = (u, v)^T$ is the velocity vector, p is the pressure, $\boldsymbol{f} = (f_x, f_y)$ is the body force vector, ν is the kinematic viscosity. The boundary condition on the boundary $\Gamma = \partial\Omega$ are

$$\boldsymbol{u}|_\Gamma = \boldsymbol{g}. \tag{3}$$

3. Finite element approximation of the Navier–Stokes equations

Discretization of the domain Ω in the finite elements Ω^h is assumed. The finite element space for the velocity is defined by $\boldsymbol{V}^h \subset H_0^1(\Omega) \times H_0^1(\Omega)$ and that for the pressure— by $P^h \subset L_2(\Omega)$, where $H_0^1(\Omega) = \{v \in H^1(\Omega), v|_\Gamma = 0\}$. Using the test functions $\boldsymbol{v}^h \in \boldsymbol{V}^h$, $q^h \in P^h$, one can obtain the discrete Galerkin equations (a weak formulation) for the Navier–Stokes equations:

Find $\boldsymbol{u}^h \in \boldsymbol{V}^h$, $p^h \in P^h$ such that

$$a(\boldsymbol{u}^h, \boldsymbol{v}^h) + c(\boldsymbol{u}^h, \boldsymbol{u}^h, \boldsymbol{v}^h) + b(\boldsymbol{v}^h, p^h) = (\boldsymbol{f}, \boldsymbol{v}^h), \quad \boldsymbol{v}^h \in \boldsymbol{V}^h, \tag{4}$$

$$b(\boldsymbol{u}^h, q^h) = 0, \quad q^h \in P^h, \tag{5}$$

with

$$a(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \nabla\boldsymbol{u}\cdot\nabla\boldsymbol{v}\, d\Omega, \qquad b(\boldsymbol{u}, q) = -\int_\Omega (\nabla\cdot\boldsymbol{u}) q\, d\Omega,$$

$$c(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = \int_\Omega [(\boldsymbol{u}\cdot\nabla)\boldsymbol{v}]\cdot\boldsymbol{w}\, d\Omega, \qquad (\boldsymbol{f}, \boldsymbol{v}) = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\, d\Omega.$$

The incompressibility term (5) is penalized by the pressure in the form $b(\boldsymbol{u}^h, q^h) = \varepsilon(p^h, q^h)$, where $\varepsilon > 0$ is an arbitrary penalty parameter. Substituting it in the discretized Navier–Stokes equations (4) eliminate the pressure from the problem thus reducing the total number of degrees of freedom of the problem. The nonlinearity in the momentum equations is performed by the Picard procedure:

$$c(\boldsymbol{u}^{n-1}, \boldsymbol{v}^n, \boldsymbol{w}) = \int_\Omega [(\boldsymbol{u}^{n-1} \cdot \nabla)\boldsymbol{v}^n] \cdot \boldsymbol{w}\, d\Omega,$$

where n is a step of the iterative procedure. This will yield a global matrix system for the calculation of the velocity and the pressure.

Let the initial velocity $\boldsymbol{u}^0$ and the initial pressure p^0 be guessed.

In Uzawa algorithm, the velocity and the pressure at $(n+1)$-th iteration are computed from

$$\Big(D + C + \frac{1}{\varepsilon} Q M_p^{-1} Q^T\Big)\boldsymbol{u}^{n+1} = F + Qp^n,$$
$$p^{n+1} = p^n - \frac{1}{\varepsilon} M_p^{-1} Q^T \boldsymbol{u}^{n+1}.$$

In the mixed method, they are computed from

$$\begin{pmatrix} D + C & -Q \\ Q^T & \varepsilon M_p \end{pmatrix} \begin{pmatrix} \boldsymbol{u}^{n+1} \\ p^{n+1} \end{pmatrix} = \begin{pmatrix} F \\ \varepsilon M_p p^n \end{pmatrix} \tag{6}$$

The notations used are the following:

- the diffusion matrix D is assembled with

$$[D_k]_{ij} = \int_{\Omega_k} \left(\frac{\partial w_i}{\partial x}\frac{\partial w_j}{\partial x} + \frac{\partial w_i}{\partial y}\frac{\partial w_j}{\partial y} \right) d\Omega, \quad i, j = 1, \ldots, e_u;$$

- the convection matrix C:

$$[C_k]_{ij} = \int_{\Omega_k} \left(u_l w_l \frac{\partial w_j}{\partial x} + v_l w_l \frac{\partial w_j}{\partial y} \right) w_i\, d\Omega, \quad i, j, l = 1, \ldots, e_u;$$

- the matrix associated with the gradient operator Q:

$$[Q_k]_{ij} = \left(\int_{\Omega_k} q_i \frac{\partial w_j}{\partial x}\, d\Omega, \int_{\Omega_k} q_i \frac{\partial w_j}{\partial y}\, d\Omega \right)^T,$$
$$i = 1, \ldots, e_p, \;\; j = 1, \ldots, e_u;$$

- the mass matrix for the pressure M_p:

$$[(M_p)_k]_{ij} = \int_{\Omega_k} q_i q_j\, d\Omega, \quad i, j = 1, \ldots, e_p;$$

F is the source terms; w_i, q_i are interpolating functions for the velocity and the pressure, respectively; Ω_k is element of the triangulation; e_u, e_p depend on order of interpolating functions for the velocity and the pressure, respectively.

The velocity and the pressure are computed in the hypervector $(\boldsymbol{u}, p)^T$ and thus is a peculiarity of the mixed method (6). The LBB condition or the inf-sup condition for a pair of discrete finite element spaces $(\boldsymbol{V}, p)$ is a sufficient condition to ensure the existence of a finite element solution for the Navier–Stokes equations. The Taylor–Hood elements for approximation of the velocity and the pressure variables are selected as follows: 1) linear interpolating functions for the velocity and piecewise constant functions for the pressure; 2) quadratic interpolating functions for the velocity and linear functions for the pressure. The Taylor–Hood elements satisfy the discrete LBB condition [2].

4. Numerical solution of the Navier–Stokes equations

Consider the Navier–Stokes equations (1)–(3) with $\nu = 1$. The source term is chosen such that the exact velocity and pressure are given by

$$\boldsymbol{u}(x, y) = (x^2 y + y^3,\ -y^2 x - x^3)^T, \tag{7}$$

$$p(x, y) = x^3 + y^3 - 0.5. \tag{8}$$

The problem is solved in the domain $\Omega = [0,1]^2$ with boundary conditions for the velocity according to equation (7), (8). The algebraic linear system obtained after assembling local matrices is solved by the BiCGStab accurate to 10^{-9}. Global matrices are performed by a compressed sparse row involving a lower triangle, a diagonal, a coefficient for each triangle. The size of an unknown velocity vector is $2N_{\boldsymbol{u}}$, size of unknown pressure is N_p, the size of the matrices D, $C - 2N_{\boldsymbol{u}} \times 2N_{\boldsymbol{u}}$, $Q - 2N_{\boldsymbol{u}} \times N_p$, $M_p - N_p \times N_p$. The finite element grid is constructed dividing each rectangular element into two triangles by diagonal.

Table 1 shows the number of nodal values of velocity components and pressure on different grids.

The results of testing the mixed method and the Uzawa algorithm for calculating the Navier–Stokes equations are illustrated for different values of penalty parameters in Table 2.

The numerical values of velocity and pressure converge to the exact solutions. Table 3 show the results of testing algorithms for calculation of the

Table 1. A number of nodal values of unknowns

Method	Interpolating functions	Grid / number of triangles		
		10×10 / 200	20×20 / 800	30×30 / 1800
Mixed	1–0	1003	3803	9242
Uzawa	1–0	882	3362	7442
Mixed	2–1	442	1682	3722
Uzawa	2–1	242	882	1922

Table 2. The results of testing the mixed method and Uzawa algorithm for calculating the Navier–Stokes equations

Number of triangles	ε	Uzawa algorithm		Mixed method	
		$\|\boldsymbol{u}-\boldsymbol{u}^h\|_{L_2}$	$\|p-p^h\|_{L_2}$	$\|\boldsymbol{u}-\boldsymbol{u}^h\|_{L_2}$	$\|p-p^h\|_{L_2}$
Interpolating functions 1–0					
200	10^{-1}	$1.247\cdot10^{-3}$	0.0689	$1.094\cdot10^{-3}$	0.0675
	10^{-2}	$1.108\cdot10^{-3}$	0.0684	$1.025\cdot10^{-3}$	0.0541
	10^{-3}	$0.891\cdot10^{-3}$	0.0597	$0.866\cdot10^{-3}$	0.0395
800	10^{-1}	$0.884\cdot10^{-3}$	0.0477	$0.611\cdot10^{-3}$	0.0405
	10^{-2}	$0.835\cdot10^{-3}$	0.0462	$0.548\cdot10^{-3}$	0.0371
	10^{-3}	$0.672\cdot10^{-3}$	0.0347	$0.491\cdot10^{-3}$	0.0297
1800	10^{-1}	$0.748\cdot10^{-3}$	0.0463	$0.734\cdot10^{-3}$	0.0441
	10^{-2}	$0.621\cdot10^{-3}$	0.0438	$0.619\cdot10^{-3}$	0.0437
	10^{-3}	$0.618\cdot10^{-3}$	0.0341	$0.595\cdot10^{-3}$	0.0322
Interpolating functions 2–1					
200	10^{-1}	$0.751\cdot10^{-3}$	0.0496	$0.745\cdot10^{-3}$	0.0463
	10^{-2}	$0.747\cdot10^{-3}$	0.0439	$0.638\cdot10^{-3}$	0.0409
	10^{-3}	$0.581\cdot10^{-3}$	0.0375	$0.617\cdot10^{-3}$	0.0206
800	10^{-1}	$0.612\cdot10^{-3}$	0.0324	$0.736\cdot10^{-3}$	0.0364
	10^{-2}	$0.548\cdot10^{-3}$	0.0317	$0.519\cdot10^{-3}$	0.0253
	10^{-3}	$0.497\cdot10^{-3}$	0.0302	$0.475\cdot10^{-3}$	0.0139
1800	10^{-1}	$0.548\cdot10^{-3}$	0.0415	$0.531\cdot10^{-3}$	0.0358
	10^{-2}	$0.497\cdot10^{-3}$	0.0394	$0.474\cdot10^{-3}$	0.0207
	10^{-3}	$0.481\cdot10^{-3}$	0.0358	$0.462\cdot10^{-3}$	0.0124

Stokes equations using different values of penalty parameters. The influence of the penalty parameters on convergence of the velocity and the pressure for both algorithms is essential: smaller value of the penalty parameter corresponds to increasing the accuracy of solution. Quadratic interpolating functions for the velocity and linear functions for the pressure allow us to find a more accurate solution than the one for interpolating functions of lower order.

The results of calculations of a fluid flow in the channel with a stream bending by an angle of 180° are presented [1]. The stationary Euler equation is written down as:

$$(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}+\nabla p=\boldsymbol{f}, \quad \nabla\cdot\boldsymbol{u}=0 \tag{9}$$

in $\Omega \in \boldsymbol{R}^2$, $\boldsymbol{x}=(x,y)\in\Omega$. The boundary condition on the boundary $\Gamma=\partial\Omega : \boldsymbol{u}|_{\Gamma_1}=(0,1)$, $\boldsymbol{u}|_{\Gamma_2}=(0,-1)$, $\boldsymbol{u}\cdot\boldsymbol{n}|_{\Gamma_0}=0$.

Equations (9) are solved with $\varepsilon=0.25$ on two different grids using the mixed finite element method and interpolating functions 1–0. Figure 1 shows the velocity vector field of a two-dimensional flow.

Table 3. The results of testing the mixed method and Uzawa algorithm for calculating the Stokes equations

Number of triangles	ε	Uzawa algorithm		Mixed method	
		$\Vert\boldsymbol{u}-\boldsymbol{u}^h\Vert_{L_2}$	$\Vert p-p^h\Vert_{L_2}$	$\Vert\boldsymbol{u}-\boldsymbol{u}^h\Vert_{L_2}$	$\Vert p-p^h\Vert_{L_2}$
Interpolating functions 1–0					
200	10^{-1}	$1.088 \cdot 10^{-3}$	0.0624	$1.076 \cdot 10^{-3}$	0.0573
	10^{-2}	$1.095 \cdot 10^{-3}$	0.0637	$1.013 \cdot 10^{-3}$	0.0466
	10^{-3}	$0.753 \cdot 10^{-3}$	0.0591	$0.742 \cdot 10^{-3}$	0.0256
800	10^{-1}	$0.704 \cdot 10^{-3}$	0.0443	$0.609 \cdot 10^{-3}$	0.0402
	10^{-2}	$0.689 \cdot 10^{-3}$	0.0426	$0.525 \cdot 10^{-3}$	0.0370
	10^{-3}	$0.620 \cdot 10^{-3}$	0.0339	$0.483 \cdot 10^{-3}$	0.0286
1800	10^{-1}	$0.724 \cdot 10^{-3}$	0.0442	$0.711 \cdot 10^{-3}$	0.0439
	10^{-2}	$0.617 \cdot 10^{-3}$	0.0425	$0.612 \cdot 10^{-3}$	0.0423
	10^{-3}	$0.605 \cdot 10^{-3}$	0.0340	$0.592 \cdot 10^{-3}$	0.0318
Interpolating functions 2–1					
200	10^{-1}	$0.749 \cdot 10^{-3}$	0.0493	$0.742 \cdot 10^{-3}$	0.0467
	10^{-2}	$0.746 \cdot 10^{-3}$	0.0434	$0.634 \cdot 10^{-3}$	0.0403
	10^{-3}	$0.575 \cdot 10^{-3}$	0.0371	$0.611 \cdot 10^{-3}$	0.0201
800	10^{-1}	$0.602 \cdot 10^{-3}$	0.0319	$0.729 \cdot 10^{-3}$	0.0354
	10^{-2}	$0.539 \cdot 10^{-3}$	0.0316	$0.504 \cdot 10^{-3}$	0.0246
	10^{-3}	$0.492 \cdot 10^{-3}$	0.0301	$0.436 \cdot 10^{-3}$	0.0131
1800	10^{-1}	$0.544 \cdot 10^{-3}$	0.0413	$0.528 \cdot 10^{-3}$	0.0353
	10^{-2}	$0.496 \cdot 10^{-3}$	0.0390	$0.443 \cdot 10^{-3}$	0.0204
	10^{-3}	$0.475 \cdot 10^{-3}$	0.0352	$0.426 \cdot 10^{-3}$	0.0116

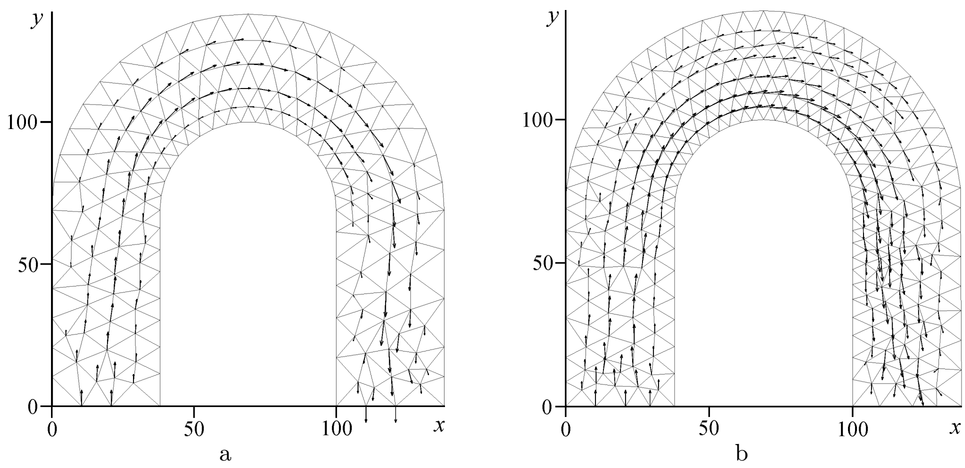

Figure 1. The fluid flow in the channel with a stream bending by an angle of 180°: a) 322 and b) 586 triangles

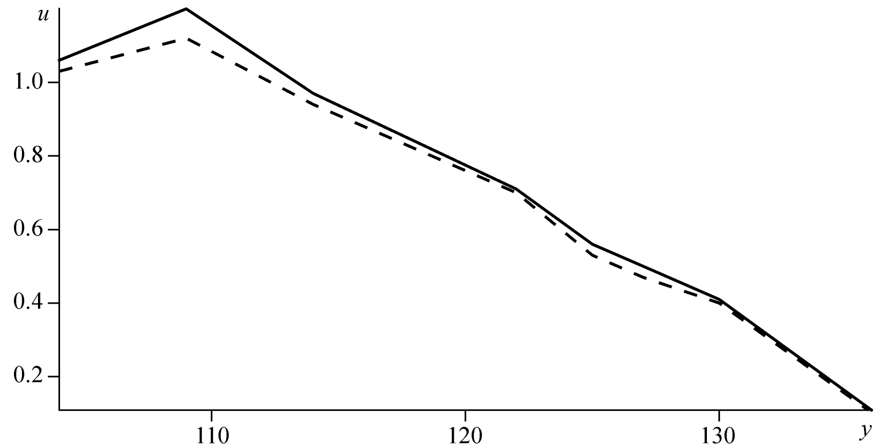

Figure 2

Figure 2 shows x-component of the velocity on the cut $x = 69$ using interpolating functions 1–0 (solid line) and 2–1 (dashed line).

5. Conclusions

The mixed FEM and the Uzawa algorithm using a variable penalty parameter allow us to build numerical schemes to satisfy the incompressibility constraint. For the finite element approximation, the discrete spaces, ensuring the LBB condition are used. The mixed FEM, using a variable penalty parameter, gives the best results for the modeling problems in the square and curvilinear areas. Quadratic interpolating functions for the velocity and linear functions for the pressure allow us to solve the Navier–Stokes equations rather than interpolating functions of lower degrees.

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