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Richardson's extrapolation without interpolation in problems of advective–diffusive transport*

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It is offered to use Richardson's extrapolation on the basis of the up-wind scheme. Four difference problems with double grid steps at each grid point are solved simultaneously in one iteration process, which allows one to obtain a difference solution with the double grid step for all the grid points. The effect of this approach is illustrated on examples of solution of the two problems: of advection and of advective–diffusive transport.

1. Introduction

Advective processes are of central importance in the geophysical fluid dynamics and their treatment is crucial in numerical modeling of the transport of tracers constituents in the ocean models. During the past decades, a wide variety of finite difference methods have been proposed for the numerical solution to the advection equation.

In the global ocean climate simulation, when advection dominates over diffusion, it is difficult to obtain a sufficient resolution to use symmetric differences for approximating advections. In case of a coarse dimensional resolution, the obtained difference solution can qualitatively vary when reducing a grid step [1]. The applied upwind scheme with large computational viscosity, to be exact, computational diffusion, will probably, result in a temperature trend, when the deep ocean temperature slowly increases with time. For overcoming this problem it is offered to use Richardson's extrapolation based on the up-wind scheme, when at each time step the problem is solved on two embedded grids with steps h and $2h$, [2]. A linear combination of two poor solutions of first accuracy order allows us to obtain more precise solution with second accuracy order. The deficiency of this method is deriving the improved solution on a coarse grid with a step $2h$, that results in inevitable interpolation.

Four difference problems with a double grid step at every grid point is simultaneously solved in one iteration process, which allows us to obtain a

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difference solution with a double grid step for all the grid points. The effect of this approach is illustrated on examples of solution of two problems of advection and advective–diffusive transport.

2. The advection-diffusion transport problem

We first consider the two-dimensional advection–diffusion equation in the domain G with the boundary Γ ([2])

$$-\varepsilon\Delta\psi + a(x,y)\frac{\partial\psi}{\partial x} + b(x,y)\frac{\partial\psi}{\partial y} = f(x,y), \qquad \psi(x,y)|_{\Gamma} = 0, \tag{1}$$

where $a(x,y) = y - 2.5$, $b(x,y) = 1$, $f(x,y) = 1$, $\varepsilon = 0.01$. The differential operator of the equation for dimensionless variables describes the behavior of an integrated stream function.

In the domain $G = \{0 \le x \le 1,\ 0 \le y \le 5\}$, construct the uniform grid G^h with the steps $\Delta x = 1/N$, $\Delta y = 1/M$, where N, M are the number of points. On the grid G^h we approximate equation (1) by the upwind difference scheme

$$\begin{aligned}
&-\varepsilon\frac{\psi_{i,j-1} - 2\psi_{ij} + \psi_{i,j+1}}{\Delta x^2} - \varepsilon\frac{\psi_{i-1,j} - 2\psi_{ij} + \psi_{i+1,j}}{\Delta y^2} + \\
&\frac{a_{i,j-1/2} + |a_{i,j-1/2}|}{2}\frac{\psi_{ij} - \psi_{i,j-1}}{\Delta x} + \frac{a_{i,j+1/2} - |a_{i,j+1/2}|}{2}\frac{\psi_{i,j+1} - \psi_{ij}}{\Delta x} + \\
&\frac{b_{i-1/2,j} + |b_{i-1/2,j}|}{2}\frac{\psi_{ij} - \psi_{i-1,j}}{\Delta y} + \frac{b_{i+1/2,j} - |b_{i+1/2,j}|}{2}\frac{\psi_{i+1,j} - \psi_{ij}}{\Delta y} = f_{ij}.
\end{aligned} \tag{2}$$

This scheme has the first order of approximation, is monotonic and has large computational viscosity that exceeds physical one. Alongside with the grid G^h we introduce a grid G^{2h} with the steps $2\Delta x$, $2\Delta y$ such, that all its points coincide with the points G^h. On the grid G^{2h} we also approximate equation (1) by the upwind difference scheme

$$\begin{aligned}
&-\varepsilon\frac{\psi_{i,j-2} - 2\psi_{ij} + \psi_{i,j+2}}{4\Delta x^2} - \varepsilon\frac{\psi_{i-2,j} - 2\psi_{ij} + \psi_{i+2,j}}{4\Delta y^2} + \\
&\frac{a_{i,j-1} + |a_{i,j-1}|}{2}\frac{\psi_{ij} - \psi_{i,j-2}}{2\Delta x} + \frac{a_{i,j+1} - |a_{i,j+1}|}{2}\frac{\psi_{i,j+2} - \psi_{ij}}{2\Delta x} + \\
&\frac{b_{i-1,j} + |b_{i-1,j}|}{2}\frac{\psi_{ij} - \psi_{i-2,j}}{2\Delta y} + \frac{b_{i+1,j} - |b_{i+1,j}|}{2}\frac{\psi_{i+2,j} - \psi_{ij}}{2\Delta y} = f_{ij}.
\end{aligned} \tag{3}$$

Further it is proposed to solve simultaneously four difference problems with the double grid step at every grid point in one iteration Gauss–Zeidel process, which allows one to come to the difference solution ψ^{2h} with the double grid step for all the points of the grid G^h.

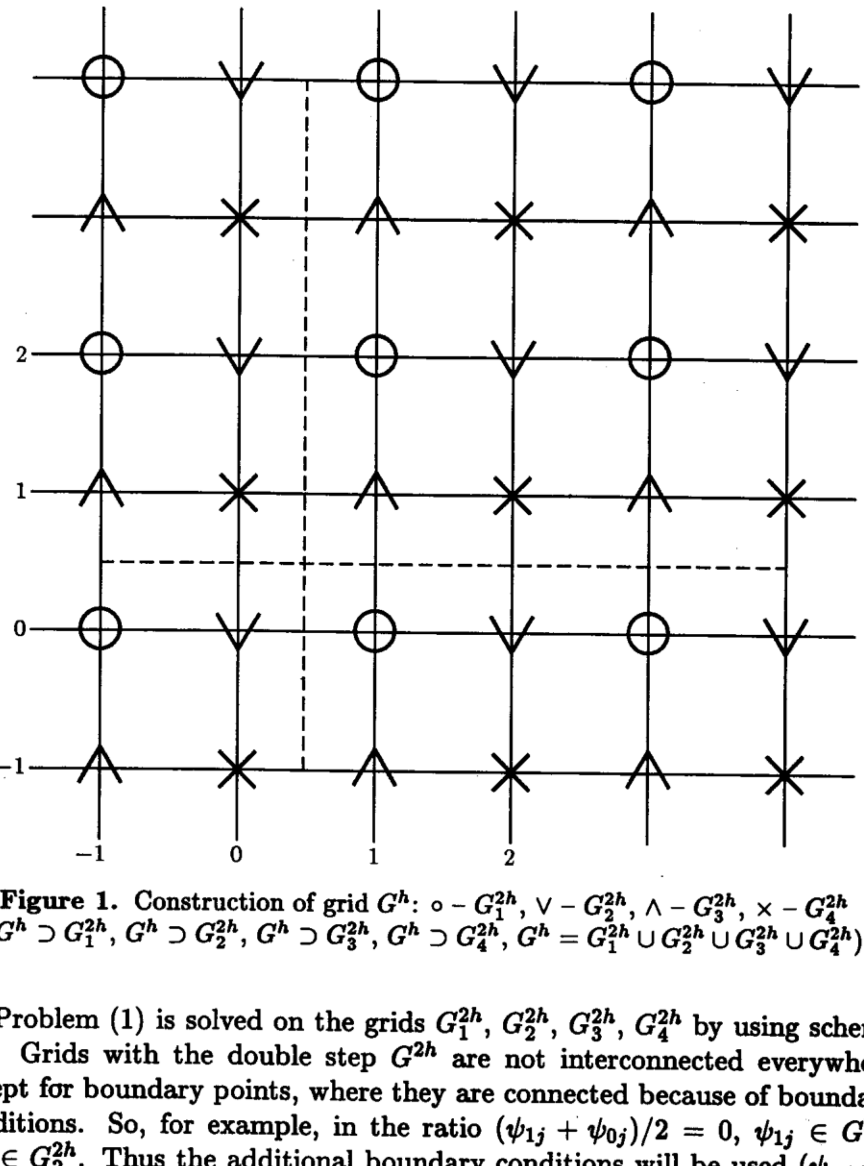

Figure 1. Construction of grid G^h: $\circ - G_1^{2h}$, $\vee - G_2^{2h}$, $\wedge - G_3^{2h}$, $\times - G_4^{2h}$ ($G^h \supset G_1^{2h}$, $G^h \supset G_2^{2h}$, $G^h \supset G_3^{2h}$, $G^h \supset G_4^{2h}$, $G^h = G_1^{2h} \cup G_2^{2h} \cup G_3^{2h} \cup G_4^{2h}$)

Problem (1) is solved on the grids G_1^{2h}, G_2^{2h}, G_3^{2h}, G_4^{2h} by using scheme (3). Grids with the double step G^{2h} are not interconnected everywhere except for boundary points, where they are connected because of boundary conditions. So, for example, in the ratio $(\psi_{1j} + \psi_{0j})/2 = 0$, $\psi_{1j} \in G_1^{2h}$, $\psi_{0j} \in G_2^{2h}$. Thus the additional boundary conditions will be used $(\psi_{-1j} + \psi_{2j})/2 = 0$, $\psi_{-1j} \in G_1^{2h}$, $\psi_{2j} \in G_2^{2h}$.

Similar relations also take place on other boundaries. In the case of another boundary value problem, for example, zero flux through boundary, additional computational boundary conditions linking grids are a natural approximation of zero flux with the step $3h$.

After obtaining the difference solutions ψ^h and ψ^{2h} of problem (1) it is possible to arrive at a solution with second accuracy order using a simple extrapolation formula

$$\overline{\psi}^h = 2\psi^h - \psi^{2h}. \tag{4}$$

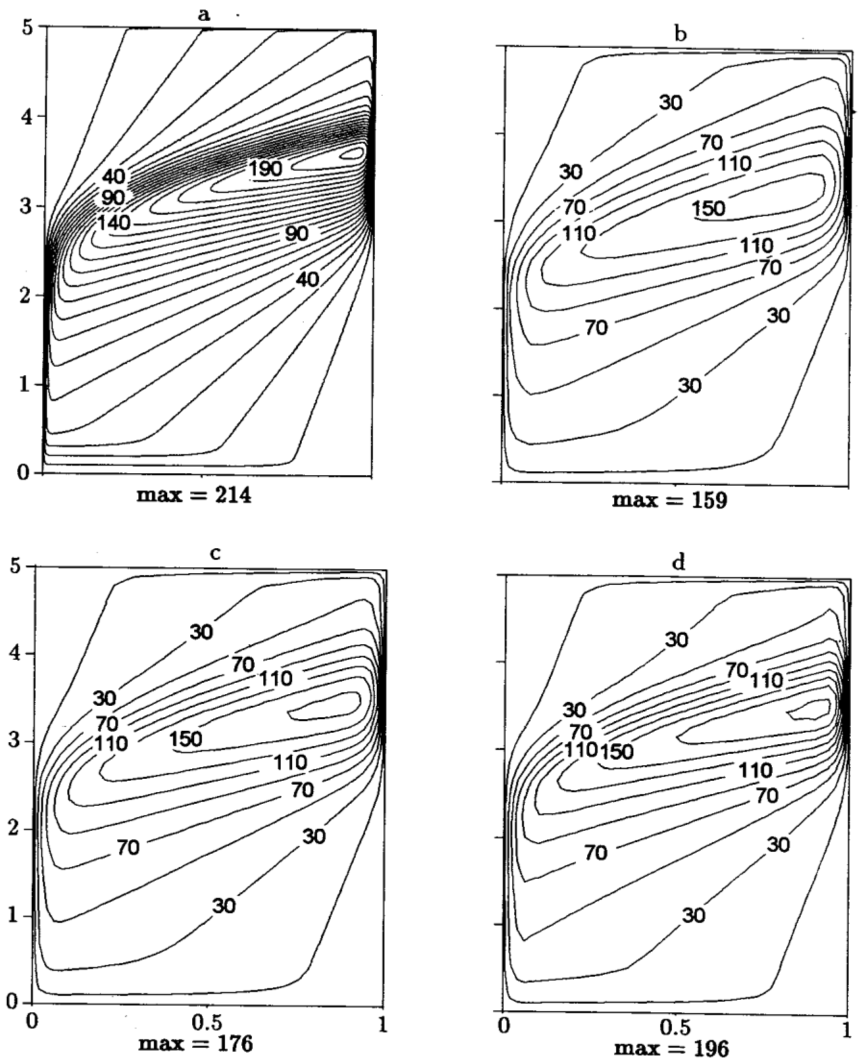

Figure 2. Isolines of difference solution (1): a) by the symmetric scheme for $N = 300$, b) step $2h$ for $N = 40$, c) step h for $N = 40$, d) Richardson's extrapolation by formula (4)

Let us consider the results of numerical experiments. First of all, Figure 2 shows the difference solution of problem (1), obtained on the grid G^h with thin resolution at $N = 300$, $M = 300$ by the symmetric central-difference scheme along both coordinates. It is possible to assume that the solution is close enough to the precise one, as it is obtained with second order of accuracy on a sufficiently fine grid. Further, we shall call it "precise".

In Figure 2, the isolines of the obtained solutions ψ^{2h}, ψ^h, $\overline{\psi}^h$ for $N = 40$ are presented. The internal boundary layer becomes intense with increasing

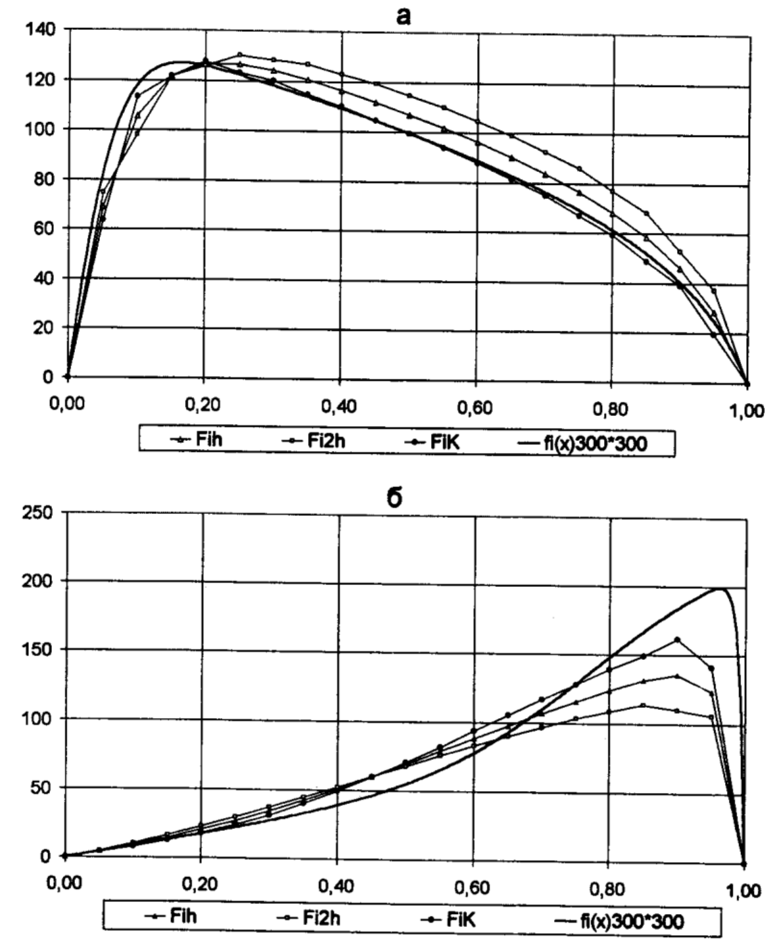

Figure 3. The difference solution (1) for $N = 20$: a) for $y = 2.5$, b) for $y = 3.5$

approximation order. The maximum values of the function are 159, 176, 196, respectively. The difference solution $\psi(x, 2.5)$, $\psi(x, 3.5)$, obtained at $N = 20$ is shown in Figure 3. Thus, the solution with second order of accuracy $\overline{\psi}^h$ significantly improves the solution ψ^h.

3. The advection transport problem

Let us dwell on an exception of numerical diffusion in an evolution problem on an example of solution of the following advection equation ([5])

$$\frac{\partial \varphi}{\partial t} + u\frac{\partial \varphi}{\partial x} + v\frac{\partial \varphi}{\partial y} = 0, \quad \varphi = \varphi(t, x, y). \tag{5}$$

For equation (10), the Cauchy problem with the initial condition and zero boundary conditions is considered:

$$\varphi(x,y)|_{t=0} = \begin{cases} 100\cos\gamma & \text{for } |\gamma| \le \pi/2, \\ 0 & \text{for } |\gamma| > \pi/2, \end{cases} \tag{6}$$

$$\gamma = \frac{\pi}{2R}\sqrt{(x-x_0)^2 + (y-y_0)^2}.$$

Here x_0, y_0 are the coordinates of the impulse center, R is radius of the initial impulse. Problem (5), (6) describes the rotating impulse with the constant angular velocity Ω in $L \times L$ square with period $T = 2\pi/\Omega$. The center of rotation is in the center of the square.

Let us introduce a regular grid with a step $\Delta x = \Delta y = L/M$: $G_h = \{(\Delta x(i-1), \Delta y(j-1)) \;:\; i = 1,\dots,M,\; j = 1,\dots,M\}$. The time step is $\Delta t = 120$ s, and the full cycle occurs for 208 steps. Equation (5) is approximated by the implicit upwind scheme with respect to x and y:

$$\begin{aligned} &\frac{\varphi_{ij}^{n+1} - \varphi_{ij}^{n}}{\Delta t} + u_{ij}^{+}\frac{\varphi_{ij}^{n+1} - \varphi_{i-1j}^{n+1}}{\Delta x} + u_{ij}^{-}\frac{\varphi_{i+1j}^{n+1} - \varphi_{ij}^{n+1}}{\Delta x} + \\ &v_{ij}^{+}\frac{\varphi_{ij}^{n+1} - \varphi_{ij-1}^{n+1}}{\Delta y} + v_{ij}^{-}\frac{\varphi_{ij+1}^{n+1} - \varphi_{ij}^{n+1}}{\Delta y} \doteq 0, \end{aligned} \tag{7}$$

the boundary conditions: $\varphi(1,j) = \varphi(i,1) = \varphi(M,j) = \varphi(i,M) = 0$.

Run	Experiment	$\max\varphi$	$\min\varphi$	$\sum\varphi_i^n/\sum\varphi_i^0$	$\sum(\varphi_i^2)^n/\sum(\varphi_i^2)^0$
R1	step $2\Delta x$	5.26	0	0.60	0.03
	step Δx	9.58	0	0.82	0.06
	correction	14.15	−1	0.99	0.12
R2	step $2\Delta x$	8.12	−0.01	0.74	0.05
	step Δx	8.14	−0.01	0.76	0.05
	correction	8.16	−0.01	0.75	0.05
R3	step $2\Delta x$	33.37	−5.09	1.05	0.33
	step Δx	33.96	−5.33	1.05	0.34
	correction	34.54	−5.61	1.05	0.35

In the first experiment R1, solutions are defined on coarse and fine grids through a full period of time (208 steps), and then formula (4) is once applied. The first solution is defined on 18×18 grid with the step $2\Delta x$, the second – on 35×35 grid with the step Δx, and the third solution with second order of accuracy is obtained by extrapolation formula (4) on the 18×18 grid. The results of calculations are contained in the table. The upwind scheme has large scheme viscosity, which results, as well as in [6], in decreasing maxima of an impulse for one period by 5–10% depending on a grid step. Decreasing $\sum\varphi_i^n/\sum\varphi_i^0$ up to 60–82% is explained by using zero

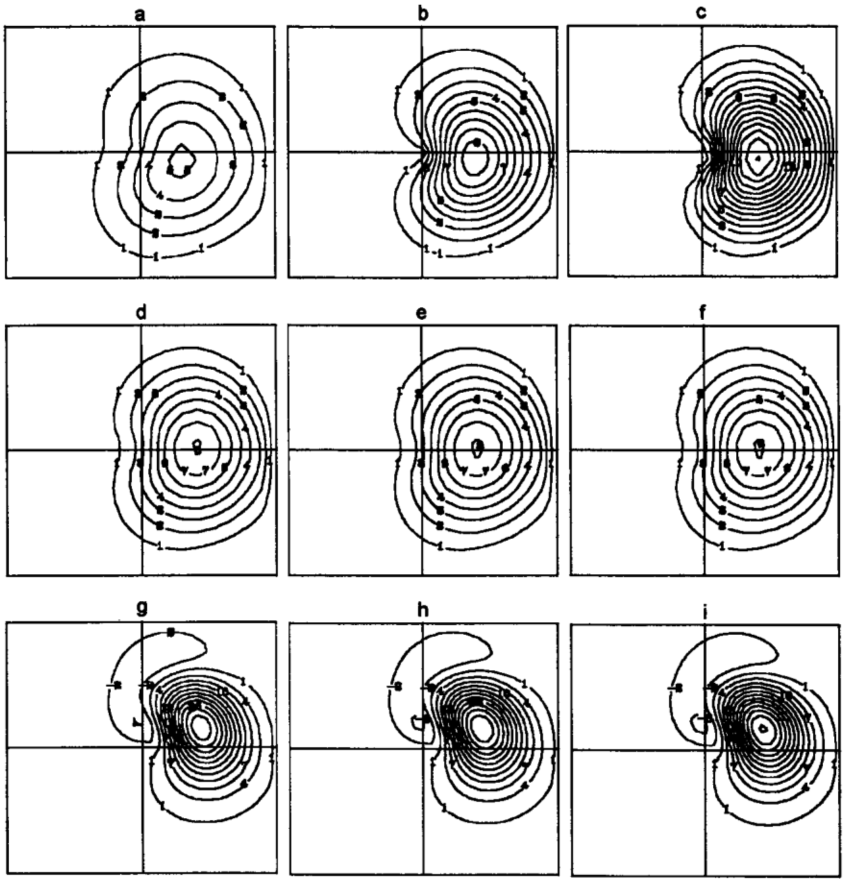

Figure 4. Difference solution of problem (5), (6) in one time period: a), b), c) obtained in experiment R1 with steps $2\Delta x$, Δx and Richardson's extrapolation, respectively; d), e), f) in experiment R2; and g), h), i) in experiment R3

boundary conditions. Richardson's extrapolations improve the solution, but only insignificantly.

In addition, Figure 4 a–c demonstrates that isolines of all the solutions are distorted, are lengthened forward stream.

In the second experiment R2, the scheme viscosity is eliminated not only with respect to space, but also with respect to time. The problem is solved on two the above-described grids simultaneously, but in this case Richardson's extrapolation by formula (4) is made in one step with respect to time $2\Delta t$ on 18×18 grid and in two steps with respect to time Δt for a fine grid. The improved solution obtained on 18×18 grid will be employed as previous approximation for both problems, interpolating it for 35×35 grid. The results of Run 2 are also presented in one full time period. The table and Figure 4 d–f demonstrate that the use of interpolation results in

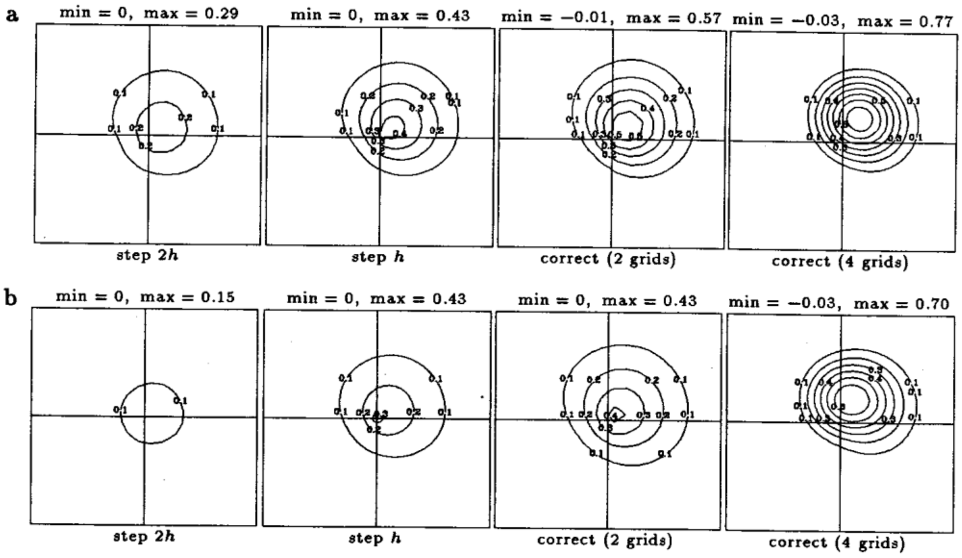

Figure 5. Difference solution of problem of the cone transport: a) in one time period, b) in two time periods

making the solution worse on a fine grid, and all clarifications disappear, as consequence application of Richardson's extrapolation is not reasonable.

In experiment R3, the problem of interpolation is solved with the use of four embedded grids, as is depicted in the second partition. In remaining this experiment retries R2.

All the four grids with twin steps are inter-connected at the expense of adding computational boundary conditions. The solution with a step $2\Delta x$ is obtained for each point of a fine 35×35 grid. The maximum impulse is 34% in one time period, the waveform kept better as well. However negative values of the function are generated, although insignificant in magnitude. Also, there appeared a phase error, the result of which this solution is 10 time steps per rotation ahead of the analytical one.

Another experiment was made with an other waveform – the cone, [7]. The experiments R1 and R3 were repeated, the result being shown in Figure 5. The waveform has essentially influenced on the result. Since in the case of the cone the scheme viscosity is absent everywhere except for its edge and base, as the second derivative is equal to zero, then on the whole both experiments have appeared more effective as compared to calculations with an impulse as cosine (6). Richardson's extrapolation in Run 1 in one time period keeps 57% of amplitude of a precise impulse, instead of 14% for cosine, and the use of the four embedded grids save 77% of an impulse, not 35% as earlier. It is interesting to emphasize that the second time period decreases the amplitude of a signal only by 7% when using four grids, while using one embedded grid – by 14%.

The presented results indicate to good prospects of the proposed method as applied to solution of advective–diffusive equations.

References

[1] Weaver A.J., Sarachik E.S. On the importance of the vertical resolution in certain ocean general circulation models // J. of Phys. Ocean. – 1990. – Vol. 20. – P. 600–609.

[2] Kochergin V.P., Scherbakov A.V. Study of difference schemes for the elliptic equation with small parameter in higher derivatives // Chislennye Modeli Okeanicheskikh Tsirkulyatsiy. – Novosibirsk: VTs SOAN SSSR. – 1972. – P. 7–24 (in Russian).

[3] Scherbakov A.V. Experiments with numerical climate model of the World Ocean. – Novosibirsk, 1989. – (Preprint / Computing Center, USSR; 859) (in Russian).

[4] Scherbakov A.V., Sinyagovskaya V.V. Richardson's extrapolation on irregular grid in problem of the advection-diffusion transport // Proceeding VTs SO RAN. Ser. Chislennoe modelirovanie v zadachakh atmosfery, okeana i okruzhayuschei sredy. – Novosibirsk, 1993. – Iss. 1. – P. 47–59 (in Russian).

[5] Crowley W.P. Numerical advection experiments // Mon. Weather Rev. – 1968. – Vol. 96, № 1.

[6] Ilyin V.O. Analysis of finite-difference schemes for numerical solution of the advection equation // Meteorologiya i Gidrologiya. – 1983. – № 6. – P. 13–23 (in Russian).

[7] Shokin Yu.I. Yanenko N.N. Method of Differential Approximation. – Novosibirsk: Nauka, 1985 (in Russian).