Behaviour analysis of parametric time Petri nets*

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We introduce a new notion of parametric time Petri nets, an extension of time Petri nets [10] whose transitions are labelled with parametric time restrictions. Further, a time behaviour analysis algorithm for the real time branching time temporal logic TCTL [1] and a one-safe parametric time Petri net is proposed. The result of the algorithm is a set of conditions on parameter variables which is sufficient for the property expressed as a TCTL-formula being satisfied for a given parametric time Petri net. Some remarks about complexity of the algorithm are also given.

Introduction

The verification problem of real-time systems is one of the main research directions of modern programming. Traditional verification methods are suitable for checking only qualitative timing properties of the systems. However, this is unsufficient for a correctness analysis of real-time systems whose behaviour crucially depends on quantitative properties. Several temporal logics for quantitative analysis of systems [3] have been proposed.

Concurrent real-time systems are often modelled by using time automata with a finite set of clocks [2], timed transition systems [4] and time process algebras (see, for example, [8]). However, all these formalisms are based on interleaving semantics and therefore information about concurrency is lost. On the other hand, time Petri nets [6] have been proposed as a suitable formalism for modelling concurrent real-time systems.

Model-checking is an effective verification tool. In [9], deductive and algorithmic verification methods based on temporal language CTL were proposed. In [5], a temporal logic was introduced for analysis of ‘fairness’ conditions of labelled Petri nets. However, model-checking algorithms usually require redundantly detailed specifications of systems, which can leave a user in repetitive trial-and-error cycles to select a parameter valuation. It should be useful to have a tool for a less-detailed specification of real-time systems and their properties.

In this paper we introduce a behaviour analysis algorithm of real-time systems represented as parametric time Petri nets. This kind of nets is an extension of time Petri nets [10] by means of parametric time constraints

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*This work is supported in part by the INTAS-RFBR (grant No 95-0378).
with character strings which represent unspecified timing constants as parameter variables. We assume that the value of each parameter variable of a parametric time Petri net does not exceed some fixed bound. Further we restrict ourselves to one-safe parametric time Petri nets. The result of the algorithm is a set of conditions on parameter variables which is sufficient for the property expressed as a TCTL-formula true for the given parametric time Petri net.

The rest of the paper is organized as follows. The basic definitions concerning parametric time Petri nets are presented in Section 1. Section 2 recalls the syntax and semantics of TCTL. In Section 3 we define a notion of a region for a time Petri net and construct the region graph in order to get a finite representation of the net. A time behaviour analysis algorithm is provided in Section 4. Some remarks about the complexity of the algorithm are finally given.

1. Parametric time Petri nets

Let $N$ be the set of natural numbers and $\mathbb{R}^+$ be the set of nonnegative real numbers. Let $\text{Var}$ be a set of parameter variables which are character strings representing unspecified timing constants and $A$ be an arbitrary set.

The syntax of a time predicate $\eta$ over $\text{Var}$ and $A$ is defined as follows:

$$\eta ::= x \sim \theta \mid \eta_1 \wedge \eta_2,$$

where $x \in A \cup \mathbb{N}$, $\theta \in \text{Var} \cup \mathbb{N}$, $\eta_1$ and $\eta_2$ are time predicates over $\text{Var}$ and $A$, and $\sim$ stands for one of the binary relations $\{<, \leq, =, \geq, >\}$. Let $B^A_{\text{Var}}$ be the set of all time predicates over $\text{Var}$ and $A$. For a time predicate $\eta \in B^A_{\text{Var}}$, the notation $[\eta]_{a_1, \ldots, a_l}$ is the result of substitution of values $a_1, \ldots, a_l$ instead of every occurrence of $y_1, \ldots, y_l$ in $\eta$.

**Definition 1.1.** A parametric time Petri net is a tuple $\mathcal{N} = (P, T, F, \text{Var}, \tau, m_0)$, where

- $P = \{p_1, p_2, \ldots, p_m\}$ is a finite set of places;
- $T = \{t_1, t_2, \ldots, t_n\}$ is a finite set of transitions ($P \cap T = \emptyset$);
- $F \subseteq (P \times T) \cup (T \times P)$ is a flow relation;
- $\text{Var} = \{\theta_1, \theta_2, \ldots, \theta_k\}$ is a finite set of parameter variables;
- $\tau : T \rightarrow B^T_{\text{Var}}$ is a function that associates a time predicate from $B^T_{\text{Var}}$ with each transition from $T$;
- $m_0 \subseteq P$ is the initial marking.

For $t \in T$, $\cdot t = \{p \in P \mid (p, t) \in F\}$ and $t^* = \{p \in P \mid (t, p) \in F\}$ denote the preset and postset of $t$, respectively. To simplify the presentation, we
assume that \( t \cap t^* = \emptyset \) for every transition \( t \). For the sake of convenience, we fix a parametric time Petri net \( \mathcal{N} = (P, T, F, \text{Var}, \tau, m_0) \) and work with it in what follows.

The parametric time Petri net \( \mathcal{N}_1 \) is shown in Figure 1, where the table contains the time predicates \( \eta_i \ (1 \leq i \leq 5) \) representing time constraints on transitions \( t_i \).

A marking \( m \) of \( \mathcal{N} \) is any subset of \( P \). A transition \( t \) is enabled in a marking \( m \) if \( t \subseteq m \) (all its input places have tokens in \( m \)), otherwise it is disabled. Let \( \text{enable}(m) \) be the set of transitions enabled in \( m \).

Let \( \mathcal{V} = [T \rightarrow \mathbb{R}^+ \cup \{\#\}] \) be the set of time assignments for transitions from \( T \) where \( \# \) is a special symbol for labelling disabled transitions such that \( \# \sim \theta = \text{true} \) for each \( \theta \in \text{Var} \cup \mathbb{N} \) and \( \sim \in \{<, \leq, =, \geq, >\} \). Assume that \( \nu \in \mathcal{V} \) and \( \delta \in \mathbb{R}^+ \). Then

\[
(\nu + \delta)(t) = \begin{cases} 
\nu(t) + \delta, & \text{if } \nu(t) \neq \#, \\
\nu(t), & \text{otherwise}.
\end{cases}
\]

A state \( q \) of \( \mathcal{N} \) is a pair \( \langle m, \nu \rangle \), where \( m \) is a marking and \( \nu \in \mathcal{V} \). The initial state in \( \mathcal{N} \) is a pair \( q_0 = \langle m_0, \nu_0 \rangle \), where

- \( m_0 \) is the initial marking in \( \mathcal{N} \);
- \( \forall t \in T, \nu_0(t) = \begin{cases} 
0, & \text{if } t \in \text{enable}(m_0), \\
\#, & \text{otherwise}.
\end{cases} \)

Let \( S \) denote the set of states of \( \mathcal{N} \).

A parameter valuation \( \chi \) of \( \mathcal{N} \) is a mapping from \( \text{Var} \) into \( \mathbb{N} \). Given a time assignment \( \nu \in \mathcal{V} \) and a transition \( t \in T \), we use the following notations:

\[
\tau^\nu(t)^\chi(t_1, \ldots, t_n, \theta_1, \ldots, \theta_k) = [\tau(t)^\chi(t_1, \ldots, t_n, \theta_1, \ldots, \theta_k); \tau^\nu(t)^\chi(t_1, \ldots, t_n, \theta_1, \ldots, \theta_k)] = (P, T, F, \text{Var}, \tau^\chi, m_0)
\]
There are two causes of state changes in \( \mathcal{N} \), i.e. (1) firing of transition, and (2) time passage.

In a state \( q = \langle m, \nu \rangle \) of \( \mathcal{N} \), a transition \( t \in T \) is fireable, if \( t \in enable(m) \) and there exists a parameter valuation \( \chi \) such that \( \tau^{\nu, \chi}(t) = \text{true} \). In this case, the state \( q' = \langle m', \nu' \rangle \) of \( \mathcal{N} \) is obtained from \( q \) by firing \( t \) (written \( q \overset{t}{\Rightarrow} q' \)), if

- \( m' = (m \setminus t) \cup t^* \), and 
  
- \( \forall t' \in T. \nu'(t') = \begin{cases} 
 0, & \text{if } t' \in enable(m') \setminus enable(m), \\
 0, & \text{if } t' \in enable(m) \setminus enable(m'), \\
 0, & \text{otherwise.}
\end{cases} \)

In a state \( q = \langle m, \nu \rangle \) of \( \mathcal{N} \), time \( \delta \in \mathbb{R}^+ \) can pass, if for all \( t \in enable(m) \) there exist \( \delta' \geq \delta \) and a parameter valuation \( \chi \) such that \( \tau^{\nu + \delta', \chi}(t) = \text{true} \). In this case, the state \( q' = \langle m', \nu' \rangle \) of \( \mathcal{N} \) is obtained from \( q \) by passing \( \delta \) (written \( q \overset{\delta}{\Rightarrow} q' \)), if

- \( m' = m \), and 
- \( \nu' = \nu + \delta \).

A \( q \)-run \( r \) of \( \mathcal{N} \) is an infinite sequence of states \( q_i \in S \) and time values \( \delta_i \in \mathbb{R}^+ \) of the form:

\[
q = q_1 \overset{\delta_1}{\Rightarrow} q_2 \overset{\delta_2}{\Rightarrow} \ldots \Rightarrow q_n \overset{\delta_n}{\Rightarrow} \ldots,
\]

satisfying the progress condition: for each \( w \in \mathbb{R}^+ \) there is \( n \in \mathbb{N} \) such that \( \sum_{1 \leq i \leq n} \delta_i \geq w \). We denote the expression \( \sum_{1 \leq i \leq n} \delta_i \) by \( time(r, n) \).

A state \( q \) is reachable if it belongs to some \( q_0 \)-run. Let \( RS(\mathcal{N}) \) denote the set of all reachable states of \( \mathcal{N} \).

\( \mathcal{N} \) is one-safe, if for every \( \langle m, \nu \rangle \in RS(\mathcal{N}) \) and for every \( t \in enable(m) \) it holds that \( t^* \cap m = \emptyset \).

A transition \( t \in T \) is time bounded if for each parameter valuation \( \chi \) and for each parameter variable \( \theta \) from \( \tau(t) \) there exists a constant \( c \) such that \( \chi(\theta) < c \). \( \mathcal{N} \) is time bounded if each of its transition is time bounded. The set of all such constants \( c \) for the given \( \mathcal{N} \) is denoted by \( C(\mathcal{N}) \).

Further \( \mathcal{N} \) will always denote a one-safe time bounded parametric time Petri net.

2. TCTL: syntax and semantics

Timed Computation Tree Logic (TCTL) was introduced by R. Alur, C. Courcoubetis, D. Dill [1] as a specification language for real time systems. We now review the syntax and semantics of TCTL.
Let $AP$ be a set of atomic propositions. For our purpose, it is convenient to take $AP = P$.

**Definition 2.1.** The formula $\phi$ of TCTL is inductively defined as follows:

$$
\phi ::= p \mid \neg \phi_1 \mid \phi_1 \land \phi_2 \mid \forall \phi_1 U_{\leq} \phi_2 \mid \exists \phi_1 U_{=} \phi_2,
$$

where $p \in AP$, $c \in N$, $\phi_1$ and $\phi_2$ are formulas of TCTL, $\sim$ stands for one of the binary relations $\{<, \leq, =, \geq, >\}$.

Informally, $\exists \phi_1 U_{<} \phi_2$ means that for some computation path there exists an initial prefix of time length less than $c$ such that $\phi_2$ holds in the last state of the prefix, and $\phi_1$ holds in all its intermediate states.

We define the derived connectives of the propositional calculus, such as $\lor$ and $\rightarrow$, in terms of $\neg$ and $\land$ in the usual way. In addition, some of the commonly used abbreviations are:

- $\forall \diamondsuit_{\leq} \phi \equiv \forall \text{true } U_{\leq} \phi,$
- $\exists \diamondsuit_{\leq} \phi \equiv \exists \text{true } U_{\leq} \phi,$
- $\forall \diamondsuit_{=} \phi \equiv \neg \exists \diamondsuit_{=} \neg \phi,$
- $\exists \diamondsuit_{=} \phi \equiv \neg \forall \diamondsuit_{=} \neg \phi.$

The unrestricted temporal operators correspond to TCTL-operators subscripted by $'\geq 0'$.

**Definition 2.2.** Given a parameter valuation $\chi$, a TCTL-formula $\phi$ and a state $q = (m, \nu) \in RS(N^x)$, we define the satisfaction relation $q \models \phi$ inductively as follows:

- $q \models p$ $\iff$ $p \in m$;
- $q \models \neg \phi_1$ $\iff$ $q \not\models \phi_1$;
- $q \models \phi_1 \rightarrow \phi_2$ $\iff$ $q \models \phi_1$ and $q \models \phi_2$;
- $q \models \exists \phi_1 U_{\leq} \phi_2$ $\iff$ for some $q$-run of $N^x$, $r \models \phi_1 U_{\leq} \phi_2$;
- $q \models \forall \phi_1 U_{=} \phi_2$ $\iff$ for every $q$-run of $N^x$, $r \models \phi_1 U_{=} \phi_2$.

For a $q$-run $r = (q = q_1 \xrightarrow{\delta_1} q_2 \xrightarrow{\delta_2} \ldots)$ in $N^x$ the relation $r \models \phi_1 U_{=} \phi_2$ holds iff there exist $k$ and $\delta \leq \delta_k$ such that:

1. $(\delta + \text{time}(r, k)) \sim c$;
2. $(m_k, \nu_k + \delta) \models \phi_2$;
3. $\forall 1 \leq i < k \forall 0 \leq \delta' < \delta_i \cdot (m_i, \nu_i + \delta') \models \phi_1$;
4. $\forall 0 \leq \delta' < \delta \cdot (m_k, \nu_k + \delta') \models \phi_1$.

$N^x$ satisfies a TCTL-formula $\phi$ (written $N^x \models \phi$) iff $q_0 \models \phi$. A TCTL-formula $\phi$ is satisfiable iff there is $N^x$ such that $N^x \models \phi$. 

Theorem 2.1. The satisfiability question for TCTL is $\Sigma_1^1$-hard.

Proof. Follows from Definition 2.2 and the corresponding theorem from [1].

A TCTL-formula containing $U$-operator is said to be an $U$-formula.

Further we fix a TCTL-formula $\phi$. $\chi$ is a satisfiable valuation of $\mathcal{N}$ for $\phi$ if $\mathcal{N}^\chi \models \phi$. To analyze $\mathcal{N}$ for $\phi$ means to find the set of satisfiable valuations of $\mathcal{N}$ for $\phi$.

3. Region graph

Since a parametric time Petri net constitutes a dense time model, the number of its states is infinite. In order to get a finite representation of the behaviour of a parametric time Petri net, we define a notion of a region [1]. Two states of a parametric time Petri net are in the same region iff their markings coincide and the corresponding time assignment values agree on the integral parts and on the ordering of the fractional parts.

We extend a parametric time Petri net with a clock tick indicator $\kappa$ which is conceptually a time assignment of a transition $t^\kappa$ that can not be reset and whose actual value is of no concern. But we can test whether it is an integer or not at any moment.

Let $c_\phi$ be the maximal constant appearing in $\phi$.

Then $K_{\mathcal{N};\phi} = \max(\{c_\phi\} \cup \{c \mid \theta \in C(\mathcal{N})\})$. For any $\delta \in \mathbb{R}^+ \cup \{\#\}$, $\text{frac}(\delta)$ denotes the fractional part of $\delta$, and $\lfloor \delta \rfloor$ denotes the integral part of $\delta$. Assume that $\lfloor \# \rfloor = \#$, $\text{frac}(\#) = 0$.

Definition 3.1. Given $\nu, \nu' \in \mathcal{V}$, $\nu \simeq_\phi \nu'$ iff the following conditions are met:

- for each $x \in T$ if either $\nu(x) \leq K_{\mathcal{N};\phi}$ or $\nu'(x) \leq K_{\mathcal{N};\phi}$, then $\lfloor \nu(x) \rfloor = \lfloor \nu'(x) \rfloor$;
- for each $x, x' \in T \cup \{t^\kappa\}$
  \begin{align*}
    - \text{frac}(\nu(x)) &\leq \text{frac}(\nu'(x')) \text{ iff } \text{frac}(\nu'(x)) \leq \text{frac}(\nu'(x')); \\
    - \text{frac}(\nu(x)) &\leq 0 \text{ iff } \text{frac}(\nu'(x)) = 0.
  \end{align*}

Given $\nu \in \mathcal{V}$, let us denote $[\nu]_\phi = \{\nu' \mid \nu' \simeq_\phi \nu\}$. A region of $\mathcal{N}$ with respect to $\phi$ is a pair $[q]_\phi = \langle m, [\nu]_\phi \rangle$, where $\langle m, \nu \rangle \in RS(\mathcal{N})$. We denote $m(\langle m, [\nu]_\phi \rangle) = m$.

Figure 2 shows the set of regions of the parametric time Petri net $\mathcal{N}_1$.

Lemma 3.1. Let $\langle m, \nu \rangle, \langle m, \nu' \rangle \in RS(\mathcal{N})$ with $\nu \simeq_\phi \nu'$. For every TCTL-formula $\psi$ such that $K_{\mathcal{N};\phi} \geq K_{\mathcal{N};\psi}$ and a parameter valuation $\chi$ such that $\langle m, \nu \rangle \in RS(\mathcal{N}^\chi)$ the following holds: $\langle m, \nu \rangle \models \psi \iff \langle m, \nu' \rangle \models \psi$.  


<table>
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<th>$m$</th>
<th>$t_1$</th>
<th>$t_2$</th>
<th>$t_3$</th>
<th>$t_4$</th>
<th>$t_5$</th>
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Figure 2. The regions of $N_1$

**Proof.** Before proving the lemma, we need to show that there exist $(m_0, \nu_0)$-paths $r$ and $r'$ in $N$ containing $(m, \nu)$ and $(m, \nu')$, respectively, which are similar in following sense. Let us consider a $(m_0, \nu_0)$-path $r$: $(m_0, \nu_0) \xrightarrow{\delta_0} (m_1, \nu_1) \xrightarrow{\delta_1} \ldots (m_k, \nu_k) = (m, \nu) \xrightarrow{\delta_k} (m_{k+1}, \nu_{k+1}) \ldots$ We show that there exists a $(m_0, \nu_0')$-path $r': (m_0, \nu_0') \xrightarrow{\delta'_0} (m_1, \nu_1') \xrightarrow{\delta'_1} \ldots (m_k, \nu_k') = (m, \nu') \xrightarrow{\delta'_k} (m_{k+1}, \nu_{k+1}') \ldots$ with the same sequence of markings as $r$ taken at "almost" the same time moments. Formally, for every $i \geq 1$ the following correspondences between $(\nu_i, time(r, i))$ and $(\nu_i', time(r', i))$ are met (written $(\nu_i, time(r, i)) \simeq (\nu_i', time(r', i))$):

- the time assignments $\nu_i$ and $\nu_i'$ are equivalent: $\nu_i \simeq \nu_i'$;
- the time values $time(r, i)$ and $time(r', i)$ agree on integer parts:
  - $|time(r, i)| = |time(r', i)|$,
  - $\text{frac}(time(r, i)) = 0 \iff \text{frac}(time(r', i)) = 0$;
- the time values $time(r, i)$ and $time(r', i)$ are in the same order with time assignments $\nu(t)$ and $\nu'(t)$, correspondingly: for each transition $t \in T$ it holds:
Given \( r \), we construct the desired path \( r' \) in two steps:

1. First, we construct a prefix \( \rho' \) (ending in \( \langle m, \nu' \rangle \)) of \( r' \) corresponding to the prefix \( \rho \) of \( r \): \( \langle m_0, \nu_0 \rangle \xrightarrow{\delta} \langle m_1, \nu_1 \rangle \xrightarrow{\delta} \ldots \langle m_k, \nu_k \rangle = \langle m_0, \nu_0 \rangle \). Let us construct \( \rho' \) step by step from \( \langle m, \nu' \rangle \) to \( \langle m, \nu \rangle \). Suppose that \( (k-i) \) states of \( \rho' \) have already been constructed. Given \( \delta_i \), we try to find \( \delta'_i \) so that we can extend \( \rho' \) with \( \langle m_{i-1}, \nu_{i-1} \rangle \) so that the equivalence is maintained at the \( (k-i+1) \)th step.

   Let \( \Delta \) be the set \( \{ \text{time}(r, i) \} \cup \{ \nu_i(t) \mid t \in T \} \). The set \( \Delta' \) is defined analogously. Because of the equivalence at the \( (k-i) \)th step, the necessary and sufficient requirement of the desired \( \delta'_i \) is such that each element of \( \Delta' \) should cross the same number of integer boundaries (due to the addition of \( \delta'_i \)) as the corresponding element of \( \Delta \) does (due to the addition of \( \delta_i \)). The reader can convince himself that the existence of \( \delta'_i \) meeting the constraints depends only on the ordering of the fractional parts of the elements of \( \Delta \). Hence, the existence of \( \delta_i \) guarantees the existence of \( \delta'_i \). Let \( \langle m, \nu \rangle \in RS(A^\times) \). For each \( i \) the constraint \( \eta \) on the time of reaching a state \( \langle m_{i+1}, \nu_{i+1} \rangle \) from the state \( \langle m_i, \nu_i \rangle \) depends only on \( \nu_i \) and contains only integer constants. Using \( \nu_i \simeq \nu'_i \) we get \( |\nu_i| = |\nu'_i| \). So, the constraint \( \eta' \) on the time of obtaining a state \( \langle m_{i+1}, \nu'_{i+1} \rangle \) from the state \( \langle m_i, \nu_i \rangle \) is satisfied iff \( \eta \) is satisfied. Then \( \rho' \) is a prefix in \( A^\times \) and, hence, \( \langle m, \nu \rangle \in RS(A^\times) \).

2. Secondly, we consider the path \( r' \) as a concatenation of the prefix \( \rho' \) and some \( \langle m, \nu' \rangle \)-path constructed step by step, starting from the state \( \langle m, \nu' \rangle \) for a given \( \langle m, \nu \rangle \)-path. The construction of the \( \langle m, \nu' \rangle \)-path is similar to the construction of \( \rho' \).

Now we show that the paths \( r \) and \( r' \) are similar. The base case follows since \( \nu \simeq \nu' \) and \( \text{time}(r, 1) = \text{time}(r', 1) = 0 \). Suppose that we have constructed \( r' \) correctly up to \( i \) steps. It is easy to see that the choice of \( \delta'_i \) depends only on an ordering of the fractional parts of elements on \( \Delta' \) which is the same as in \( \Delta \). Hence, the existence of \( \delta_i \) guarantees the existence of \( \delta'_i \) satisfying the above conditions.

We shall proceed by induction on the structure of \( \psi \). The base case and the cases corresponding to the logical connectives follow immediately. We will consider the case \( \psi = \exists \phi_1 \cup_{<} \phi_2 \). Suppose, \( \langle m, \nu \rangle \models \psi \). There exists an \( \langle m, \nu \rangle \)-path \( r \) such that \( r \models \phi_1 \cup_{<} \phi_2 \), i.e. there exist \( k \) and \( \delta \leq \delta_k \) such that:

1. \( (\delta + \text{time}(r, k)) \sim c_i \);
2. \( \langle m_k, \nu_k + \delta \rangle \models \phi_2 \);
3. \( \forall 1 \leq i < k \forall \delta'. (\delta' = 0 \lor 0 < \delta' < \delta_i) \langle m_i, \nu_i + \delta' \rangle \models \phi_1; \)
4. \( \forall 0 \leq \delta' < \delta. \langle m_k, \nu_k + \delta' \rangle \models \phi_1. \)

Let us construct an \( \langle m, \nu' \rangle \)-path \( r' \) as above. We have to show that \( r' \models \phi_1 \mathcal{U} \circ \phi_2. \)

1. We know that \( (\nu_{k-1}, \text{time}(r, k - 1)) \simeq (\nu_{k-1}', \text{time}(r', k - 1)). \) Because of the construction of \( r' \) for a given \( \delta \) we can find \( \delta' \) such that \( (\nu_k, \text{time}(r, k)) \simeq (\nu_k', \text{time}(r', k)). \) Using the fact that \( (\delta + \text{time}(r, k)) \sim_c \) \( (\delta' + \text{time}(r', k)) \sim_c. \)
2. We have \( \langle m_k, \nu_k + \delta \rangle \models \phi_2 \) and, by induction hypothesis, \( \langle m_k, \nu_k' + \delta' \rangle \models \phi_2. \)
3. Let \( i < k \) and \( 0 \leq \delta' < \delta_i. \) Now using the equivalence of \( (\nu_i, \text{time}(r, i)) \) and \( (\nu_i', \text{time}(r', i)) \), we can find \( 0 \leq \delta \leq \delta_i \) such that \( (\nu_{i+1}, \text{time}(r, i + 1)) \simeq (\nu_{i+1}', \text{time}(r', i + 1)). \) Since \( \langle m_i, \nu_i + \delta \rangle \models \phi_1, \) by induction hypothesis, we get \( \langle m_k, \nu_k' + \delta' \rangle \models \phi_1; \)
4. Similar to item 3.

Thus, \( r' \models \phi_1 \mathcal{U} \circ \phi_2 \) and \( \langle m, \nu' \rangle \models \psi. \)

\[ \square \]

**Lemma 3.2.** The number of equivalence classes of \( \mathcal{V} \) induced by \( \simeq \phi \) is bounded by \( |T| \cdot 2^{2|T|} \cdot (K_{\mathcal{N}; \phi} + 1)|T|. \)

**Proof.** We can represent an equivalence class \( [\nu]_\phi \) of \( \mathcal{V} \) induced by \( \simeq \phi \) by a triple of arrays \( \langle \alpha, \beta, \gamma \rangle \) as follows:

The array \( \alpha \) is a \( T \)-indexed array associating one of the intervals from \( [0, 0], (0, 1], [1, 1], \cdots, [K_{\mathcal{N}; \phi}, K_{\mathcal{N}; \phi}], (K_{\mathcal{N}; \phi}, \infty) \) with each transition \( t \in T. \)
The array \( \alpha \) represents a time assignment \( \nu \) iff \( \alpha(t) = \nu(t) \) for each \( t \in T. \)

Let \( T_\alpha \) be the set of transitions \( t \) such that \( \alpha(t) \) is not of the form \( [i, i] \) for some \( i < K_{\mathcal{N}; \phi}. \) Thus \( T_\alpha \) is a set of transitions for which an order of fractional parts of time assignments is essential.

The array \( \beta : T_\alpha \rightarrow \{1, \cdots, |T_\alpha|\} \) is a permutation of \( T_\alpha. \) It gives the ordering of fractional parts of time assignments corresponding to the transition from \( T_\alpha \) with respect to \( \leq. \) The array \( \beta \) represents a time assignment \( \nu \) iff for each pair \( t_1, t_2 \in T_\alpha \) we have: if \( \beta(t_1) \leq \beta(t_2), \) then \( \text{frac}(\nu(t_1)) \leq \text{frac}(\nu(t_2)). \)

The array \( \gamma \) is a boolean \( T_\alpha \)-indexed array, and is used to specify which transition in \( T_\alpha \) has the same fractional parts of corresponding time assignments. For each transition \( t \in T_\alpha, \gamma(t) \) tells whether or not the fractional part of \( \nu(t) \) equals to the fractional part of its \( \beta \)-predecessor. The array \( \gamma \) represents a time assignment \( \nu \) iff for each \( t_1 \in T_\alpha \) it holds that \( \gamma(t_1) = 0 \) iff there is a transition \( t_2 \in T_\alpha \) such that \( \beta(t_2) = \beta(t_1) + 1 \) and \( \text{frac}(\nu(t_1)) = \text{frac}(\nu(t_2)). \)
It is easy to see that the number of equivalence classes of \( \mathcal{V} \) induced by \( \simeq_{\phi} \) is bounded by the number of triples \( (\alpha, \beta, \gamma) \) of the desired form. The number of ways to choose \( \alpha \) is \( (K_{N;\phi} + 1)^{|T|} \). For a given \( \alpha \), the number of ways to choose \( \beta \) is bounded by the number of permutations over \( T_{\alpha} \) which is bounded by \(|T|!\), and the number of ways to choose \( \gamma \) is bounded by the number of \( k^{|T|} \).

**Definition 3.2.** The region graph of \( \mathcal{N} \) and \( \phi \) is defined to be a labelled graph \( G(\mathcal{N}, \phi) = (V, E, l) \). The set of vertices \( V \) is the set of regions of \( \mathcal{N} \) with respect to \( \phi \). The set of arcs \( E \subseteq V \times V \) consists of two types of arcs:

- The arc \( ([q], [q']) \) may represent firing of a transition in \( \mathcal{N} \): \( q \xrightarrow{\delta} q' \).
- The arc \( ([q], [q']) \) may represent the passage of time:
  - \( q' = q + \delta \) for some \( \delta \in \mathbb{R}^+ \);
  - there are no \( q \) and \( \delta \in \mathbb{R}^+ \), \( 0 < \delta < \delta \) such that \( [q] \neq [q] \), \( [q] \neq [q'] \), \( q + \delta = q' \), and \( q + \delta - \delta = q' \).

The function \( l \) labels an arc either with a transition \( t \in T \) (if the arc represents firing of \( t \)) or with the sign \( \delta' \) (if the arc represents the passage of time). For the sake of convenience, we fix a region graph \( G(\mathcal{N}, \phi) = (V, E, l) \) and work with it in what follows.

Figure 3 shows the region graph of the parametric time Petri net \( \mathcal{N}_1 \) (see Figure 1).

**Lemma 3.3.** The number of regions of \( \mathcal{N} \) for \( \phi \) is bounded by

\[
|T|! \cdot 2^{|P|} \cdot (K_{N;\phi} + 1)^{|T|}.
\]

**Proof.** Assume \( \langle m, [\nu] \rangle \) to be a region of \( \mathcal{N} \). Since \( m \) is an array of the length \(|P|\) consisting of '0' and '1', the number of different markings of \( \mathcal{N} \) is \( 2^{|P|} \). Moreover, the number of equivalence classes of time assignments is bounded by \( |T|! \cdot 2^{|P|} \cdot (K_{N;\phi} + 1)^{|T|} \), due to Lemma 3.2. Hence, the number of regions of \( \mathcal{N} \) for \( \phi \) is \( |T|! \cdot 2^{|P|} \cdot (K_{N;\phi} + 1)^{|T|} \), by Lemma 3.2.

For each arc \( ([q], [q']) \) from \( E \), we let \( \varepsilon([q], [q']) = \uparrow \) if going from \( q \) to \( q' \) the value of \( \kappa \) increases from a noninteger to an integer; \( \varepsilon([q], [q']) = \downarrow \) if going from \( q \) to \( q' \) the value of \( \kappa \) increases from an integer to a noninteger; otherwise \( \varepsilon([q], [q']) = 0 \).

Given a vertex \( v \) in \( V \), a path starting from \( v \) (written \( \Gamma^v \)) is a sequence of vertices \( \langle v_1v_2 \ldots \rangle \) from \( V \), where \( v_1 = v \) and for every \( i \geq 1 \) \( (v_i, v_{i+1}) \in E \) if \( v_{i+1} \) exists. We denote by \( l(\Gamma^v) \) a sequence of labels corresponding to \( \Gamma^v \). Then \( tr(\Gamma^v) \) is a restriction of \( l(\Gamma^v) \) to \( T \). Given a transition \( t \in tr(\Gamma^v) \) and \( v', v'' \in \Gamma^v \) such that \( l(t, \Gamma^v) = t \), we denote \( ln(t, \Gamma^v) = v' \).

Given \( v, v' \) from \( V \), we use \( \Gamma_{v'}^{v} \) to denote the finite path starting from \( v \) and ending in \( v' \). We define the time of \( \Gamma_{v'}^{v} = \langle v = v_1v_2 \ldots v_k = v' \rangle \) (written
time($\Gamma_v^w$)) as the number of arcs $(v_i,v_{i+1})$ such that $\varepsilon(v_i,v_{i+1}) = \uparrow$ for $1 \leq i < k - 1$. Let Simple be the set of all simple paths in $G(\mathcal{N}, \phi) = (V, E, l)$.

A cycle is a finite path $\langle v_1 \ldots v_m \rangle$ such that $m \geq 2$ and $v_1 = v_m$. $\Gamma^w$ is a short path if each simple cycle is traversed at most once along $\Gamma^w$. $\Gamma^w$ is a slim path if each cycle of zero time is traversed at most once along $\Gamma^w$.

**Lemma 3.4.** Given two vertices $v$, $v' \in V$ and $d \in \mathbb{N}$, there is a path from $v$ to $v'$ of time $d$ iff there is a slim path from $v$ to $v'$ of time $d$.

**Proof** follows from the fact that a path which is not slim can be reduced to a slim one by deleting duplicate zero cycles. □

Given a simple path $\Gamma_{v_k}^w = (v_1v_2 \ldots v_k)$ and a finite set $H$ of simple cycles in $G(\mathcal{N}, \phi)$, we call $(\Gamma_{v_k}^w, H)$ a cactus structure iff for each $\Omega \in H$ there exists a finite sequence $\Omega_1, \ldots, \Omega_m$ ($\Omega_i = (v_i^1v_i^2 \ldots v_i^k)$, $1 \leq i \leq m$) of simple cycles in $H$ such that
• $\Omega_1 = \Omega$, and

• for each $1 \leq i < m$ there is $1 \leq j \leq k_{i+1}$ such that $v_j^{i+1} = v_i^i$, and

• for some $1 \leq i \leq k$ $v_i = v_i^m$ holds.

Given a set of nonnegative integers $r_1, \ldots, r_m$, let $gcd(r_1, \ldots, r_m)$ and $lcm(r_1, \ldots, r_m)$ be, respectively, the greatest common divisor and the least common multiple of the nonzero elements in $r_1, \ldots, r_m$.

Given a cactus structure $(\Gamma_v^v, \{\Omega_1, \ldots, \Omega_m\})$. Let us denote $r_i = time(\Omega_i)$ for each $1 \leq i \leq m$. $(\Upsilon, \Psi)$ is an offset-period pair of $\Gamma_v^v$, where

$$\Upsilon = time(\Gamma_v^v) + \sum_{1 \leq i \leq m} r_i + m \cdot lcm(r_1, \ldots, r_m), \Psi = gcd(r_1, \ldots, r_m).$$

Let $Path(\Gamma_v^v, \{\Omega_1, \ldots, \Omega_m\})$ be the set of paths induced by the cactus structure $(\Gamma_v^v, \{\Omega_1, \ldots, \Omega_m\})$.

**Lemma 3.5.** Given $v, v' \in V$ and $d \in N$ such that

$$d \geq \max_{\Gamma_v^v \in Simple} \{\Upsilon \mid (\Upsilon, \Psi) \text{ is an offset-period pair of } \Gamma_v^v\}.$$

**Proof** follows from the Definition of cactus structure and Lemma 3 [11]. □

There exists a path $\Gamma_v^{v'}$ of time $d$ iff there is $i \geq 0$ and offset-period pair $(\Upsilon, \Psi)$ of $\Gamma_v^v$ such that $d = \Upsilon + i \cdot \Psi$.

Given a cactus structure $(\Gamma_v^v, H)$ with the offset-period pair $(\Upsilon, \Psi)$. A short path $\Gamma_v^v$ corresponds to the constraint $\sim c'$ if either there exists a slim path $\tilde{\Gamma}^v \in Path(\Gamma_v^v, H)$ such that $time(\tilde{\Gamma}^v) < \Upsilon$ and $time(\tilde{\Gamma}^v) \sim c$, or there exists $i > 0$ such that $\Upsilon + i \cdot \Psi \sim c$.

4. Labelling algorithm

To decide a time behaviour analysis problem, we need to label a pair consisted of a vertex of $G(N, \phi)$ and a TCTL-formula by a first-order-logic formula (called a condition) with parameter variables as free variables.

Informally, a condition is a constraint on parameter variables on which the given TCTL-formula is true on the states corresponding to the given vertex.

Suppose, we want to analyze $\mathcal{N}$ with respect to $\phi$, i.e. to label the initial vertex $v_0$ with a condition $L_{v_0}^\phi$. We label the vertices of $G(N, \phi)$ with subformulas of $\phi$ or its negation starting from the subformulas of length 1, then of length 2, and so on; then we construct conditions $L_v^\psi$ for all subformulas $\psi$ of $\phi$ and for all vertices of $G(N, \phi)$.
Given a vertex \( v = (m, [\nu]_\phi) \) of \( G(N, \phi) \) and a time predicate \( x \sim \theta \), the notation \([\nu]_\phi(x) \sim \theta\) means \( \dot{\nu}(x) \sim \theta \) for all \( \dot{\nu} \in [\nu]_\phi \). Then \( \tau^v = [\tau][\nu]_\phi(t_1), \ldots, [\nu]_\phi(t_n) \).

A function \( \text{before} : B^T_{\forall \text{ar}} \to B^T_{\forall \text{ar}} \) is inductively defined as follows:

- Case \( \eta = x \sim \theta \). Then \( \text{before}(\eta) = (x \leq \theta \lor x \sim \theta) \);
- Case \( \eta = \eta_1 \land \eta_2 \). Then \( \text{before}(\eta) = \text{before}(\eta_1) \land \text{before}(\eta_2) \).

Given a TCTL-formula \( \psi \), we define a proposition \( a^\psi \) which is true on the given vertex \( v \) if \( v \) is labelled with \( \psi \).

Let \( \psi \) be a subformula of \( \phi \). Assume that the vertices are already labelled with each subformula \( \psi' \) of \( \psi \) and the condition \( L^v_{\psi'} \) is already constructed for each vertex \( v' \in V \). We shall label a vertex \( v \) with a formula \( \psi \) or its negation and construct a condition \( L^v_{\psi} \). Let us consider the structure of \( \psi \).

Case \( \psi \in P \). If \( \psi \in m \), then label \( v \) with \( \psi \), else with \( \neg \psi \). \( L^v_{\psi} = \text{true} \).

Case \( \psi = \neg \psi_1 \). If \( v \) is labelled with \( \psi_1 \), then label it with \( \neg \psi \), else with \( \psi \).

If \( \psi \) is a \( U \)-formula, then \( L^v_{\psi} = \neg L^v_{\psi} \), else \( L^v_{\psi} = \text{true} \).

Case \( \psi = \psi_1 \land \psi_2 \). If \( v \) is labelled with \( \neg \psi_1 \) and with \( \psi_2 \), then label it with \( \psi \), else with \( \neg \psi \).

If \( \psi_1 \) or \( \psi_2 \) is a \( U \)-formula, then \( L^v_{\psi} = L^v_{\psi_1} \land L^v_{\psi_2} \), else \( L^v_{\psi} = \text{true} \).

Case \( \psi = Q \psi_1 \land \psi_2 \), where \( Q \) is either an existential or universal quantifier.

A \( \psi \)-path is a short path \( \Gamma^v \) for which there exists \( i \geq 1 \) such that for all \( j < i \), \( v_j \) is labelled with \( \psi_1 \), \( v_i \) is labelled with \( \psi_2 \) and \( \Gamma_{v_i}^v \) corresponds to the constraint \( ' \sim c' \). The set of such \( v_i \) for the given \( \Gamma^v \) is called \( R(\Gamma^v) \). The set of short paths of \( G(N, \phi) \) starting from \( v \) is denoted by \( F^v(\psi) \). A vertex \( v \) should be labelled with \( \psi \) if there exists a \( \psi \)-path \( \Gamma^v \) of \( G(N, \phi) \) (all \( \Gamma^v \) are \( \psi \)-paths of \( G(N, \phi) \) depending upon \( Q \)).

Given \( \psi \)-path \( \Gamma^v \) and a vertex \( \bar{v} \) from \( \Gamma^v \),

\[
L^v_{\psi}(\bar{v}, \Gamma^v) = L^\psi_0 \land \bigwedge_{v' \in \Gamma^v_v} L^\psi_{v'} \land \bigwedge_{t \in \text{enable}(m(v))} \tau^t(t) \land \bigwedge_{t \in \text{tr}(\Gamma^v_{v'})} (\tau^{\text{in}(t, \Gamma^v_{v'})(t)} \land \ldots)
\]

\[
\land \bigwedge_{t' \in \text{enable}(m(\in(t, \Gamma^v_{v'})))} \text{before}(\tau^{\text{in}(t, \Gamma^v_{v'})(t')}));
\]

\[
L^v_{\psi} = \bigvee_{\Gamma^v \in F^v(\psi)} (\bigvee_{\theta \in R(\Gamma^v)} L^\psi_{(\bar{v}, \Gamma^v)} \rightarrow a^\psi) \text{ if } Q = \exists;
\]

\[
L^v_{\psi} = \bigwedge_{\Gamma^v \in F^v(\psi)} (\bigvee_{\theta \in R(\Gamma^v)} L^\psi_{(\bar{v}, \Gamma^v)} \rightarrow a^\psi) \text{ if } Q = \forall.
\]
As an example, we consider an application of the above algorithm to the parametric time Petri net \( N_1 \) (see Figure 1) and a TCTL-formula \( \phi = \exists true \mathcal{U}_2 (p_4 \land p_5) \), which uniformly means that the transition \( t_5 \) fires not earlier than in time 2. Using the algorithm, we obtain the condition on the parameter variables \( (\theta_1 = \theta_2 = \theta_3 = 1) \lor (0 \leq \theta_2 \leq \theta_3 \leq 2 \land \theta_1 = 2) \).

**Theorem 4.1.** Given a TCTL-formula \( \psi \) with \( K_{N_1, \phi} \geq K_{N_1, \psi} \), a vertex \( v = \langle m, [\nu]_\phi \rangle \) of \( G(N_1, \phi) \) and a parameter valuation \( \chi \) such that \( \langle m, \nu \rangle \in RS(N_1, \chi) \). \( \chi \) satisfies the condition \( L^\psi \) constructed by the above labelling algorithm iff \( \langle m, \nu \rangle \models \psi \).

**Proof.** The proof is conducted by induction on the structure of \( \psi \). The base case and the cases corresponding to the logical connectives follow immediately. We prove that \( \chi \) satisfies \( L^\psi \) iff \( \langle m, \nu \rangle \models \psi \), where \( \psi = \exists \psi_1 \mathcal{U}_{\leq c} \psi_2 \). The other case \( \psi = \forall \psi_1 \mathcal{U}_{\leq c} \psi_2 \) is similar.

(\( \Leftarrow \)) Suppose \( \langle m, \nu \rangle \models \psi \). This means that there is an \( \langle m, \nu \rangle \)-path \( r \) in \( N_1 \times : \langle m, \nu \rangle = \langle m_1, \nu_1 \rangle \overset{\delta_1}{\rightarrow} \langle m_2, \nu_2 \rangle \overset{\delta_2}{\rightarrow} \ldots \) for which \( r \models \phi_1 \mathcal{U}_{\leq c} \phi_2 \). This implies that there exist \( k \) and \( \delta \leq \delta_k \) such that:

1. \( (\delta + time(r, k)) \sim c \);  
2. \( \langle m_k, \nu_k + \delta \rangle \models \phi_2 \);  
3. \( \forall 1 \leq i < k \ \forall \delta'. (\delta' = 0 \lor 0 < \delta' < \delta_i) \langle m_i, \nu_i + \delta' \rangle \models \phi_1 \);  
4. \( \forall 0 \leq \delta' < \delta \ . \langle m_k, \nu_k + \delta' \rangle \models \phi_1 \).  

For a given \( r \) in \( N_1 \times \) we shall construct the corresponding path in \( G(N_1, \phi) \). For each \( i \geq 1 \) we can find a finite path \( \Gamma^\langle m_i, [\nu_i]_\phi \rangle_{m_{i+1}, [\nu_{i+1}]_\phi} \) in \( G(N_1, \phi) \) in the following way. If \( m_{i+1}, \nu_{i+1} \) is obtained from \( m_i, \nu_i \) by passing \( \delta_i \), then \( \Gamma^\langle m_i, [\nu_i]_\phi \rangle_{m_{i+1}, [\nu_{i+1}]_\phi} = \{ \langle m_i, [\nu_i]_\phi \rangle, \langle m_i, [\nu_i^0]_\phi \rangle, \langle m_i, [\nu_i^1]_\phi \rangle, \ldots, \langle m_i, [\nu_i^l]_\phi \rangle \} \) for some \( j_i \) (where \( m_i, [\nu_i^l]_\phi = succ \langle m_i, [\nu_i]_\phi \rangle \) for each \( 0 \leq l < j_i \)). Otherwise \( \Gamma^\langle m_i, [\nu_i]_\phi \rangle_{m_{i+1}, [\nu_{i+1}]_\phi} = \{ \langle m_i, [\nu_i]_\phi \rangle, \langle m_{i+1}, [\nu_{i+1}]_\phi \rangle \} \) where \( \langle m_{i+1}, [\nu_{i+1}]_\phi \rangle = succ \langle m_i, [\nu_i]_\phi \rangle \) for some \( t \in fireable(m_i, [\nu_i]_\phi) \). Note that for all \( 0 \leq \delta' < \delta_i \) there exists \( 0 \leq l < j_i \) such that \( \nu_i + \delta' \sim c \nu_i^l \).

Let \( \Gamma^\langle m, [\nu]_\phi \rangle \) be a path in \( G(N_1, \phi) \) obtained as a concatenation of paths \( \Gamma^\langle m_i, [\nu_i]_\phi \rangle_{m_{i+1}, [\nu_{i+1}]_\phi} \).

Now we show that \( \Gamma^\langle m, [\nu]_\phi \rangle \) satisfies the formula \( \phi_1 \mathcal{U}(\phi_2 \land p_{\leq c}) \), i.e. there are \( k \) and \( l < j_k \) such that: a) \( \Gamma^\langle m_k, [\nu_k]_\phi \rangle_{m_{k+1}, [\nu_{k+1}]_\phi} \) corresponds to the constraint \( c \sim \nu \); b) \( \langle m_k, [\nu_k]_\phi \rangle \) is labelled with \( \phi_2 \) and c) \( \langle m_{k'}, [\nu_{k'}]_\phi \rangle \) appearing in \( \Gamma^\langle m, [\nu]_\phi \rangle \) before \( \langle m_k, [\nu_k]_\phi \rangle \) is labelled with \( \phi_1 \). Using the construction of \( \Gamma^\langle m_k, [\nu_k]_\phi \rangle_{m_{k+1}, [\nu_{k+1}]_\phi} \), we obtain that there is \( l \leq j_k \) such that \( \nu_k + \delta \sim \nu_k^l \).
a) Since \(\text{time}(r, k) = \text{time}(\Gamma^{(m, [\nu])}_{\phi})\), we have \(\delta + (\Gamma^{(m, [\nu])}_{\phi}) \sim c\). Hence \(\text{time}(\Gamma^{(m, [\nu])}_{\phi}) \sim c\). For a cactus structure \((\Gamma^{(m, [\nu])}_{\phi}, H)\), either \(\text{time}(\Gamma^{(m, [\nu])}_{\phi}) \sim \Psi\), or by Lemma 3.5 there exists \(i > 0\) such that \(\Psi + i \sim c\). Thus \(\Gamma^{(m, [\nu])}_{\phi}\) corresponds to the constraint \(\sim c\).

b) Using the fact that \(\langle m, [\nu]_k + \delta \rangle \models \phi_2\), we have \(\langle m, [\nu]_k \rangle \models \phi_2\). Then by induction hypothesis \(\langle m, [\nu]_k \rangle\) is labelled with \(\phi_2\).

c) Let us consider \(\langle m', [\nu']_{k'} \rangle\) appearing in \(S\) before \(\langle m, [\nu]_k \rangle\) (i.e. either \(k' < k\), or \(k' = k\) and \(l' < l\)). By the construction of \(\Gamma^{(m', [\nu']_{k'})}_{\phi}\), there is \(\delta' \leq \delta_{k'}\) such that \(\nu'_{k'} \sim \nu'_{k'}\). Then \(\langle m', [\nu']_{k'} \rangle \models \phi_1\) implies \(\langle m', [\nu']_{k'} \rangle \models \phi_1\) by Lemma 3.1. Hence, by induction hypothesis \(\langle m', [\nu']_{k'} \rangle\) is labelled with \(\phi_1\).

Note that \(\Gamma^{(m, [\nu])}_{\phi}\) is a slim path down to the progress condition of \(r\). Hence, \(\Gamma^{(m, [\nu])}_{\phi}\) is a \(\psi\)-path. We show that \(\chi\) satisfies the condition \(L^\psi_{\langle m, [\nu]_k \rangle}_{\phi}(\langle m, [\nu']_{k'} \rangle, \Gamma^{(m, [\nu])}_{\phi})\). Really, by induction hypothesis, \(\chi\) satisfies the conditions \(L^\psi_{\langle m, [\nu]_k \rangle}_{\phi}\) and \(L^\psi_{\langle m, [\nu]_k \rangle}_{\phi}\) for all \(\langle m, [\nu]_k \rangle\) from \(\Gamma^{(m, [\nu])}_{\phi}\). And using the firing rules in \(N\), we obtain that \(\chi\) satisfies the condition

\[
\bigwedge_{t \in \text{enable}(\langle m, [\nu]_k \rangle)} \tau^t(\Phi) \wedge \bigwedge_{t' \in \text{tr}(\langle m', [\nu']_{k'} \rangle)} \tau^{t'}(\Phi) \wedge \bigwedge_{t' \in \text{enable}(\langle m, [\nu']_{k'} \rangle)} \text{before}(\tau^{t'}(\Phi))
\]

Thus \(\chi\) satisfies \(I^\psi_{\langle m, [\nu]_k \rangle}_{\phi}\).

(\(\Rightarrow\)) Suppose that \(\chi\) satisfies \(I^\psi_{\langle m, [\nu]_k \rangle}_{\phi}\). Then, according to the above algorithm, there is a \(\psi\)-path \(\Gamma^{(m, [\nu])}_{\phi}\): \(\{\langle m, [\nu]_i \rangle = \langle m_1, [\nu]_1 \rangle, \langle m_2, [\nu]_2 \rangle, \ldots \}\) in \(G(N, \phi)\), i.e. there is \(n \in N\) such that \(\langle m, [\nu]_i \rangle\) is labelled with \(\phi_1\) for each \(1 \leq i < n\), \(\langle m, [\nu]_n \rangle\) is labelled with \(\phi_2\) and \(\Gamma^{(m, [\nu])}_{\phi}\) corresponds to the constraint \(\sim c\). We shall construct the corresponding \(\langle m, [\nu] \rangle\)-path \(r\) in \(N\): \(\langle m, [\nu] \rangle = \langle m_1, [\nu]_1 \rangle \sim \langle m_2, [\nu]_2 \rangle \sim \ldots \) such that for each \(i \geq 1\) there is \(\delta_i \in \mathbb{R}^+\) for which \(\langle m_i, [\nu]_i \rangle \sim \langle m_{i+1}, [\nu]_{i+1} \rangle\) and \(\nu_i + \delta_i \in [\nu_{i+1}]_\phi\). Note that, by the construction of \(L^\psi_{\langle m, [\nu]_k \rangle}_{\phi}\), \(r\) is a path in \(N\). Then, similar to the first part of the proof, we can show that \(\langle m, [\nu] \rangle \models \psi\).

**Theorem 4.2.** Given a TCTL-formula \(\phi\), there is a procedure of analysis of \(N\) with respect to \(\phi\) bounded by:

\[
O(|\phi| \cdot 2|T| \cdot 2^{P} + 2|T| \cdot (K_{\phi} + 1)^{|T|}).
\]

**Proof.** From the definition of the region graph and from Lemma 3.3, it follows that \(|V| = O(|T| \cdot 2^{P} + 2|T| \cdot (K_{\phi} + 1)^{|T|})\). For a vertex
$v$ of $G(N, \phi)$ there are at most $|T|$ output edges representing the firing of transitions and an edge representing time passage. Hence $|E| = O(|V| \cdot 2^{|P| \cdot 2^{|T|}} \cdot (K_{N, \phi} + 1)^{|T|})$. The region graph $G(N, \phi)$ can be constructed in time $O(|V| + |E| + |G|)$ (written $|G|$). When we construct a condition for a given formula $\phi$ and a vertex $v$ of $G(N, \phi)$, we need to consider all short paths in $G(N, \phi)$ beginning from this vertex. The number of simple cycles is $|G|!$. So, the number of short paths is $(|G|!)^2$. The time of considering a short path is $|G| + (|G| + 1)!$. Then, to construct a condition for $L_{\phi}^v$, we need the time $O(|\phi| \cdot (|G|!)^3)$. The complexity follows from the bounds on the size of $V$ and $E$.

\[\square\]

**Conclusion**

In this paper we introduce a notion of a parametric time Petri net and propose a TCTL-based algorithm for behaviour analysis of it, which provides a technique for adjustment of timing limitations with respect to the system properties.

In future we suppose to optimize the proposed algorithm and to construct an algorithm for behaviour analysis of parametric time Petri nets in terms of parametric TCTL [11], an extension of TCTL obtained by using parameter variables in $U$-operators.

**References**


