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Numerical solution of Volterra integral equations of the second kind by implicit Runge–Kutta method

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An effective multistep algorithm for numerical solution of Volterra integral equations of the second kind, based on the implicit Runge–Kutta (RK) method, is constructed. The choice of the Gauss scheme for the implicit RK method allows to obtain algorithm, having a higher approximation order for one-step method and maintaining the same accuracy for most calculations in multistep method. The comparison of solutions computed by algorithm, presented in this paper, with the analogues ones, obtaining earlier by explicit RK method, reveales more precise results for the first of them and possibility to integrate with much more longer steps without essential loss of accuracy.

1. The implicit Runge–Kutta (RK) methods, developed by Butcher and his followers more than 20 years ago [1], presented itselves in a good light in numerical solution for systems of differential equations.

A comparising with the more known explicit RK methods, shows that the implicit ones possesses the following properties:

- absence of considerable restrictions on their stability [2], which allows to integrate them with more long steps;
- an algorithmical possibility for construction of unrestricted approximation order schemes;
- choosing of more convenient integration scheme for different tasks of mathematical physics.

2. Consider the implicit RK scheme for the solution of differential equation

$$\frac{\partial y}{\partial t} = f(t, y), \quad 0 \le t \le T, \quad y(0) = y_0. \tag{1}$$

In assumption that solution in point t_n, $y_n = y(t_n)$ is already known, we integrate equation (1) over interval $[t_n, \xi_i]$, $t_n \le \xi_i \le t_{n+1}$, and obtain:

$$y(\xi_i) = y_n + \int_{t_n}^{\xi_i} f(t, y(t))\, dt. \tag{2}$$

Substitute unknown values $y(t)$ in integrand function in the points ξ_j, $j = 1, \ldots, m+1$ on some values η_j and construct the quadrature formula with weights β_{ij} in nodes ξ_j, where

$$\tau = t_{n+1} - t_n, \quad \xi_j = t_n + \lambda_j \tau, \quad \lambda_0 = 0, \quad \lambda_{m+1} = 1, \quad \lambda_j \leq \lambda_{j+1}.$$

Then we obtain the system of nonlinear algebraic equations for determination of sought function values on the next time step:

$$\eta_i = y_n + \tau \sum_{j=1}^{m} \beta_{ij} f(\xi_j, \eta_j), \quad i = 1, \ldots, m+1, \quad y_{n+1} = \eta_{m+1}. \tag{3}$$

In monograph [3] is shown that a consistent and complete system of linear algebraic equations for determination of quadrature weights β_{ij}, may be obtained by the use of sampling functions method, combined with weak approximation principle. Also, the order of approximation will be defined by a number of stages m in (3).

The implicit RK method possesses properties, which create a possibility of its application not only for the solution of differential equations.

3. On a basis of above mentioned discussion, we construct at first the implicit one-step RK scheme for the solution of the uniform nonlinear Volterra integral equation of the second kind:

$$y(t) = \int_{t_0}^{t} K(t, s, y(s))\, dx, \quad t_0 \leq t \leq T. \tag{4}$$

It is known that the explicit RK method is used for the solution of equation (4). It was obtained by adoption of the idea of maximum closeness for the solution to the Taylor approximation without calculation of derivatives from initial function [4]. The same approach reduces to obtaining the computational algorithms until 4-th order approximation inclusively, being an immediate analogues of explicit RK methods for the solution of differential equations.

To find the implicit RK scheme for solving (4), we consider it in the node ξ_i, where $t_0 \leq \xi_i \leq t$:

$$y(\xi_i) = \int_{t_0}^{\xi_i} K(\xi_i, s, y(s))\, ds. \tag{5}$$

Despite of the obvious similarity of formulas (2) and (5), they have the essential distinction: that is the presence of an additional variable in kernel of integral operator, equal to high integration bound ξ_i.

Next we substitute unknown function values in nodes ξ_j on variables η_j, and construct a quadrature formula for solution of (5) on these nodes. Then we obtain the following system to solve (4):

$$\eta_i = \tau \sum_{j=1}^{m} \beta_{ij} K(\xi_i, \xi_j, \eta_j), \quad i = 1, \ldots, m+1, \quad y^1 = \eta_{m+1}, \tag{6}$$

where $\tau = t - t_0$, $\xi_j = \lambda_j \tau$, $\lambda_0 = 0$, $\lambda_{m+1} = 1$, $\lambda_j \leq \lambda_{j+1}$.

Here and further the upper function index will denote the integration step for which a function is defined.

To find the quadrature coefficients β_{ij} in (6), we choose a polynomial function as kernel of integral operator, and use the above mentioned method of sampling functions and weak approximation principle. Thus, at least for the Gauss scheme (i.e., for non-singular matricies $B = \{\beta_{ij}\}$ and $\Lambda = \mathrm{diag}\{\lambda_j\}$), we obtain a system of equations for defining the weight coefficients of quadrature and integration nodes, coinciding with analogous systems for solving differential equations, and, hence, we obtain the same values of weight coefficients and nodes.

We suppose that the solution y^1 of system (6) approximates the solution of initial equation (4) in point t with order M, if for ε, where $\varepsilon = |y(t) - y^1|$, is realized $\varepsilon = \tau^M R_m$. A small value for ε may be obtained because of a smallness as time step τ, as approximation coefficient R_m. Smallness of the second one in solution of integral equations with big steps of integration, may be sufficient for obtaining a numerical solution close to the seeking one.

In assumption of existence of bounded derivatives until $2m$-th order inclusively for integral kernel, we obtain

$$\varepsilon = \tau^{2m+1} R_m, \quad R_m = \frac{1}{(2m+1)!} K^{(2m)}(t, \xi, y(\xi)) E_m,$$

where $\xi \in [t_0, t]$, and the value for functional E_m, which may be found from relations between quadrature coefficients and node values, will be as follows [3]:

$$E_m = (-1)^{2m} \frac{(m!)^4}{[(2m)!]^2}.$$

4. The multistep implicit RK method. Calculation of $y^2, y^3, \ldots, y^n$ values.

To obtain the solution of the values, following after y^1 with constant step $\tau = t_k - t_{k-1}$, one can use the method of shift for integral origin [6], and by virtue of this, equation (4) may be written in the form

$$y(t) = \Psi^n(t) + \bar{y}(t), \quad t \in [t_{n-1}, t_n], \tag{7}$$

where

$$\Psi^n(t) = \int_{t_0}^{t_{n-1}} K(t, s, y(s))\, ds, \tag{8}$$

and $\bar{y}(t)$ is the solution of the equation

$$\bar{y}(t) = \int_{t_{n-1}}^{t} K(t, s, \bar{y}(s) + \Psi^n(s))\, ds. \tag{9}$$

The system of algebraic equations for determination the solution of equation (9) one may obtain by the use the one-step implicit RK method, stated in the previous paragraph, if the function values $\Psi^n(\xi_j^n)$ are known in nodes of chosen quadrature ξ_j^n or if their suitable approximation Ψ_j^n, which will be discussed below, exists. In this case the system of equations for solving (9) will be the following:

$$\eta_i^n = \tau \sum_{j=1}^{m} \beta_{ij} K(\xi_i^n, \xi_j^n, \eta_j^n + \Psi_j^n), \quad i = 1, \ldots, m+1, \quad \bar{y}^n = \eta_{m+1}^n, \tag{10}$$

The values Ψ_j^n, where $j = 1, \ldots, m+1$ may be determined by choosing a suitable quadrature formula for calculation of integral (8).

We use the circumstance that the Gauss scheme will be used to solve equations (9) on every time step $[t_{k-1}, t_k]$, where $k = 1, \ldots, n$, that means the choice of λ_i coinciding with Gauss quadrature nodes on interval $[0, 1]$. Moreover, the values of sought function y_i^k inside every such step, where $k = 1, \ldots, n-1$, have been calculated yet in the corresponding points.

Write (8) in the form:

$$\Psi^n(t) = \sum_{k=1}^{n-1} \int_{t_{k-1}}^{t_k} K(t, s, y(s))\, ds, \tag{11}$$

then every integral under sum sign may be calculated, using the Gauss quadrature formula.

So, in assumption that the coefficients β_{ij} and λ_i are calculated earlier, before the first time step beginning, the algorithm for calculation y^n value, approximating $y(t_n)$, will be the following:

(a) calculate, using the Gauss quadrature formula with the nodes

$$\xi_i^k = \tau(\lambda_i + k - 1); \quad k = 1, \ldots, n-1; \quad i = 1, \ldots, m,$$

and taking into account (11), the values Ψ_j^n in nodes $\xi_j^n = \tau(\lambda_j + n - 1)$, where $j = 1, \ldots, m+1$;

(b) calculate the values η_i^n, $i = 1, \ldots, m+1$ and $\bar{y}^n$ by solving the system of equations (10);

(c) determine the values $y_i^n = \eta_i^n + \Psi_i^n$, which will be used in next time step;

(d) the sought value y^n will be obtained by the formula

$$y^n = \Psi_{m+1}^n + \bar{y}^n.$$

It is known that on every interval $[t_{k-1}, t_k]$ for the Gauss scheme the Butcher condition $C(m)$ is valid, i.e., the values y_i^k approximate $y(\xi_i^k)$ in nodes with order m [1]. In assumption of a boundedness of derivatives $\frac{\partial}{\partial z}K(t_l, s_l, z)$ in points t_l and s_l, corresponding to node values, and, taking into account that the approximation order of quadrature formulas for (11) and system of equations, analogues to (10) with substitution Ψ_j^n on $\Psi^n(\xi_j^n)$, will be $2m-1$ and $2m+1$ accordingly, one may conclude that the common approximation order of algorithm stated above will be not less than m.

5. In order to approbate the stated method, we concider the following integral equation, for which a solution had been sought using the explicit RK 4-th order approximation method with double-sided correction [5, 6]:

$$y(t) = \int_0^t (te^{s(t-2s)} + e^{-2s^2})(y(s) + 1 - s)^2 ds, \tag{12}$$

in points $t = 0.025$ and $t = 0.05$.

The exact solution of this equation is:

$$y(t) = e^{t^2} - 1 + t. \tag{13}$$

The first purpose of numerical experiments was the solving of equation (12) by the implicit one-step RK method in the points indicated.

The sought values were computed by 5-th stages ($m = 5$) Gauss scheme, that is according to 10-th order approximation, using the computer packet RUNKUT for computation the weight quadrature coefficients and nodes of integration [7]. Therefore, it was naturally to wait more precise solution than early obtained one is.

Indeed, the obtained results coincided with exact values of function (13) at points indicated in 5 significant digits, and improved the values, computing by explicit RK method, more than on two orders by their absolute error values.

Comparing the solutions, computed by these two methods, it is necessary to note that the approximation order in the explicit RK scheme was restricted, because of the necessity to seek a consistent system of equations to determine the scheme coefficients. On the other hand, the approximation order in the implicit RK method is obtained automatically by choosing the necessary amount of stages.

Further, the values in points, exceeding the initial ones on order, (i.e., t=0.25 and t=0.5), have been computed. The coincidence with precise values in these points was in 5 significant digits also. For example, in point t=0.5 was computed $y^1 = 0.7840245$, while the precise value is $y(0.5) = 0.7840254$.

Pass to a search of solution to equation (12) for a greater value of variable t. In this case the use of one-step RK method may not lead to a solution of

the system of equations (6) at all, and it is necessary to use the multistep algorithm, stated in previous paragraph, with the smaller value of time step τ.

The mentioned algorithm was applied for the solution of equation in point $t = 2.$ with step $\tau = 0.5$, $(\bar{n} = 4)$. In this case the approximation order is equal to m, that allowed to obtain a result similar to precise one. So, in this case $y^4 = 55.59805$ for $y(2.) = 55.59815$.

6. Thus, the stated above results allow to hope that the implicit RK method will be one of the main base algorithms for the solution of Volterra integral equations of the second kind.

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