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Two-dimensional inverse problem for wave equation, the uniqueness in regularizating statement

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Inverse problem for wave equation with instant, lagging source on segment of boundary $y = 0$ is considered. The problem of determination of sound velocity, smooth depending of both spacing coordinates by response of medium on the same segment, is posed. In regularizating statement the uniqueness of problem is proved with assumption of eikonal regularity. Regularization consists in addition to wave operator the operator of higher degree with small parameter.

Let $u(x, y, t)$ be the solution to the initial boundary problem for the wave equation

$$\Delta u - u_{tt} = 0, \tag{1}$$

$$u_y|_{y=0} = -\psi_\Gamma \delta(t - \tau_0(x)), \quad u|_{t<t_0} = 0. \tag{2}$$

Here $\Delta = c^2(x,y)(\partial^2/\partial x^2 + \partial^2/\partial y^2)$ – the Laplace–Beltrami operator, generated by the metric tensor $g_{ij} = \delta_{ij}/c^2$; $c(x, y)$ is the smooth positive function (sound velocity) (for simlicity sake smoothness is concidered to be the differentiability of class C^∞); $\delta(\cdot)$ – the Dirac function; ψ_Γ – the characteristic function of segment $\Gamma = \{(x, y) : y = 0, |x| \le a\}$ and function τ_0 is defined by the equality

$$\tau_0(x) = T(x^2 - a^2), \qquad |x| \le a, \quad T > 0.$$

Let $c_0 = c|_\Gamma$ be known function. Choose numbers a, T so that function $n_0^2(x) - 4T^2x^2$ is positive for all $x, |x| \le a$, $n = 1/c$, $n_0 = n|_\Gamma$, and denote $\lambda(x) = \sqrt{n_0^2(x) - \tau_0'^2(x)}$. Consider the eikonal $\tau(x, y)$:

$$|\nabla\tau|^2 \equiv c^2(\tau_x^2 + \tau_y^2) = 1, \quad \tau|_\Gamma = \tau_0.$$

It describes the front of falling wave in the case of retarded source (2). We consider the values τ only up to $\tau = 0$. Emit the geodesic $\gamma(x', \tau)$, $\tau_0(x') \le \tau \le 0$ from each point x', $|x'| \le a$ in the direction

$$\xi(x') = c_0^2(x')(\tau_0'(x'), \lambda(x')).$$

Let $D' = \{(x', \tau) : |x'| < a, \ \tau_0 \le \tau < 0\}$. Obviously, the mapping $\gamma : (x', \tau) \to \gamma(x', \tau)$ transforms D' to the domain D, bounded by front

$\tau = 0$ and segment Γ, $D = \{(x, y) : |x| < a, \tau_0(x) \leq \tau(x, y) < 0\}$. (Note, the edges of Γ belong to the front $\tau = 0$.) Assume that γ is diffeomorphism D' on D (calculating the Jacobian of that mapping on $\Gamma' = \{(x', \tau) : |x'| < a, \tau = \tau_0(x')\}$, it may be shown that in the small it is always true). Besides that, the function $\tau(x, y)$ will be smooth in the domain D. Henceforth we will use the coordinates (x', τ) and all functions of the kind $f \circ \gamma$ will be marked by prime.

Consider the problem of the velocity determination in the domain D by the function

$$u_0 = u|_{G_0}, \quad G_0 = \{(x, t) : \ |x| \leq a, \ \tau_0(x) \leq t \leq -\tau_0(x)\}.$$

In the similar statement the uniqueness and the local solvability was proved in [1] in the case of the analyticity of c for definite direction.

As in [1], inverse problem providing regularity of the function $\tau(x, y)$ after selecting of the singularity reduced to the Cauchy problem with respect to the wave function $u(x, y, t)$ and the eikonal

$$\Delta u - u_{tt} = 0 \ \text{ in } \ G, \tag{3}$$

$$|\nabla \tau|^2 = 1, \tag{4}$$

$$2(\nabla \tau \nabla \bar{u}) + \bar{u} \Delta \tau = 0, \quad \bar{u}(x, y) = u(x, y, \tau(x, y)), \tag{5}$$

$$u|_{G_0} = u_0, \quad u_y|_{G_0} = 0, \tag{6}$$

$$\tau|_\Gamma = \tau_0, \quad \tau_y|_\Gamma = \lambda, \tag{7}$$

where $G = \{(x, y, t) : (x, y) \in D, \ \tau(x, y) < t < -\tau(x, y)\}$. Moreover, the coordination condition is assumed

$$\lambda \bar{u}_0 = 1, \quad \bar{u}_0 = \bar{u}|_\Gamma. \tag{8}$$

The method of singularity separation in similar problems is well-known [1, 2] and we do not dwell upon it's basis. Note that, formally, equations (3), (4), (5) initial data (6), (7) and coordination condition are easily obtained by the substitution of the function $u(x, y, t)$ to $\theta(t - \tau(x, y))u(x, y, t)$ in (1), (2), where $\theta(\cdot)$ is the Heaviside function.

Since we consider the regularizating wave equation, we need additional derivatives of wave function on G_0. Assume, that equation (3) and the equation following from it by derivation on y hold up to Γ. Then obviously, values of derivatives u_{yy} and u_{yyy} on Γ are calculated. Actually, $u_{yy}|_\Gamma$ is calculated from (3) directly, further, by virtue of (6), (7) we have

$$\bar{u}_y(x, 0) - \lambda(x) u_{0t}(x, \tau_0(x)) = 0,$$

and then the values $\tau_{yy}|_\Gamma$ are found from (5). Differentiating eikonal equation with respect to y, find $c_y|_\Gamma$ and lastly, differentiating wave equation calculate $u_{yyy}|_\Gamma$.

The main result of this paper is the uniqueness theorem for inverse problem in regularizating statement, when differential operator of higher degree with small parameter $\alpha\Lambda^2$ is added to wave equation. Define Λ by the equality

$$\Lambda = \frac{1}{\bar{u}^2}\delta\Big(\frac{S}{\bar{u}^2}\Big), \quad S = \nabla - \nabla\tau(\nabla\tau, \nabla),$$

where scalar product and divergence operator δ are acting with respect to the metric tensor g_{ij}. Note, since by definition, $S\tau = 0$, and $(\nabla\tau, S) = 0$, then the operator Λ does not conclude derivatives with respect to the normal of the front.

Consider regularizing statement of the inverse problem

$$\Delta u - u_{tt} + \alpha\Lambda^2 = 0 \text{ in } G, \tag{9}$$

$$|\nabla\tau^2| = 1, \tag{10}$$

$$2(\nabla\bar{u}, \nabla\tau) + \bar{u}\Delta\tau = 0, \tag{11}$$

$$u|_{G_0} = u_0, \quad u_y|_{G_0} = 0, \quad u_{yy}|_{G_0} = u_{02}, \quad u_{yyy}|_{G_0} = u_{03}, \tag{12}$$

$$\tau|_\Gamma = \tau_0, \quad \tau_y|_\Gamma = \lambda. \tag{13}$$

Theorem. *Let γ be diffeomorphism $\bar{D}'$ on $\bar{D}$. Then problem* (9)–(13) *has not more than one smooth solution for any positive α.*

Proof. Proceed to the coordinates (x', τ) and consider the vector field $X = \partial\gamma(x', \tau)/\partial x'$. Show that it is orthogonal to the geodesic $\gamma(x', \tau)$, i.e., $\langle X, \gamma_\tau\rangle = 0$, where $\langle X, Y\rangle = n^2(\gamma)(X^1Y^1 + X^2Y^2)$. Actually, by direct calculations it is easy to verify that X is the Jacobian field, i.e.,

$$\frac{\nabla^2}{\partial\tau^2}X^i + R^i_{jkl}(\gamma)\gamma^j_\tau\gamma^i_\tau X^l = 0,$$

where

$$\frac{\nabla}{\partial\tau}X^i = X^i_\tau + \Gamma^i_{jk}(\gamma)\gamma^j_\tau X^k,$$

R^i_{jkl} and Γ^i_{jk} are the tensors of the Riemann curvature and coefficients of connectedness of the tensor g_{ij}. Moreover, it is easy to verify that for $\tau = \tau_0(x)$

$$\langle X, \xi\rangle = \Big\langle \frac{\nabla X}{\partial\tau}, \xi\Big\rangle = 0.$$

Hence it follows [3] that everywhere $\langle X, \gamma_\tau\rangle = 0$. In the coordinates (x', τ) the metric $c(x, y)\sqrt{(dx)^2 + (dy)^2}$ transforms to

$$\sqrt{\langle X, X\rangle^2(dx')^2 + (d\tau)^2}$$

with the covariant metric tensor

$$(g'_{ij}) = \begin{pmatrix} \langle X, X\rangle & 0 \\ 0 & 1 \end{pmatrix},$$

and the contravariant

$$(g'^{ij}) = \begin{pmatrix} 1/\langle X, X\rangle & 0 \\ 0 & 1 \end{pmatrix}.$$

Equations (9), (11) transform to

$$u'_{\tau\tau} - u'_{tt} + \frac{1}{g'}u'_{x'x'} + \alpha\Lambda^2 u' = -\frac{g'_\tau}{2g'}u'_\tau + \frac{g'_{x'}}{2g'^2}u'_{x'},$$

$$(x', \tau, t) \in G' = \{(x', \tau, t) : (x', \tau) \in D', \tau < t < -\tau\},$$

$$2\bar{u}'_\tau + (\ln\sqrt{g'})_\tau \bar{u}' = 0,$$

where $g' = \langle X, X\rangle$ is the Jacobian of the mapping γ. Herefrom and from (8) it follows that the functions $\bar{u}'$ connect with g' by the equality

$$g' = \frac{1}{\bar{u}'^4\lambda^2}. \tag{14}$$

We show that

$$\Lambda u' = \lambda(\lambda u'_{x'})_{x'}. \tag{15}$$

Really, since the operators S and Λ are invariantly defined, then

$$S_1 = \partial/\partial x', \quad S_2 \equiv 0,$$

$$\Lambda u' = \frac{1}{\bar{u}'^2\sqrt{g'}}\Big(\Big(\frac{\sqrt{g'}}{\bar{u}'^2}S^1 u'\Big)_{x'} + \Big(\frac{\sqrt{g'}}{\bar{u}'^2}S^2 u'\Big)_\tau\Big),$$

where

$$S^1 = g'^{11}S_1 + g'^{12}S_2, \quad S^2 = g'^{21}S_1 + g'^{22}S_2.$$

By virtue of (14) hence it follows (15).

Replacing g' by $\bar{u}'$, we obtain the one non-linear, non-local equation with respect to the function u'. Let there exist two smooth solutions u'_1, u'_2, corresponding to the same initial data, and the functions $\bar{u}'_1$, $\bar{u}'_2$ are positive in the closed domain $\bar{D}'$ (otherwise, as it follws from (14), the corresponding Jacobians g'_1, g'_2 would have the singularities). Then for its difference $w = u'_1 - u'_2$, the inequality holds

$$|Pw|^2 \le C(\bar{w}^2 + \bar{w}^2_{x'} + \bar{w}^2_\tau + w^2_{x'} + w^2_\tau), \quad (x', \tau, t) \in G', \tag{16}$$

where

$$Pw \equiv w_{\tau\tau} + \Big(\frac{1}{g'}w_{x'}\Big)_{x'} - w_{tt} + \alpha\Lambda^2 w, \quad g' = g_1', \quad \bar{w}(x',\tau) = w(x',\tau,\tau),$$

and C is the constant, depending on C^1-norm of the functions u_1', u_2', $\bar{u}_1'$, $\bar{u}_2'$. Introduce the function

$$v(x',\tau,t) \doteq w(x',\tau,t)\mathrm{e}^{-s\tau},$$

and consider the inequality, following from (16)

$$\begin{aligned} &4sv_\tau\Big(\Big(v_{\tau\tau} - v_{tt} + \Big(\frac{1}{g'}v_{x'}\Big)_{x'} + \alpha\Lambda^2 v + s^2 v\Big)\Big) \\ &= 4sv_\tau(\mathrm{e}^{-s\tau}P(\mathrm{e}^{s\tau}v) - 2sv_\tau) \le \mathrm{e}^{-2s\tau}(P(\mathrm{e}^{s\tau}v))^2 \\ &\le C(\bar{v}_{x'}^2 + 2\bar{v}_\tau^2 + (2s^2+1)\bar{v}^2 + v_{x'}^2 + v_\tau^2 + 2v_\tau^2 + 2s^2v^2). \end{aligned} \tag{17}$$

With the help of the obvious identity

$$\frac{2}{\lambda}v_\tau\Lambda^2 v = 2(\lambda v_\tau(\Lambda v)_{x'} - \lambda v_{x'\tau}\Lambda v)_{x'} + \frac{1}{\lambda}((\Lambda v)^2)_\tau, \tag{18}$$

the left-hand side of (17) after dividing on λ easily transforms to

$$\begin{aligned} &2s\frac{1}{\lambda}\Big(v_\tau^2 + v_t^2 - \frac{1}{g'}v_{x'}^2 + s^2v^2 + \alpha(\Lambda v)^2\Big)_\tau - 4s\Big(\frac{1}{\lambda}v_\tau v_t\Big)_t + \\ &4s\Big(\frac{v_{x'}v_\tau}{\lambda g'} + \alpha\lambda(v_\tau(\Lambda v)_{x'} - v_{x'\tau}\Lambda v)\Big)_{x'} + \frac{2s}{\lambda}\Big(\frac{1}{g'}\Big)_\tau v_{x'}^2 - \frac{4s}{g'}v_{x'}v_\tau\Big(\frac{1}{\lambda}\Big)_{x'}. \end{aligned} \tag{19}$$

Denote by $C_1 > 0$ the strict constant of the equation

$$\Big|\Big(\frac{1}{g'}\Big)_\tau\Big|\xi^2 + \Big|\frac{\lambda_{x'}}{\lambda g'}\Big|(\xi^2+\eta^2) \le C_1(\xi^2+\eta^2).$$

Then from (17), (18) we obtain

$$\operatorname{div} V \le 2\frac{C_1 s + C}{\lambda}(v_{x'}^2 + v_\tau^2) + \frac{C}{\lambda}(\bar{v}_{x'}^2 + 2\bar{v}_\tau^2 + (2s^2+1)\bar{v}^2 + 2s^2v^2).$$

Here the divergence terms in (19) are denoted by $\operatorname{div} V$. Multiplying the latter inequality on $\exp(-C_2\tau)$, $C_2 = C + C_1$ we obtain

$$\begin{aligned} &\operatorname{div}(\mathrm{e}^{-C_2\tau}V) + \frac{C_2 s}{\lambda}\mathrm{e}^{-C_2\tau}\Big(s^2v^2 + 2\alpha(\Lambda v)^2 - 2\frac{1+g'}{g'}v_{x'}^2\Big) + \\ &\frac{C_2 s^2}{\lambda}\mathrm{e}^{-C_2\tau}\Big(s - \frac{2C}{C_2}\Big)v^2 \le \frac{C}{\lambda}\mathrm{e}^{-C_2\tau}(\bar{v}_{x'}^2 + 2\bar{v}_\tau^2 + (2s^2+1)\bar{v}^2). \end{aligned} \tag{20}$$

It is easy to see that the second term in the left-hand side for sufficiently large s within divergence term is nonnegative. Actually

$$\frac{1}{\lambda}(s^2v^2+2\alpha(\Lambda v)^2) \equiv \frac{1}{\lambda}(sv+\sqrt{2\alpha}\Lambda v)^2-2s\sqrt{2\alpha}(\lambda v v_{x'})_{x'}+2s\lambda\sqrt{2\alpha}v_{x'}^2, \quad (21)$$

and consequently if

$$s > \frac{1}{2\alpha}\left\|\frac{1+g'}{\lambda^2 g'}\right\|_{C^2(D')},$$

then from (20) follows the inequality

$$2s\Big(\frac{e^{-C_2\tau}}{\lambda}\Big(v_\tau^2+v_t^2-\frac{1}{g'}v_{x'}^2+s^2v^2+\alpha(\Lambda v)^2\Big)\Big)_\tau - 4se^{-C_2\tau}\Big(\frac{1}{\lambda}v_\tau v_t\Big)_t +$$
$$4se^{-C_2\tau}\Big(\frac{v_\tau v_{x'}}{\lambda g'}+\alpha\lambda(v_\tau(\Lambda v)_{x'}-v_{x'\tau}\Lambda v)-2C_2s^2\sqrt{2\alpha}\lambda v v_{x'}\Big)_{x'}+$$
$$\frac{C_2s^2}{\lambda}e^{-C_2\tau}\Big(s-\frac{2C}{C_2}\Big)v^2 \le \frac{C}{\lambda}e^{-C_2\tau}(\bar{v}_{x'}^2+2\bar{v}_\tau^2+(2s^2+1)\bar{v}^2).$$

Integrate the obtained inequality on the domain G'. By virtue of the zero initial data, in the left-hand side after integrating of the divergence terms there remain only integrals by characteristics $t=\tau$, $t=-\tau$

$$s\int_{D'}\frac{e^{-C_2\tau}}{\lambda}\Big(\tilde{v}_\tau^2-\frac{1}{g'}\tilde{v}_{x'}^2+s^2\tilde{v}^2+\alpha(\Lambda\tilde{v})^2\Big)dx'd\tau+$$
$$s\int_{D'}\frac{e^{-C_2\tau}}{\lambda}\Big(\bar{v}_\tau^2-\frac{1}{g'}\bar{v}_{x'}^2+s^2\bar{v}^2+\alpha(\Lambda\bar{v})^2\Big)dx'd\tau+$$
$$s^2C_2\Big(s-\frac{2C}{C_2}\Big)\int_{G'}\frac{e^{-C_2\tau}}{\lambda}v^2dx'd\tau$$
$$\le -2C\int_{D'}\tau\frac{e^{-C_2\tau}}{\lambda}(\bar{v}_{x'}^2+2\bar{v}_\tau^2+(2s^2+1)\bar{v}^2)dx'd\tau,$$

where $\tilde{v}(x',\tau)=v(x',\tau-\tau)$. Taking advantage of (21), as below, it is easy to see that for sufficiently large s hence it follows $\bar{v}=\tilde{v}=v=0$. Thus, the function $\bar{u}'(x',\tau)$, and consequently $g'(x',\tau)$, $(x',\tau)\in D'$ are determinated unequaly.

Let us find the equations for the geodesic. In order to do it, note that the coordinate functions x and y satisfy the equations $\Delta x=\Delta y=0$. But because of invariance of the Laplace–Beltrami operator the functions $\gamma^i(x',\tau)$, $i=1,2$ satisfy the equations

$$\Delta\gamma^i \equiv \frac{1}{\sqrt{g'}}\Big(\sqrt{g'}\gamma_\tau^i\Big)+\frac{1}{\sqrt{g'}}\Big(\frac{1}{\sqrt{g'}}\gamma_{x'}^i\Big)=0, \quad (x',\tau)\in D'.$$

and the uniqueness of the geodesic follows from the well-known results on uniqueness of the solution of the Cauchy problem for the elliptic equation (see, for example [4]). The geodesic is fined, the domain $D = \gamma D'$, the "wave function" $u, u' = u \circ \gamma$ and the velocity

$$c \circ \gamma = \sqrt{(\gamma_\tau^1)^2 + (\gamma_\tau^2)^2}$$

are determined. □

References

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