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On solvability of the Cauchy problem for one-dimensional system of the Hopf type equations

D.K. Salaev

Abstract. The Cauchy problem for a one-dimensional system of equations of the Hopf type is considered. A theorem on the solvability of the considered Cauchy problem in the class of analytic functions has been proved.

Introduction

Modern theories of continuum mechanics [1–3] suggest the influence of its past on the movement of a medium of the past, and in the general case, a material can have an arbitrarily long "memory". However, a long memory gives rise to significant difficulties, which can be overcome in two ways: first, to consider special classes of movements in which memory, whatever it may be, does not have the ability to significantly manifest itself (for example, viscometric flows of viscous fluids [5, Ch. V]), and second, to single out the classes of media or materials in which the stresses at any point are affected only by the prehistory of motion over an arbitrarily small time interval. Materials of this type are called infinitesimal memory materials.

The most important materials with the infinitesimal memory are materials in which the stresses at a point x at a time t^* are determined by the first n derivatives of the strain gradient $F(x)$ with respect to the time t^* at the same time t^*. Such materials are called Differential Complexity materials $n = 1, 2, \dots$. The theory of isotropic fluids of the differential complexity type was constructed by Rivlin and Erikeen [3, 4], and based on this theory Coleman and Noll [3] have constructed a simpler asymptotic theory of slow motions of fluids of order $n = 1, 2, \dots$. In the case of incompressible fluids, a zero-order fluid is an elastic fluid whose motion is described by the Euler equations

$$\frac{\partial v}{\partial t} + v_k \frac{\partial v}{\partial x_k} + \frac{1}{\rho}\nabla p = F, \quad \operatorname{div} v = 0, \tag{1}$$

the first order fluid is a Newtonian linear viscous fluid, whose motion of which is described by the Navier–Stokes equations

$$\frac{\partial v}{\partial t} + v_k \frac{\partial v}{\partial x_k} - \nu\Delta v + \frac{1}{\rho}\nabla p = F, \quad \operatorname{div} v = 0. \tag{2}$$

Here $\rho > 0$ and $\nu > 0$.

1. A system of equations of the two-velocity hydrodynamics

In the isothermal case the equations of motion of a two-velocity medium in the dissipative case with one pressure in the system has the form [4–8]:

$$\frac{\partial \rho_1}{\partial t} + \operatorname{div}(\rho_1 v_1) = 0, \quad \frac{\partial \rho_2}{\partial t} + \operatorname{div}(\rho_2 v_2) = 0, \tag{3}$$

$$\frac{\partial v_1}{\partial t} + (v_1, \nabla) v_1 + \frac{1}{\rho}\nabla p = \frac{\nu_1}{\rho}\Delta v_1 + \frac{\nu_1 + 3\mu_1}{3\rho}\nabla \operatorname{div} v_1 + \frac{\rho_2}{2\rho}\nabla (v_1 - v_2)^2 + F, \tag{4}$$

$$\frac{\partial v_2}{\partial t} + (v_2, \nabla) v_2 + \frac{1}{\rho}\nabla p = \frac{\nu_2}{\rho}\Delta v_2 + \frac{\nu_2 + 3\mu_2}{3\rho}\nabla \operatorname{div} v_2 - \frac{\rho_1}{2\rho}\nabla (v_1 - v_2)^2 + F, \tag{5}$$

where v_1 and v_2 are the velocity vectors of the subsystems that make up the two-velocity continuum with the corresponding partial densities ρ_1 and ρ_2, ν_1 (μ_1), and ν_2 (μ_2) are the corresponding shear (bulk) viscosities, $\rho = \rho_1 + \rho_2$ is the total density of the two-velocity continuum; F is the vector of the mass force per unit mass. The system of equations (3)–(5) is closed by the equation of state of the two-velocity continuum

$$p = p(\rho, (v_1 - v_2)^2).$$

2. The Hopf-type system of equations

Systems of equations (3)–(5) in the case of a constant phase saturation are a generalization of system (1) and (2) for a multiphase medium, respectively. A subclass of system (3)–(5) in the case of the constant phase saturation in the dissipative case are systems of equations of the Hopf type. In the one-dimensional case, in the absence of the mass forces, this system has the form [9, 10]

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -b(u - v), \tag{6}$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = \varepsilon b(u - v), \tag{7}$$

where $\varepsilon = \frac{\rho_1}{\rho_2}$ is the dimensionless positive constant, b is a positive constant.

System (6), (7) differs from that system of the two-velocity hydrodynamics in the dissipative case, due to the coefficient of friction, the absence of pressure, and the condition of incompressibility. For this reason, the problems associated with the Hopf-type system will sometimes be called two-velocity fluid dynamics without pressure. Also, in the case when the energy dissipation occurs only due to the interfacial friction coefficient, we will call the inviscid system of the Burgers type or the Hopf type system, or we will also call it the Riemann type system. This gives a simple quasilinear system of equations. When the friction coefficient ($b = 0$) disappears, system (6), (7) goes to the well-known Hopf equation [11].

Zeldovich has proposed to consider an inviscid free system in the one-velocity case in the absence of mass forces as an equation describing the evolution of a rarefied gas of non-interacting particles [12]. According to his idea, the pure kinematics of the underlying particles can lead to peculiarities in the distribution of mass and is responsible for the inhomogeneity of matter in the universe.

Following [13], we denote by $\mathbb{C}(0, T; X_s)$ the space of the analytic functions $u(z)$ in the disc $\mathbb{C}_T = \{z \in \mathbb{C} : \ |z| < T\}$, bounded for $|z| \leq T$ and taking values in the Banach space X_s. Having defined in it the norm

$$\|u\|_{\mathbb{C}(0,T;X_s)} = \sup_{t\in[0,T]} \|u\|_{s,t}, \quad \|u\|_{s,t} = \sup_{|z|=t} \|u(z)\|_s, \quad \|\cdot\|_s = \|\cdot\|_{X_s},$$

we obtain the Banach space.

Here X_s, $s \in [0, 1]$, is a one-parameter family (scale) of the Banach spaces such that $X_s \subseteq X_{s'}$, for $s' < s$, and the norm of the embedding operator ≤ 1, i.e. for all $u \in X_s$

$$\|u\|_{s'} \leq \|u\|_s, \quad s' < s.$$

Let for any pair of the numbers $s', s \in [0, 1]$, $s' < s$, the mapping V is defined on the ball $\mathbb{C}^{r,u_0}(0, T; X_s) = \{u \in \mathbb{C}(0, T; X_s) : \|u - u_0\|_{\mathbb{C}(0,T;X_s)} < r\}$ with the center $u_0 \in \mathbb{C}(0, T; X_1)$ and takes it to $\mathbb{C}(0, T; X_{s'})$. We call V the Volterra operator of the class $J(\alpha, \beta, \mathbb{C})$, $\alpha > 0$, $\beta \geq 0$, if there exists a number $c > 0$ such that for any $u, v \in \mathbb{C}^{r,u_0}(0, T; X_s)$, $s' < s$, $t \in [0, T]$, the following estimate is fulfilled:

$$\|Vu - Vv\|_{s',t} \leq c(s - s')^{-\beta}(J^\alpha \|u - v\|_{s,\tau})(t),$$

where J^α is the integration operator of order $\alpha > 0$,

$$(J^\alpha \varphi)(t) = \Gamma^{-1}(\alpha) \int_0^t (t - \tau)^{\alpha-1} \varphi(\tau)\, d\tau,$$

$\Gamma(\alpha)$ is the gamma function. In particular, for $\alpha = 1$

$$(J\varphi)(t) = \int_0^t \varphi(\tau) d\tau, \quad J \equiv J^1.$$

Theorem 1 [13]. *Let $V \in J(\alpha, \beta, \mathbb{C})$. Then:*

1) *the solution to the equation $u = Vu$ is unique in the ball $\mathbb{C}^{r,u_0}(0, T; X_s)$ at $s > 0$;*
2) *if $Vu_0 \in \mathbb{C}^{r,u_0}(0, T; X_s)$ for some $s \in (0, 1]$, there exists a number $c > 0$ such that for any $s' < s$ the equation $u = Vu$ has the solution*

$$u \in \mathbb{C}^{r,u_0}(0,T';X_{s'}), \quad T' < c(s-s'), \quad T' < T.$$

This theorem is the Volterra version of well-known theorems on the solvability of the abstract Cauchy problem (see [14, 15] and the literature cited therein). It is proved in essence in the same way as the Nishida theorem in [14]. The more general case of the spaces $L_p(0,T;X_s)$ $(1 \leq p \leq \infty)$ is considered in [16].

We now turn to the statement of the Cauchy problem for a Hopf-type system.

Let the Cauchy data be given for system (6), (7):

$$u_k|_{t=0} = u_k^0(x), \quad k = 1, 2, \tag{8}$$

where $u_1(t,x) = u(t,x)$, $u_2(t,x) = v(t,x)$.

Integrating systems (6), (7) and taking into account (8), we arrive to the equivalent system of equations

$$u_1(t,x) = u_1^0(x) - \int_0^t \left[u_1(\tau,x)\frac{\partial u_1(\tau,x)}{\partial x} + b(u_1(\tau,x) - u_2(\tau,x))\right]d\tau, \tag{9}$$

$$u_2(t,x) = u_2^0(x) - \int_0^t \left[u_2(\tau,x)\frac{\partial u_2(\tau,x)}{\partial x} - \varepsilon b(u_1(\tau,x) - u_2(\tau,x))\right]d\tau. \tag{10}$$

Introducing the vector functions $w = (u_1, u_2)$, this system can be written down in the form $w = Vw$, where the operator V is defined by the right-hand sides of equalities (9), (10). Let the Banach space X_s, $s \in [0,1]$, consist of analytic vector-functions $w : \Omega_s \to \mathbb{C}^2$ in the domain $\Omega_s = \{(x,t) \in \mathbb{C} : |x| < \delta(1+s),\ |t| < \delta(1+s)\}$, $\delta > 0$, such that

$$\|w\|_s = |w|_s + |w_x|_s < \infty, \tag{11}$$

where $|w|_s = \sup\{|w(x,t)|,\ (x,t) \in \Omega_s\}$. Let us set $w_0 = (u_1^0(x), u_2^0(x))$ and show that the operator V satisfies the conditions of Theorem 1. Since $w \in \mathbb{C}^{r,w_0}(0,T;X_s)$, from (9), (10) it follows that $V : \mathbb{C}^{r,w_0}(0,T;X_s) \to \mathbb{C}^{r,w_0}(0,T;X_{s'})$ for $s' < s$. It follows from the definition of the mapping V that, for $w_1, w_2 \in C^{r,w_0}(0,T;X_s)$, the difference $Vw_1 - Vw_2$ is represented as a linear combination of the vectors $w_1 - w_2$ and $D(w_1 - w_2)$ $(D = D_x)$ with operator coefficients of the type of Jb, where the function b is expressed in terms of w_1, w_2 and their first derivatives, J is the integration operator from zero to t. Notice, that

$$|Ju|_s \leq J|u|_s, \quad |bu|_s \leq |b|_s|u|_s, \quad |Du|_{s'} \leq \delta^{-1}(s-s')^{-1}|u|_{s'},\ \ s' < s. \tag{12}$$

These estimates are trivial and follow from the Cauchy formula for analytic functions. Thus, with allowance for (11)

$$|D^{\alpha}(Vw_1 - Vw_2)|_{s'} \le c(s-s')^{-|\alpha|}J|w_1 - w_2|_s,$$
$$\|Vw_1 - Vw_2\|_{s'} \le 2c(s-s')^{-|\alpha|}J\|w_1 - w_2\|_s.$$

These estimates show that $V \in J(1,1,\mathbb{C})$. If $t < T$, from the definition of V we have $\|Vw_0 - w_0\|_s = O(T)$ for sufficiently small T. Thus, Theorem 1 implies the existence of the unique analytic solution $w = (u_1, u_2)$ of system (9), (10) in some complex neighborhood of zero. Thus, we have proved the following

Theorem 2. *Let* $Y = (-\varepsilon, \varepsilon)$, $u_1^0(x), u_2^0(x) \in \mathbb{C}^{\omega}(Y)$ *be the class of real analytic functions. Then the Cauchy problem* (6)–(8) *has a unique solution* $(u_1, u_2) \in \mathbb{C}^{\omega}$ *in some neighborhood of zero.*

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