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Solvability of inverse dynamic problems for a one-dimensional system of Hopf-type equations

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Abstract. Inverse dynamic problems for a one-dimensional system of Hopf-type equations are considered. Theorems on the solvability of the considered problems in the class of analytic functions are proved.

It is known that a significant number of physical, geophysical, biological, ecological and other processes are described by nonlinear differential equations in partial derivatives. Practically important problems that arouse interest in such equations lead to various kinds of direct and inverse problems for them.

Interest in inverse problems arose at the beginning of the 20th century. The first inverse problems were associated with the research of astrophysicists and geophysicists (see [1, 2]). Later, prominent Soviet mathematicians A.N. Tikhonov, M.M. Lavrentiev, and V.G. Romanov introduced the term "inverse problem" (see [3–5]). At the same time, it turned out that most of the inverse problems are not well-posed according to Hadamard. The foundations of the theory and practice of studying ill-posed problems were laid down and developed in the works of A.N. Tikhonov, M.M. Lavrentiev and their followers.

The analysis and solution of inverse problems of many kinds has recently expanded greatly because of their relevance in many applications: elastography and medical imaging, seismology, potential theory, ion transport problems or chromatography, finance, etc.; see, for example, [6–8].

This paper considers some inverse problems for non-linear non-stationary PDEs of the Hopf type in one spatial domain of measurement.

1. Hopf type system of equations

Hopf-type systems of equations are a particular case of a system of equations for a two-fluid medium [9, 10]. In the one-dimensional case, in the presence of body forces, this system has the form [11, 12]:

$$\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} = -b(t)(u_1 - u_2) + f(x) g_1(t), \tag{1}$$

$$\frac{\partial u_2}{\partial t} + u_2 \frac{\partial u_2}{\partial x} = \varepsilon b(t)(u_1 - u_2) + f(x) g_2(t), \tag{2}$$

where $f(x)g_1(t)$ and $f(x)g_2(t)$ are body forces, $\varepsilon = \frac{\rho_1}{\rho_2}$ is a dimensionless positive constant, $b(t)$ is a positive function.

The system (1), (2) differs from the system of two-velocity hydrodynamics in the dissipative case due to the coefficient of friction, absence of pressure, and incompressibility condition. For this reason, the problems associated with a Hopf-type system will sometimes be referred to as two-velocity hydrodynamics without pressure. Also in the case when the energy dissipation occurs only due to the coefficient of interfacial friction, we will call the inviscid Burgers-type system or the Hopf-type system, or we will also call the Riemann-type system, which gives the simplest quasilinear system of equations. When the friction coefficient disappears ($b = 0$), the system (1), (2) passes to the well-known Hopf equation [13].

Zel'dovich proposed to consider an inviscid free system in the one-velocity case in the absence of body forces as an equation describing the evolution of a rarefied gas of noninteracting particles [14]. According to his idea, the pure kinematics of the underlying particles can lead to singularities in the distribution of mass and is responsible for the inhomogeneity of matter in the universe.

Following [15], we denote by $\mathbb{C}(0,T;X_s)$—the space of the analytic functions $u(z)$ in the disc $\mathbb{C}_T = \{z \in \mathbb{C} : |z| < T\}$, bounded for $|z| \le T$ and taking values in the Banach space X_s. Having defined in it the norm

$$\|u\|_{\mathbb{C}(0,T;X_s)} = \sup_{t\in[0,T]} \|u\|_{s,t},$$

$$\|u\|_{s,t} = \sup_{|z|=t} \|u(z)\|_s, \quad \|\cdot\|_s = \|\cdot\|_{X_s},$$

we obtain the Banach space.

Here X_s, $s \in [0,1]$, is a one-parameter family (scale) of the Banach spaces such that $X_s \subseteq X_{s'}$, for $s' < s$, and the norm of the embedding operator ≤ 1, i.e. for all $u \in X_s$

$$\|u\|_{s'} \le \|u\|_s, \quad s' < s.$$

Let for any pair of the numbers $s', s \in [0,1]$, $s' < s$, the mapping V is defined on the ball $\mathbb{C}^{r,u_0}(0,T;X_s) = \{u \in \mathbb{C}(0,T;X_s) : \|u - u_0\|_{\mathbb{C}(0,T;X_s)} < r\}$ with the center $u_0 \in \mathbb{C}(0,T;X_1)$ and takes it to $\mathbb{C}(0,T;X_{s'})$. We call V the Volterra operator of the class $J(\alpha,\beta,\mathbb{C})$, $\alpha > 0$, $\beta \ge 0$, if there exists a number $A > 0$ such that for any $u, v \in \mathbb{C}^{r,u_0}(0,T;X_s)$, $s' < s$, $t \in [0,T]$, the estimate is fulfilled:

$$\|Vu - Vv\|_{s',t} \le c\,(s - s')^{-\beta} (J^\alpha \|u - v\|_{s,\tau})(t),$$

where J^α is the integration operator of order $\alpha > 0$,

$$(J^{\alpha}\varphi)(t) = \Gamma^{-1}(\alpha) \int_0^t (t-\tau)^{\alpha-1}\varphi(\tau)d\tau,$$

$\Gamma(\alpha)$ is the gamma function. In particular, for $\alpha = 1$

$$(J\varphi)(t) = \int_0^t \varphi(\tau)\, d\tau, \quad J \equiv J^1.$$

Theorem 1 [15]. *Let $V \in J(\alpha, \beta, \mathbb{C})$. Then*:

1) *the solution to the equation $u = Vu$ is unique in the ball $\mathbb{C}^{r,u_0}(0,T;X_s)$ at $s > 0$;*
2) *if $Vu_0 \in \mathbb{C}^{r,u_0}(0,T;X_s)$ for some $s \in (0,1]$, then there is a number $a > 0$ such that for any $s' < s$ the equation $u = Vu$ has the solution*

$$u \in \mathbb{C}^{r,u_0}(0,T';X_{s'}), \quad T' < a(s-s'), \;\; T' < T.$$

This theorem is the Volterra version of well-known theorems on the solvability of the abstract Cauchy problem (see [16, 17] and the literature cited therein). It is proved in essence in the same way as the Nishida theorem in [16]. The more general case of the spaces $L_p(0,T;X_s)$ $(1 \le p \le \infty)$ is considered in detail in [18].

2. The inverse problems

Let us now turn to the formulation of inverse problems for a Hopf-type system. Let the Cauchy data for the system (1), (2) be given:

$$u_k|_{t=0} = u_k^0(x), \quad k = 1,2, \tag{3}$$

Let also an additional information be given:

$$u_k|_{x=0} = \varphi_k(t), \quad k = 1,2. \tag{4}$$

The inverse problem is to determine the functions $u_1(x,t)$, $u_2(x,t)$, $g_1(t)$, $g_2(t)$ from (1)–(4). In this case, the function $f(x)$ is known and differs from zero, the matching condition is satisfied

$$u_k^0(0) = \varphi_k(0), \quad k = 1,2. \tag{5}$$

Let us apply the operator $\dfrac{\partial}{\partial x} f^{-1}(x)$ to both sides of equalities (1) and (2). After simple transformations, we obtain

$$\frac{\partial u_1}{\partial t} - \varphi_1'(t) = -b(t)(u_1 - u_2) + b(t)(\varphi_1 - \varphi_2) + \\ b(t)\int_0^x \frac{f'(\xi)}{f(\xi)}(u_1 - u_2)\,d\xi + \int_0^x \frac{f'}{f}\Big(\frac{\partial u_1}{\partial t} + u_1\frac{\partial u_1}{\partial \xi}\Big)d\xi - \\ \int_0^x \Big(\Big(\frac{\partial u_1}{\partial \xi}\Big)^2 + u_1\frac{\partial^2 u_1}{\partial \xi^2}\Big)d\xi, \tag{6}$$

$$\frac{\partial u_2}{\partial t} - \varphi_2'(t) = \varepsilon b(t)(u_1 - u_2) - \varepsilon b(t)(\varphi_1 - \varphi_2) - \\ \varepsilon b(t)\int_0^x \frac{f'(\xi)}{f(\xi)}(u_1 - u_2)\,d\xi + \int_0^x \frac{f'}{f}\Big(\frac{\partial u_2}{\partial t} + u_2\frac{\partial u_2}{\partial \xi}\Big)d\xi - \\ \int_0^x \Big(\Big(\frac{\partial u_2}{\partial \xi}\Big)^2 + u_2\frac{\partial^2 u_2}{\partial \xi^2}\Big)d\xi. \tag{7}$$

Thus, studying the solvability of the inverse problem (1)–(4) was reduced to studying the solvability of the direct problem (3), (6), (7). This direct problem is equivalent to the system of integro-differential equations

$$u_1 = u_1^0(x) + \varphi_1(t) - \varphi_1(0) - \\ \int_0^t b(\tau)(u_1(x,\tau) - u_2(x,\tau) - \varphi_1(\tau) + \varphi_2(\tau))\,d\tau + \\ \int_0^t \int_0^x \frac{f'(\xi)}{f(\xi)}\Big(\frac{\partial u_1(\xi,\tau)}{\partial \tau} + u_1(\xi,\tau)\frac{\partial u_1(\xi,\tau)}{\partial \xi}\Big)d\xi d\tau - \\ \int_0^t \int_0^x \Big(\Big(\frac{\partial u_1(\xi,\tau)}{\partial \xi}\Big)^2 + u_1(\xi,\tau)\frac{\partial^2 u_1(\xi,\tau)}{\partial \xi^2}\Big)d\xi\,d\tau + \\ \int_0^t b(\tau)\int_0^x \frac{f'(\xi)}{f(\xi)}(u_1(\xi,\tau) - u_2(\xi,\tau))\,d\xi\,d\tau, \tag{8}$$

$$u_2 = u_2^0(x) + \varphi_2(t) - \varphi_2(0) + \\ \varepsilon\int_0^t b(\tau)(u_1(x,\tau) - u_2(x,\tau) - \varphi_1(\tau) + \varphi_2(\tau))\,d\tau + \\ \int_0^t \int_0^x \frac{f'(\xi)}{f(\xi)}\Big(\frac{\partial u_2(\xi,\tau)}{\partial \tau} + u_2(\xi,\tau)\frac{\partial u_2(\xi,\tau)}{\partial \xi}\Big)d\xi\,d\tau - \\ \int_0^t \int_0^x \Big(\Big(\frac{\partial u_2(\xi,\tau)}{\partial \xi}\Big)^2 + u_2(\xi,\tau)\frac{\partial^2 u_2(\xi,\tau)}{\partial \xi^2}\Big)d\xi\,d\tau - \\ \varepsilon\int_0^t b(\tau)\int_0^x \frac{f'(\xi)}{f(\xi)}(u_1(\xi,\tau) - u_2(\xi,\tau))\,d\xi\,d\tau. \tag{9}$$

Introducing the vector function $u = (u_1, u_2)$, this system can be written down in the form $u = Vu$, where the operator V is defined by the right-hand sides of equalities (8), (9). Further, repeating the reasoning in the proof of Theorem 2 [10], with allowance for the estimate

$$|D^2 u|_s \le 2\delta^{-1}|u|_s,$$

which follows from the fact that the analytic function reaches its maximum at the boundary. Hence $V \in J(1,1,\mathbb{C})$. Thus, for small T, Theorem 1 implies the existence of the unique analytic solution $u = (u_1(x,t), u_2(x,t))$ of system (8), (9) in some complex neighborhood of zero. The functions $g_1(t)$ and $g_2(t)$ are determined by the formulas

$$g_1(t) = \frac{1}{f(x)}\Big(\frac{\partial u_1}{\partial t} + u_1\frac{\partial u_1}{\partial x} + b(u_1 - u_2)\Big),$$
$$g_2(t) = \frac{1}{f(x)}\Big(\frac{\partial u_2}{\partial t} + u_2\frac{\partial u_2}{\partial x} - \varepsilon b(u_1 - u_2)\Big).$$

Thus, we have proved

Theorem 2. *Let $f(x)$, $u_1^0(x)$, $u_2^0(x) \in C^\omega(Y)$, $f(x) \neq 0$. Then there are functions $u_1(x,t)$, $u_2(x,t)$, $g_1(t)$, $g_2(t)$ that solve the inverse problem* (1)–(4)*, satisfying in some neighborhood of zero, such that in it $u_1(x,t)$, $u_2(x,t)$, $g_1(t)$, $g_2(t) \in C^\omega$.*

The next inverse problem is to determine the functions $u_1(x,t)$, $u_2(x,t)$, $b(t)$, $g_2(t)$ from (1)–(4). In this case, the function $f(x)$ is known and separated from zero and $\varphi_1(t) - \varphi_2(t) \neq 0$, the matching condition (5) is satisfied.

Further, apply to both parts of equalities (1) and (2) the operators $\frac{\partial}{\partial x}(u_1 - u_2)^{-1}$ and $\frac{\partial}{\partial x} f^{-1}(x)$, respectively. After simple transformations, we obtain

$$\begin{aligned}\frac{\partial u_1}{\partial t} &= \varphi_1'(t) + (f(x) - f(0))g_1(t) - \int_0^x \Big(\Big(\frac{\partial u_1}{\partial \xi}\Big)^2 + u_1\frac{\partial^2 u_1}{\partial \xi^2}\Big)d\xi - \\ &\int_0^x (u_1 - u_2)^{-1}\Big(\frac{\partial u_1}{\partial \xi} - \frac{\partial u_2}{\partial \xi}\Big)\Big(f(\xi)g_1(t) + \frac{\partial u_1}{\partial t} + u_1\frac{\partial u_1}{\partial \xi}\Big)d\xi, \quad (10)\\ \frac{\partial u_2}{\partial t} &= \varphi_1'(t) + \int_0^x \frac{f'}{f}\Big(\frac{\partial u_2}{\partial t} + u_2\frac{\partial u_2}{\partial \xi}\Big)d\xi + \\ &\varepsilon(f(x) - f(0))g_1(t) - \int_0^x \Big(\Big(\frac{\partial u_2}{\partial \xi}\Big)^2 + u_2\frac{\partial^2 u_2}{\partial \xi^2}\Big)d\xi + \\ &\varepsilon\int_0^x \Big((u_1 - u_2)^{-1}\Big(\frac{\partial u_1}{\partial \xi} - \frac{\partial u_2}{\partial \xi}\Big) + \frac{f'}{f}\Big)\Big(\frac{\partial u_1}{\partial t} + u_1\frac{\partial u_1}{\partial \xi}\Big)d\xi. \quad (11)\end{aligned}$$

Consequently, the study of the solvability of the inverse problem of determining functions $u_1(x,t)$, $u_2(x,t)$, $b(t)$, $g_2(t)$ from (1)–(4) has been reduced to the study of the solvability of the direct problem (3), (10), (11). This problem is equivalent to the system of integro-differential equations

$$u_1 = u_1^0(x) + \varphi_1(t) - \varphi_1(0) + (f(x) - f(0)) \int_0^t g_1(\tau)\, d\tau -$$
$$\int_0^t \int_0^x \Big(\Big(\frac{\partial u_1}{\partial \xi}\Big)^2 + u_1 \frac{\partial^2 u_1}{\partial \xi^2} \Big) d\xi\, d\tau -$$
$$\int_0^t \int_0^x (u_1 - u_2)^{-1} \Big(\frac{\partial u_1}{\partial \xi} - \frac{\partial u_2}{\partial \xi}\Big) \Big(f(\xi) g_1(t) + \frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial \xi}\Big) d\xi\, d\tau,$$
$$u_2 = u_2^0(x) + \varphi_2(t) - \varphi_2(0) + \int_0^t \int_0^x \frac{f'}{f} \Big(\frac{\partial u_2}{\partial t} + u_2 \frac{\partial u_2}{\partial \xi}\Big) d\xi\, d\tau +$$
$$\varepsilon (f(x) - f(0)) \int_0^t g_1(\tau)\, d\tau - \int_0^t \int_0^x \Big(\Big(\frac{\partial u_2}{\partial \xi}\Big)^2 + u_2 \frac{\partial^2 u_2}{\partial \xi^2} \Big) d\xi\, d\tau +$$
$$\varepsilon \int_0^t \int_0^x \Big((u_1 - u_2)^{-1} \Big(\frac{\partial u_1}{\partial \xi} - \frac{\partial u_2}{\partial \xi}\Big) + \frac{f'}{f} \Big) \Big(\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial \xi}\Big) d\xi\, d\tau.$$

Repeating the arguments in the proof of Theorem 2, we have proved

Theorem 3. *Let $f(x)$, $u_1^0(x)$, $u_2^0(x) \in C^\omega(Y)$, $f(x) \neq 0$ and $\varphi_1(t) - \varphi_2(t) \neq 0$. Then there are functions $u_1(x,t)$, $u_2(x,t)$, $b(t)$, $g_2(t)$ solving the inverse problem (1)–(4), satisfying in some neighborhood of zero, such that in it $u_1(x,t)$, $u_2(x,t)$, $b(t)$, $g_2(t) \in C^\omega$. In this case, the unknown functions $b(t)$ and $g_2(t)$ are determined by the formula*

$$b(t) = \frac{1}{u_1 - u_2} \Big(f(x) g_1(t) - \frac{\partial u_1}{\partial t} - u_1 \frac{\partial u_1}{\partial x}\Big),$$
$$g_2(t) = \frac{1}{f(x)} \Big(\varepsilon \Big(\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x}\Big) + \frac{\partial u_2}{\partial t} + u_2 \frac{\partial u_2}{\partial x} \Big) - \varepsilon g_1(t).$$

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