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On one inverse source problem for a poroelasticity system

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Abstract. The inverse problem of determining a distributed source from a porodynamic system described by three elastic parameters in a reversible hydrodynamic approximation is considered. A theorem on solvability in the class of twice continuously differentiable in time and having Fourier transforms in the spatial variable is proved.

Keywords: porous medium, Cauchy problem, direct problem, inverse problem, slow wave, porosity.

1. Introduction

In applied problems of wave dynamics, there is often a need to take into account the porosity, fluid saturation of the medium and the hydrodynamic background. In particular, such questions arise in exploration geophysics during exploration of oil layers and the choice of wave action on oil and gas fields for the purpose of intensifying production. Similar questions also arise in seismology during geophysical monitoring of the properties of the focal zone for earthquake prediction [1–3].

In geophysics, the dynamic and kinematic characteristics of elastic waves propagating in fragmented fluid-saturated rocks contain information about the structure, composition and conditions of occurrence of rocks, they also contain data on the lithology of rocks and the nature of their boundaries, fracturing, porosity, the presence of various types of disturbances and local inclusions, as well as the composition and phase state of fluids filling the pore space of reservoirs. Mathematical models in wave theory provide a tool for determining the numerical values of the propagation velocities and absorption coefficients of elastic seismic waves depending on the material composition of the fluid-filled reservoir, its structure and the influence of the environment. The more realistic and adequate the mathematical model, the more accurate the determined values of the propagation velocity and absorption coefficient of elastic seismic waves.

The revealed features of seismic wave absorption in fractured-porous media with simultaneous manifestation of multiple electroseismic effects cannot be reconciled with the simplest models of an ideally elastic isotropic medium and a Biot medium. Real geological media are multiphase, electrically conductive, fractured, porous, etc. [4–11].

In [12], a nonlinear mathematical model of a porous elastically deformable medium saturated with liquid is constructed. The model is based on three main principles: the fulfillment of conservation laws, the Galilean principle of relativity, and the consistency of the equations of motion of the saturating liquid with the conditions of thermodynamic equilibrium:

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \boldsymbol{j} = 0, \quad \frac{\partial S}{\partial t} + \operatorname{div}\Big(\frac{S}{\rho}\boldsymbol{j}\Big) = 0, \quad \boldsymbol{j} = \rho_s \boldsymbol{u}_1 + \rho_l \boldsymbol{u}_2,$$

$$\frac{\partial \rho_l}{\partial t} + \operatorname{div}(\rho_l \boldsymbol{u}_2) = 0, \quad \frac{\partial g_{ik}}{\partial t} + g_{kj}\partial_i u_{1j} + g_{ij}\partial_k u_{1j} + u_{1j}\partial_j g_{ik} = 0,$$

$$\frac{\partial e}{\partial t} + \operatorname{div} \boldsymbol{Q} = 0, \quad \rho_s = \mathrm{const}\sqrt{\det(g_{ik})},$$

$$\frac{\partial j_i}{\partial t} + \partial_k(\rho_s u_{1i} u_{1k} + \rho_s u_{2i} u_{2k} + p\delta_{ik} + h_{ij} g_{jk}) = 0,$$

$$Q_k = \Big(\hat{\mu} + \frac{\boldsymbol{u}_2^2}{2} + \frac{TS}{\rho}\Big) j_k + \rho_s(\boldsymbol{u}_1, \boldsymbol{u}_1 - \boldsymbol{u}_2) u_{1k} + u_{1i} h_{km} g_{mi},$$

$$\frac{\partial \boldsymbol{u}_2}{\partial t} + (\boldsymbol{u}_2, \nabla)\boldsymbol{u}_2 = -\frac{\nabla p}{\rho} + \frac{\rho_s}{2\rho}\nabla(\boldsymbol{u}_1 - \boldsymbol{u}_2)^2 - \frac{h_{ik}}{2\rho}\nabla g_{ik}.$$

Here $\boldsymbol{u}_1$ is the velocity of the elastic porous medium; $\boldsymbol{u}_2$ is the velocity of the saturating liquid, $\rho = \rho_l + \rho_s$, ρ_s, ρ_l are the continuum density, the partial density of the porous body, and the partial density of the liquid, respectively; g_{ik} is the metric tensor of elastic deformation; h_{ik} is the stress tensor; e, S are the energy and entropy of a unit volume; $\hat{\mu}$ is the chemical potential; T is the temperature; p is the pressure, $\boldsymbol{j}$ is the relative momentum. In this case, the first law of thermodynamics is satisfied for the system under consideration:

$$de_0 = TdS + \hat{\mu} d\rho + (\boldsymbol{u}_1 - \boldsymbol{u}_2, d\boldsymbol{j}) + \frac{1}{2} h_{ik} dg_{ik}.$$

2. Direct problem

Let us consider the problem of wave propagation on a straight line for a porous medium in a reversible approximation, described by a one-dimensional homogeneous system of equations for $0 \le t \le T$ [1–11]:

$$\frac{\partial^2 u_1}{\partial t^2} - a_{11}\frac{\partial^2 u_1}{\partial x^2} - a_{12}\frac{\partial^2 u_2}{\partial x^2} = F_1(t,x), \tag{1}$$

$$\frac{\partial^2 u_2}{\partial t^2} - a_{21}\frac{\partial^2 u_1}{\partial x^2} - a_{22}\frac{\partial^2 u_2}{\partial x^2} = F_2(t,x), \tag{2}$$

where

$$a_{11} = \frac{\lambda + 2\mu}{\rho_s} + \Big(\rho\alpha_3 + \frac{\lambda + 2\mu/3}{\rho^2}\Big)\rho_s - \frac{\lambda + 2\mu/3}{\rho},$$

$$a_{12} = \Big(\rho^2\alpha_3 + \frac{\lambda + 2\mu/3}{\rho^2} - \frac{\lambda + 2\mu/3}{\rho_s}\Big)\frac{\rho_l}{\rho},$$
$$a_{21} = \Big(\rho^2\alpha_3 + \frac{\lambda + 2\mu/3}{\rho}\Big)\frac{\rho_l}{\rho} - \frac{\lambda + 2\mu/3}{\rho},$$
$$a_{22} = \Big(\rho^2\alpha_3 + \frac{\lambda + 2\mu/3}{\rho}\Big)\frac{\rho_l}{\rho},$$

$\rho_s = \rho_s^f(1 - d_0)$ and $\rho_l = \rho_l^f d_0$ are partial densities, ρ_s^f and ρ_l^f are the physical densities of the elastic porous body and liquid, respectively, d_0 is the porosity, $F = (F_1, F_2)$ is the vector of mass forces, α_3, λ and μ are the elastic parameters of the porous medium.

Let us consider the Cauchy problem for the system of poroelasticity equations (1) and (2) with the following Cauchy data [10]:

$$u_1|_{t=0} = \varphi_1(x), \quad \frac{\partial u_1}{\partial t}\Big|_{t=0} = \psi_1(x), \quad -\infty < x < \infty, \tag{3}$$
$$u_2|_{t=0} = \varphi_2(x), \quad \frac{\partial u_2}{\partial t}\Big|_{t=0} = \psi_2(x), \quad -\infty < x < \infty. \tag{4}$$

Here $\varphi_1(x)$, $\varphi_2(x)$, $\psi_1(x)$, $\psi_2(x)$, $F_1(t,x)$, and $F_2(t,x)$ are given functions. The problem of determining $u_1(t,x)$, $u_2(t,x)$ from (1)–(4) is usually called the direct problem for differential equations [13–15].

3. Inverse problem

In this paper, we study the issue of simultaneously determining the solution $u_1(t,x)$, $u_2(t,x)$ from the system of equations (1), (2) and the function $g_1(t)$, $g_2(t)$ ($F_1(t,x) = g_1(t)f(x)$, $F_2(t,x) = g_2(t)f(x)$), if the constants α_3, λ, μ, ρ_s^f, ρ_l^f, d_0 and the functions in conditions (3), (4) are given and

$$u_1|_{x=0} = \nu_1(t), \quad u_2|_{x=0} = \nu_2(t), \quad t > 0. \tag{5}$$

In other words, the problem is to determine the function $u_1(t,x)$, $u_2(t,x)$ and volume forces of the form $F_1(t,x) = g_1(t)f(x)$, $F_2(t,x) = g_2(t)f(x)$ based on the information given above. Such problem is usually called an inverse problem for differential equations [13, 14, 16].

In what follows we assume that all the functions and vector functions under consideration are twice continuously differentiable with respect to the variable t and have Fourier transforms with respect to x. In addition, we consider the known function $f(x) \neq 0$ for $-\infty < x < \infty$. We also assume that the data matching conditions are satisfied.

Let the functions $u_1(t,x)$, $u_2(t,x)$ be sufficiently smooth solutions to problem (1)–(4). If we put (5) in (1) and (2), we obtain

$$\frac{\partial^2 \nu_1}{\partial t^2} - a_{11}\frac{\partial^2 u_1}{\partial x^2}\Big|_{x=0} - a_{12}\frac{\partial^2 u_2}{\partial x^2}\Big|_{x=0} = g_1(t)f(0),$$
$$\frac{\partial^2 \nu_2}{\partial t^2} - a_{21}\frac{\partial^2 u_1}{\partial x^2}\Big|_{x=0} - a_{22}\frac{\partial^2 u_2}{\partial x^2}\Big|_{x=0} = g_2(t)f(0).$$

Hence

$$g_1(t) = \big(\nu_1''(t) - a_{11}u_{1xx}(t,0) - a_{12}u_{2xx}(t,0)\big)/f(0), \tag{6}$$
$$g_2(t) = \big(\nu_2''(t) - a_{21}u_{1xx}(t,0) - a_{22}u_{2xx}(t,0)\big)/f(0), \tag{7}$$

where $u_{kxx}(t,0) = \frac{\partial^2 u_k}{\partial x^2}\big|_{x=0}$, $k = 1,2$.

Suppose that there exists a Fourier transform of $u_1(t,x)$, $u_2(t,x)$ with respect to x. Let

$$\tilde{u}_k(t,\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u_k(t,x)e^{-ix\xi}dx, \quad u_k(t,x) = \int_{-\infty}^{\infty} \tilde{u}_k(t,\xi)e^{ix\xi}d\xi, \tag{8}$$

be respectively the direct and inverse Fourier transforms of function $u_k(t,x)$, $k = 1,2$, defined at $G_T = \{(t,x):\ t > 0,\ -\infty < x < \infty\}$ with respect to the variable x.

We apply the Fourier transform to (1), (2) and obtain a system of ODEs

$$\tilde{u}_{1tt} + a_{11}\xi^2\tilde{u}_1 + a_{12}\xi^2\tilde{u}_2 = g_1(t)\tilde{f}(\xi), \tag{9}$$
$$\tilde{u}_{2tt} + a_{21}\xi^2\tilde{u}_1 + a_{22}\xi^2\tilde{u}_2 = g_2(t)\tilde{f}(\xi), \tag{10}$$

where

$$\tilde{f}(\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-ix\xi}dx$$

is the Fourier transform of the variable function $f(x)$. Using (8) for $u_1(t,x)$ and $u_2(t,x)$, from (6) and (7) we obtain the following:

$$g_1(t) = \Big(\nu_1''(t) + a_{11}\int_{-\infty}^{\infty}\xi^2\tilde{u}_1(t,\xi)\,d\xi + a_{12}\int_{-\infty}^{\infty}\xi^2\tilde{u}_2(t,\xi)\,d\xi\Big)/f(0), \tag{11}$$
$$g_2(t) = \Big(\nu_2''(t) + a_{21}\int_{-\infty}^{\infty}\xi^2\tilde{u}_1(t,\xi)\,d\xi + a_{22}\int_{-\infty}^{\infty}\xi^2\tilde{u}_2(t,\xi)\,d\xi\Big)/f(0). \tag{12}$$

We have obtained representations (11) and (12) of the function g_1 and g_2 assuming that the functions $u_1(t,x)$ and $u_2(t,x)$ take Fourier transforms with respect to x ($\tilde{u}_k = F(u_k)$, $k = 1,2$) and the image takes the inverse Fourier transform ($u_k = F^{-1}(\tilde{u}_k)$, $k = 1,2$). Substituting (9), (10) into (1), (2) we obtain a system of ordinary integro-differential equations

$$\frac{d^2\tilde{u}_1}{dt^2} + a_{11}\xi^2\tilde{u}_1 + a_{12}\xi^2\tilde{u}_2$$
$$= \frac{\tilde{f}(\xi)}{f(0)}\Big(\nu_1''(t) + a_{11}\int_{-\infty}^{\infty}\xi^2\tilde{u}_1(t,\xi)\,d\xi + a_{12}\int_{-\infty}^{\infty}\xi^2\tilde{u}_2(t,\xi)\,d\xi\Big), \quad (13)$$
$$\frac{d^2\tilde{u}_2}{dt^2} + a_{21}\xi^2\tilde{u}_1 + a_{22}\xi^2\tilde{u}_2$$
$$= \frac{\tilde{f}(\xi)}{f(0)}\Big(\nu_2''(t) + a_{21}\int_{-\infty}^{\infty}\xi^2\tilde{u}_1(t,\xi)\,d\xi + a_{22}\int_{-\infty}^{\infty}\xi^2\tilde{u}_2(t,\xi)\,d\xi\Big), \quad (14)$$

with Cauchy data

$$\tilde{u}_k|_{t=0} = \tilde{\varphi}_k(\xi), \quad \frac{d\tilde{u}_k}{dt}\Big|_{t=0} = \tilde{\psi}_k(\xi), \quad -\infty < \xi < \infty, \quad k = 1,2, \quad (15)$$

where $\tilde{\varphi}_k = F(\varphi_k)$, $\tilde{\psi}_k = F(\psi_k)$.

Let us prove the unique solvability of the nonlocal direct problem (13)–(15). Our proof will be based on the weak approximation method [17–22].

We weakly approximate problem (13)–(15) by the problem

$$\frac{d^2\tilde{u}_1^\tau}{dt^2} + 2a_{11}\xi^2\tilde{u}_1^\tau + 2a_{12}\xi^2\tilde{u}_2^\tau = 0, \quad (16)$$
$$\frac{d^2\tilde{u}_2^\tau}{dt^2} + 2a_{21}\xi^2\tilde{u}_1^\tau + 2a_{22}\xi^2\tilde{u}_2^\tau = 0, \quad (17)$$
$$n\tau < t \le (n+1/2)\tau,$$
$$\frac{d^2\tilde{u}_1^\tau}{dt^2} = \frac{2\tilde{f}(\xi)}{f(0)}\Big(\nu_1''(t) + a_{11}\int_{-\infty}^{\infty}\xi^2\tilde{u}_1^\tau(t-\tau/2,\xi)\,d\xi +$$
$$a_{12}\int_{-\infty}^{\infty}\xi^2\tilde{u}_2^\tau(t-\tau/2,\xi)\,d\xi\Big), \quad (18)$$
$$\frac{d^2\tilde{u}_2^\tau}{dt^2} = \frac{2\tilde{f}(\xi)}{f(0)}\Big(\nu_2''(t) + a_{21}\int_{-\infty}^{\infty}\xi^2\tilde{u}_1^\tau(t-\tau/2,\xi)\,d\xi +$$
$$a_{22}\int_{-\infty}^{\infty}\xi^2\tilde{u}_2^\tau(t-\tau/2,\xi)\,d\xi\Big), \quad (19)$$
$$(n+1/2)\tau < t \le (n+1)\tau,$$
$$\tilde{u}_k^\tau|_{t=0} = \tilde{\varphi}_k(\xi), \quad \frac{d\tilde{u}_k^\tau}{dt}\Big|_{t=0} = \tilde{\psi}_1(\xi), \quad k = 1,2, \quad (20)$$

where $n = 0, 1, \ldots, N-1$; $\tau N = T$; N is an integer.

Note that with fixed $\tau > 0$ at each fractional step we solve standard problems: at the first fractional step we solve the Cauchy problem for a system of linear ordinary differential equations, at the second fractional step we solve the Cauchy problem for a system of ordinary linear differential equations of the second order, since the integrands on the right-hand side are taken from the previous fractional step and, therefore, the right-hand side of this equation is a known function.

From [23–25] follows the existence and uniqueness of the solution of problem (16)–(20) in the class of functions twice continuously differentiable with respect to t. The unknown functions $g_1(t)$, $g_2(t)$ are found by formulas (6), (7).

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