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On existence of optimal double-loop computer networks*

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A class of two-dimensional regular graphs called circulants and its special case of the double-loop networks are considered. Such graphs provide a practical interest as reliable interconnection networks for the multimodule supercomputer systems. A solution to the problem of determining the best double-loop networks with the minimum of a diameter and mean distance for the structures of computer systems is considered. A new method of geometrical representation (visualization) of the optimal circulants at the plane and connected with it complete design of transformations and movements generating the optimal graphs are obtained. Some new classes of analytical representation of optimal circulants are proposed. New results on solution of a problem of existence for the optimal two-dimensional loop networks are presented.

1. Introduction

In this work we consider an important class of regular graphs known as circulant networks [1–7, 9, 12]. Particular cases of these graphs are realized as interconnection networks in ICL DAP, ILLIAC IV, MPP, CRAY T3D, Intel Paragon etc.

In general case a circulant is defined as the network $G(N; s_1, s_2, \dots, s_n)$ with N nodes, labeled as $0, 1, 2, \dots, N-1$, having $i \pm s_1, i \pm s_2, \dots, i \pm s_n \pmod N$ nodes adjacent to each node i. The sequence $S = (s_i)$ $(0 < s_1 < s_2 < \dots < s_n < (N+1)/2)$ is a sequence of the generators of the finite Abelian automorphism group associated to a graph. The degree $2n$ (n is dimension) of a node in an undirected graph G is the number of edges incident to it. In what follows we will apply and distinguish a representation of self graph $G(N; S)$ and its description $\{N; S\}$.

The graph $G(N; 1, s_2, \dots, s_n)$ when $s_1 = 1$ is a particular case of the circulants. Such circulants known as loop networks have been studied in [3, 4, 6–14, 16]. The synthesis of optimal circulants is a fundamental problem of graph theoretical optimization representing the generalization of the (d, k)-graph problem. This problem is in a search for a graph with the minimum of a diameter and a mean distance among all circulants having N nodes

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and dimension n. Let this set be $T(N,n)$. The diameter of $G \in T(N,n)$ is defined by $d = \max_{ij} d_{ij}$, where d_{ij} is the length of shortest path from a node i to a node j. The average distance of G is defined as $\bar{d} = \frac{1}{N(N-1)} \sum_{ij} d_{ij}$. Let $d(N) = \min_S \{d(G(N;S))\}$.

Let for any graph $G \in T(N,n)$ $K_{n,m}$ denote the number of nodes to be attained from the node 0 by using at most m generators, and let $K^*_{n,m}$ be the upper bound for $K_{n,m}$. Let $L_{n,m} = K_{n,m} - K_{n,m-1}$, $L^*_{n,m}$ be the upper bound for $L_{n,m}$ on set $T(N,n)$. The values of $K^*_{n,m}$, $L^*_{n,m}$ for any n,m were determined by Korneev [15], Wong and Don Coppersmith [7]. For $n = 2$ $K^*_{2,m} = 2m^2 + 2m + 1$, $L^*_{2,m} = 4m$.

The graphs achieving the upper bounds are called optimal, namely, a graph $G \in T(N,n)$ is optimal, if $L_{n,m} = L^*_{n,m}$ for any $0 \le m \le d^* - 1$ and $L_{n,d^*} = N - K^*_{n,d^*-1}$, where an optimal diameter d^* is given from the correlation $K^*_{n,d^*-1} < N \le K^*_{n,d^*}$. The optimal diameter $d^* = \text{ulb}(N)$ is the exact lower bound of $d(N)$. As it is shown [15], the optimal graph has a minimum d and $\bar{d}$, and a maximum of reliability and connectivity among all graphs from $T(N,n)$. Note that a term "optimal" is used in the literature in different senses. For example, in [8–11] a graph is optimal if $d(G) = d(N)$ and tight optimal if $d(G) = \text{ulb}(N)$. The term "optimal" used here is more stronger because it implies also the minimum of mean distance, namely, the coincidence of mean distance with its exact lower bound.

The diameters of optimal circulants are computed from the expression for $K^*_{n,m}$. The general dependence is in that a diameter is proportional to $\sqrt[n]{N}$. Wong and Coppersmith [7] gave the lower bound for $d(N)$ equal $(\sqrt{2N}-3)/2$ in the case of $n = 2$. The exact lower bound for $d(N)$ in the case when $n = 2$ should be $\text{ulb}(N) = \lceil (\sqrt{2N-1} - 1)/2 \rceil$, where $\lceil x \rceil$ denotes the smallest integer greater than x or equal to it. This lower bound was pointed out by Monakhova [6], Boesch and Wang [2] and Bermond et al. [4].

For circulant graphs with $n = 2$ Monakhova [6], Boesch and Wang [2], Bermond et al. [4], Beivide et al. [5] showed that for any $N > 4$ exists the optimal graph $G(N; s_1, s_2)$ for the values of $s_1 = \text{ulb}(N)$, $s_2 = \text{ulb}(N) + 1$. The exact formulation of this result is the following [6].

Theorem 1. *For any natural $N > 4$ the optimal circulant $G(N; s_1, s_2)$ exists and has a description in the form*

$$\{N; s, s+1\}, \quad s = \lfloor (\sqrt{2N-1} - 1)/2 \rfloor,$$

where $\lfloor x \rfloor$ is the nearest integer to x.

The problem of finding of the optimal two-dimensional circulants with $s_1 = 1$ or double-loop networks is more difficult because such graphs exist not for all values of N. For example, Monakhova [13] and Bermond et al. [4] proved that for $N = 2t^2+2t+1$, $d(N; 1, 2t+1) = \text{ulb}(N)$, but for $N = 2t^2+2t$

Du et al. [8] and Tzviely [10] showed that $d(N) = \text{ulb}(N) + 1$. Du, Hsu, Li and Xu [8] obtained new classes of values of N for which double-loop networks can be found, that achieve lower bound $\text{ulb}(N)$. In addition to these 16 infinite classes of values N, Monakhova [3] indicated some new such classes.

In [10] Tzvieli has identified several optimal families of networks, each of which intersects each $R[d] = \{N_{d-1} + 1, \ldots, N_d\}$, where $N_d = 2d^2 + 2d + 1$, in a set of cardinality $O(\sqrt{d})$. Results in [10] also include bounds of optimal generators and an algorithm to compute those generators whenever they exist. Though this gives a much wider coverage than the classes defined in [8], many values of N remain yet to be classified.

In [3, 11] some necessary and sufficient conditions of existence of the optimal double-loop networks were pointed out. Bermond and Tzvieli [11], Monakhova [3], Mukhopadhyaya and Sinha [9] determined new dense infinite families of values of N, that are optimal. These families cover almost all elements of $R[d]$ except one or two values, if d or $d + 1$ is prime [3, 11], and cover 92% of all values of N up to 10^6 [11].

In these works the following problem remained unsolved for further investigation: obtain the necessary and sufficient conditions of the existence of two-dimensional optimal loop networks and classify those N's for which the tight optimal double-loop networks can be found.

2. Equivalence classes of optimal circulants

Henceforward we will use the knowledge how one may get other descriptions of a circulant, if one of its descriptions is known [3].

Let us have some description of the circulant graph $G(N; S)$. Multiply all s_i by an element t of leaden system of deductions (mod N). As a new s'_i we take the residues of the division ts_i to N, if they are less than $[N/2]$ or take the additions of these residues to N, if they are greater than $[N/2]$. The given transformation transferring all s_i into s'_i is called equivalent transformation and the relation between sets S and S' and also between the graphs $G(N; S)$ and $G(N; S')$ – the relation of equivalence. If now t runs the leaden system of deductions (mod N), then all graphs $G(N; S')$ given from $G(N; S)$ form the equivalence class. The equivalence of graphs $G(N; S)$ and $G(N; S')$ has, as a consequence, their isomorphism. The reverse property, generally, does not hold. For circulants with the prime number of nodes the notions of equivalence and isomorphism coincide.

Taking into consideration the above-said one may determine the value of s for the optimal double-loop networks $G(N; 1, s)$ using the description of a graph from Theorem 1. For example, the optimal graph with $N = 36$ has $\text{ulb}(N) = 4$ and $s_1 = 4$, $s_2 = 5$. The leaden system of deductions (mod 36) is 1, 5, 7, 11, 13, 17. Transferring given generators 4, 5 into s_1', s_2' gives a new equivalent optimal graph $G(36;1,8)$. Under realization of equivalent transformations we use the next result from [13]. The abbreviation $\gcd(a,b)$ denotes the greatest common divisor of the integers a and b.

Lemma 1. *Let N, a and $s < (N+1)/2$ be the natural numbers, a/N and $\gcd(N,s) = a$. Then an element q of the leaden system of deductions* (mod N) *exists such that $sq \equiv a \pmod N$.*

But for some N's this method cannot be used because the optimal circulant graphs with description in the form of $\{N;1,s\}$ do not exist. Let $E(N)$ denote the number of equivalence classes of optimal two-dimensional circulants for given N. In [3] two open questions for further investigation were offered: 1) obtain $E(N)$ for any N; 2) classify all N's for which $E(N) = 1$. A solution to these questions gives the key to a problem of existence of optimal double-loop networks.

3. Geometrical visualization of the optimization problem for two-dimensional circulants

Following [10] we use the notations: $N_d = 2d^2 + 2d + 1$, $R[d] = \{N_{d-1}+1, \ldots, N_d\}$, $d > 0$. Thus, all natural numbers are partitioned into intervals $R[d]$, $d > 0$, and $|R[d]| = 4d$. In the range $R[d]$ we distinguish the following points: $q_1[d] = 2d^2 - d$, $q_2[d] = 2d^2$, $q_3[d] = 2d^2 + d$, $q_4[d] = 2d^2 + 2d$.

Denote four sectors (quartiles) by

$$Q_1[d] = \{N_{d-1}+1, \ldots, q_1[d]\}, \quad Q_2[d] = \{q_1[d]+1, \ldots, q_2[d]\},$$
$$Q_3[d] = \{q_2[d]+1, \ldots, q_3[d]\}, \quad Q_4[d] = \{q_3[d]+1, \ldots, N_d\}.$$

Consider the following method of geometrical visualization of optimal circulants at the plane.

A circulant $G(N;s_1,s_2)$ may be constructed as a rhombus-similar frame of lattice unit squares in $\mathbf{Z}^2$ in the following way. Label each lattice point (i,j) by $(s_1 i + s_2 j) \pmod N$. As a result every label $0 \le m \le N-1$ is repeated in rhombuses infinitely many times, resulting in a tessellation of $\mathbf{Z}^2$. In Figure 1 a tessellation of the plane is presented by the rhombuses relating to the circulant with $d = 3$, $s_1 = 1$, $s_2 = 7$ and $N = 25$. The optimal graph $G(25;1,7)$ is represented with $N = N_d$. Note any optimal circulant forms dense tessellation out of rhombuses at the plane **without blanks** between them but with possible coverings of nodes (under $N < N_d$) inside the rhombuses (Figures 1 and 2). In Figure 3 an example of packing at the plane nonoptimal circulant is presented.

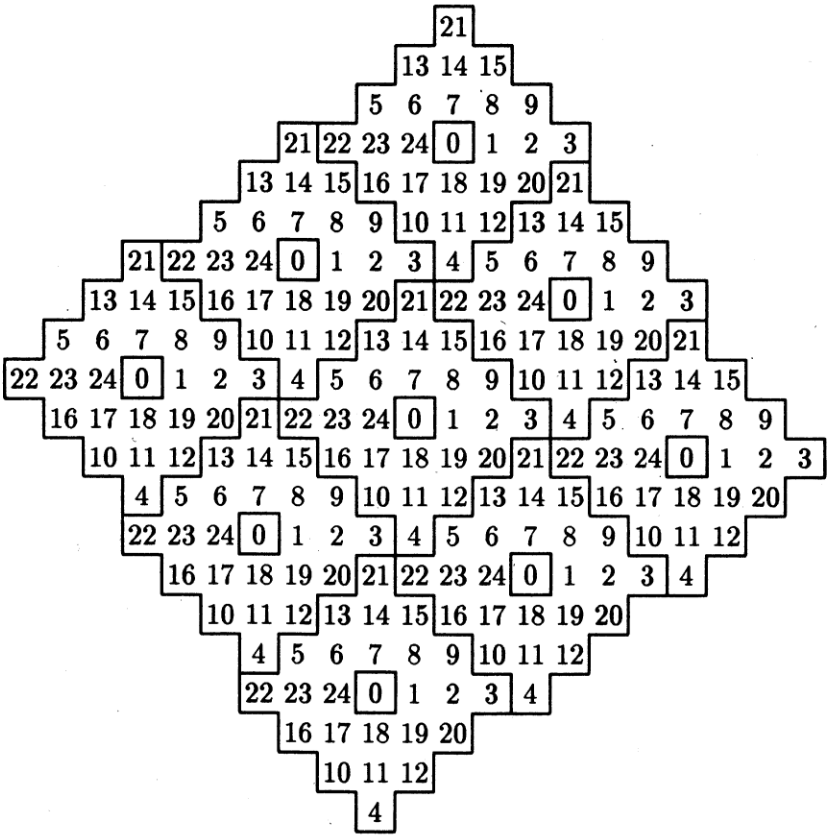

Figure 1. The optimal tessellation of the plane

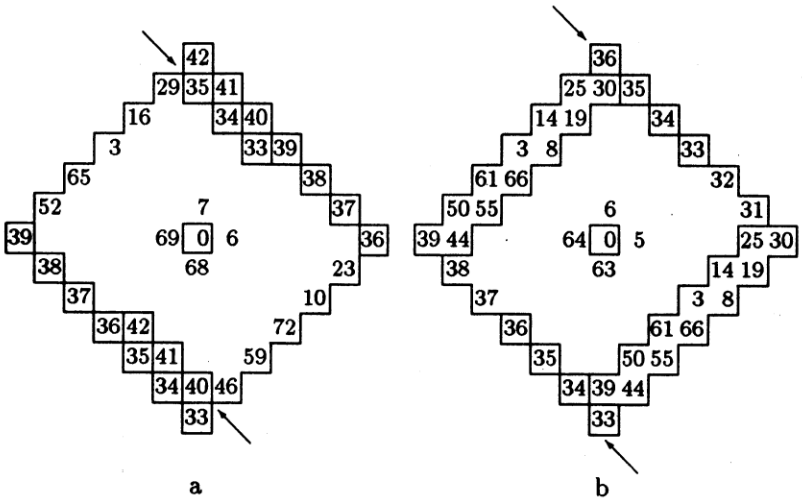

Figure 2. The optimal circulants

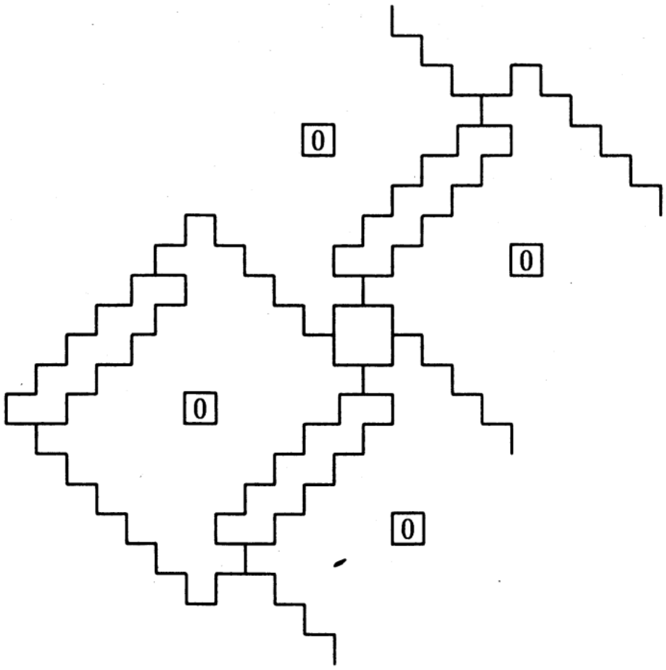

Figure 3. Nonoptimal tessellation of the plane

The optimization problem consists in finding a rhombus-similar frame out of N unit squares with the minimum of diameter and minimum of mean distance which periodically tessellates the plane without blanks with possible symmetric coverings of nodes of rhombuses at most at two latter layers of graph nodes. For a given frame of rhombus the second problem consists in finding the values of s_1 and s_2 that enable this optimal construction. In so doing we will be interested in the case of $s_1 = 1$ that corresponds to a double-loop network.

A mechanism of optimal constructions obtaining for any N is the following. In Figure 4 an initial position of dense tessellation of circulants with $N = N_d$ and the scheme of further movements at the plane are presented. Let us sequentially reduce the number of nodes in the graph beginning with $N = N_d$ to $N = N_{d-1} + 1$, getting in so doing the optimal constructions. This is reached by means of a movement the layers of tessellation A and B (or C and D) in two opposite directions along an axis X (or Y) on equal number of steps. In Figure 5 all four possible (intermediate) states of rhombus with $N \in Q_4[d]$ are presented. These states corresponds to the optimal graphs. The optimal constructions of another form in the sector $Q_4[d]$ are impossible because the covering nodes must be placed symmetrically at the opposite edges of the rhombus and situated only at the latter layer of graph nodes.

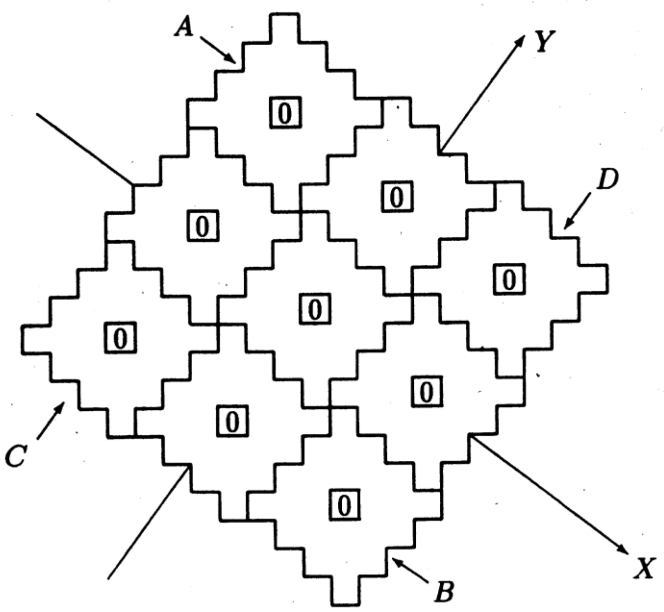

Figure 4. The scheme of movements

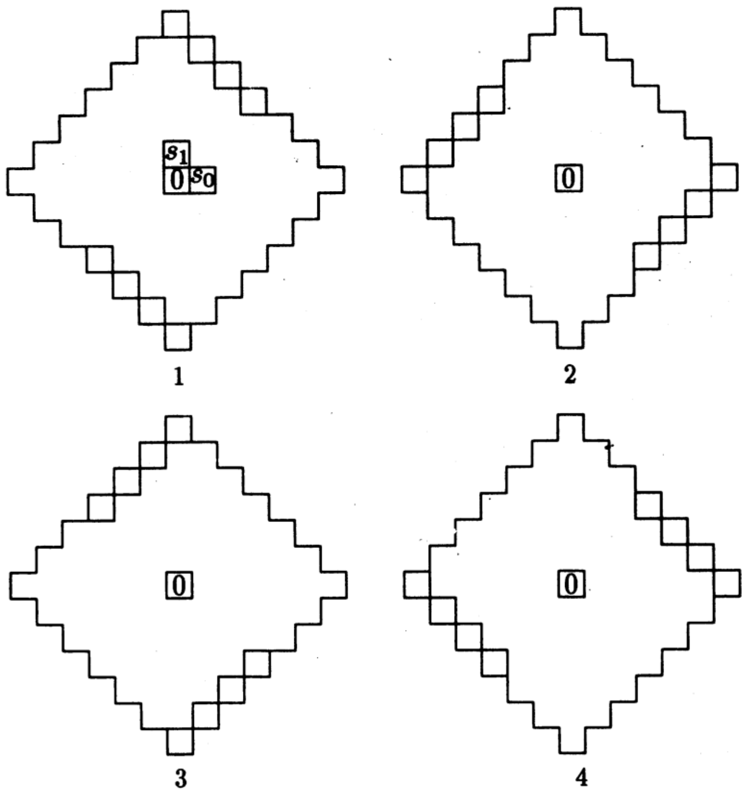

Figure 5. The sector $Q_4[d]$

The following position is an initial one for the sector $Q_3[d]$: two opposite edges of rhombus are filled completely covering unit squares. Reducing the number of uncovering squares (nodes in a graph) layers of tessellation A and B (or C and D) may be moved by two means:

1. The movement along the same axis along which the layers have been just moved.

   This case is presented in Figure 2a and it corresponds to the equivalence class of optimal descriptions of circulants of the form $\{N; d, d+1\}$. Such movement is possible for $N \in Q_3[d]$. For $N \in Q_2[d]$ the view of a movement and an example of relevant optimal graph are shown in Figure 2b. This case corresponds to the equivalence class of optimal descriptions of circulants of the form $\{N; d-1, d\}$.

2. The movement perpendicular that axis along which the layers have been just moved.

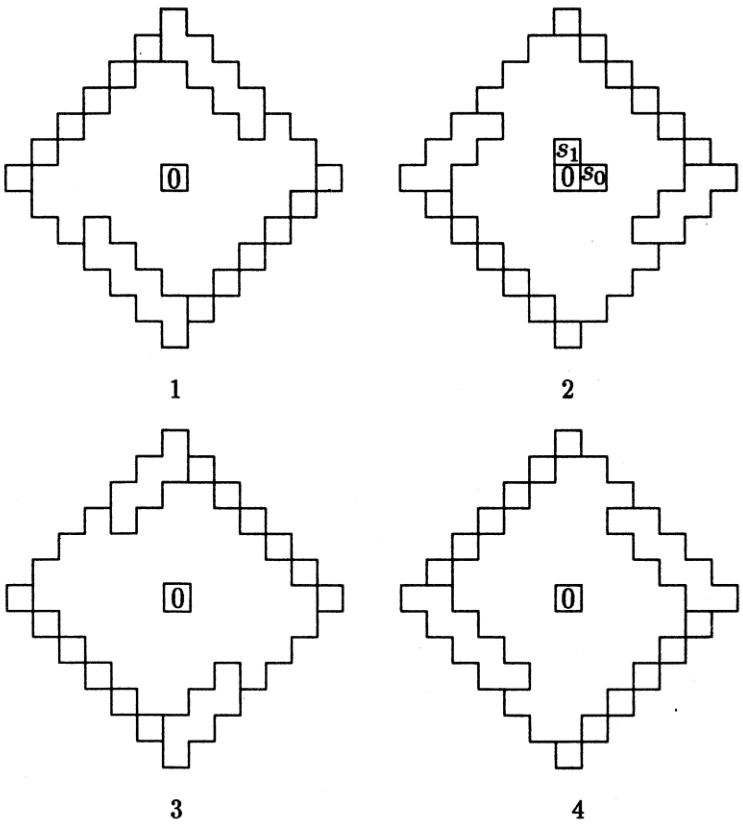

Figure 6. The sectors $Q_2[d]$ and $Q_3[d]$

These optimal constructions are presented in Figure 6: all four possible states. Note the descriptions of these optimal constructions are not equivalent to the description $\{N; d, d+1\}$ for the sector $Q_3[d]$ and to the description $\{N; d-1, d\}$ for the sector $Q_2[d]$. So if the values of s_1 and s_2 determining the optimal constructions in Figure 6 exist, then the description $\{N; s_1, s_2\}$ will form a new equivalence class of optimal descriptions different from known.

The values of s_1 and s_2 for an optimal description of a graph, if they exist, relating to the optimal constructions (1)–(4) in Figure 6 must satisfy respectively the following congruences:

$$(d+1)s_1 \equiv (d-1)s_2 \pmod N, \quad ks_1 + (2d-k-1)s_2 \equiv 0 \pmod N, \quad (1)$$

$$(d-1)s_1 + (d+1)s_2 \equiv 0 \pmod N, \quad (2d-k-1)s_1 \equiv ks_2 \pmod N, \quad (2)$$

$$(d+1)s_1 + (d-1)s_2 \equiv 0 \pmod N, \quad ks_1 \equiv (2d-k-1)s_2 \pmod N, \quad (3)$$

$$(d-1)s_1 \equiv (d+1)s_2 \pmod N, \quad (2d-k-1)s_1 + ks_2 \equiv 0 \pmod N, \quad (4)$$

where k is calculated from the equality $N = N_d - d - 2(k+1)$.

The following position is an initial one for the sector $Q_1[d]$: all edges of rhombus are filled completely covering unit squares, but two opposite edges are filled still completely covering unit squares at the layer of graph nodes $d-1$.

Reducing the number of uncovering squares (nodes in a graph) we move those layers of tessellation A and B (or C and D) which are perpendicular to ones that have been just moved. All four possible states of optimal constructions are presented in Figure 7 (1–4). The variants of such movements in this sector are the same as in the sector $Q_4[d]$ and correspond to an equivalence class of optimal descriptions of the form $\{N; d-1, d\}$.

4. The conditions of existence and analytical descriptions for optimal graphs

4.1. The sector $Q_4[d]$

Theorem 2. *Let $N \in Q_4[d]$, $d > 0$. If $\gcd(N, d) > 1$ and $\gcd(N, d+1) > 1$, then the optimal double-loop network $G(N; 1, s)$ does not exist.*

Proof. From consideration of the optimal constructions (1)–(4) in Figure 5 it should be that a value of s for optimal double-loop network $G(N; 1, s)$, if it exists, must satisfy one of four following congruences: $ds \equiv (d+1) \pmod N$, $(d+1)s \equiv (N-d) \pmod N$, $ds \equiv (N-d-1) \pmod N$, or $(d+1)s \equiv d \pmod N$.

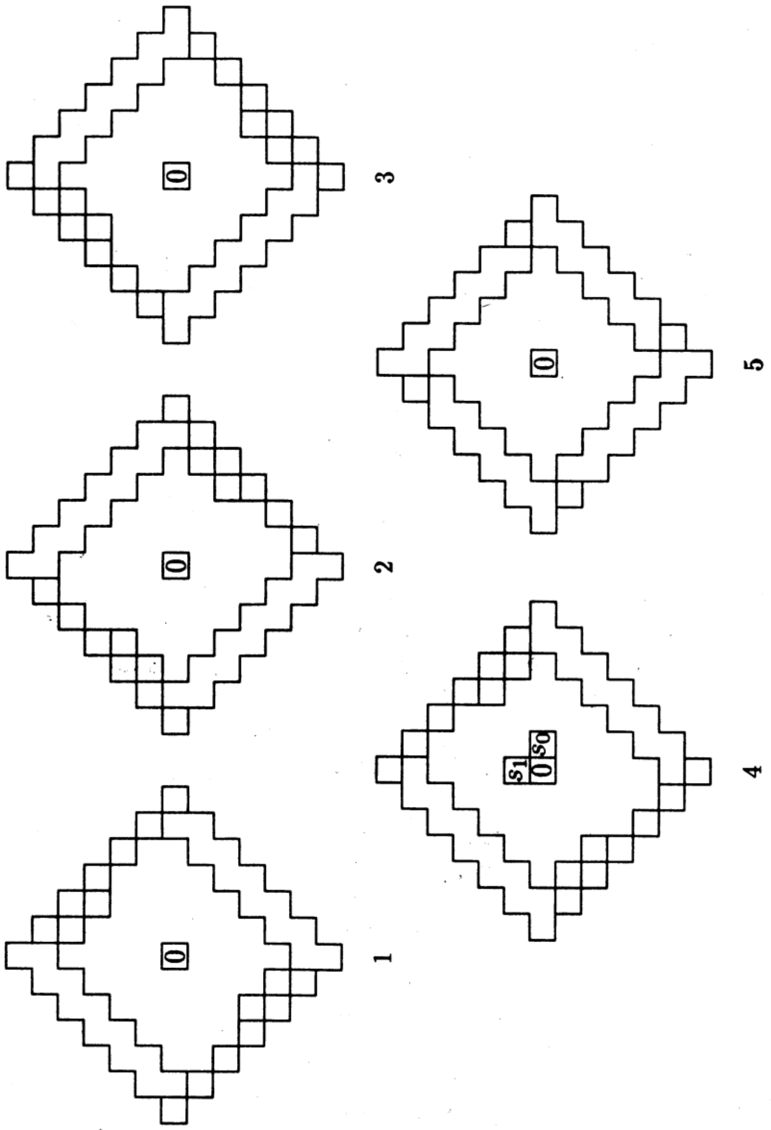

Figure 7. The sector $Q_1[d]$

Consider a possibility of integer-valued solutions of the first congruence. As $\gcd(d, N) > 1$, then integer-valued solutions exist, if the number $d+1$ is divided on $\gcd(d, N)$, but this does not take place because $\gcd(d, d+1) = 1$. The analogous results may be proved for all other congruences. Thus if $N \in Q_4[d]$ and $\gcd(N, d) > 1$ and $\gcd(N, d+1) > 1$, then the optimal double-loop network $G(N; 1, s)$ does not exist. □

In the considered sector it should be a presence of only one equivalence class of optimal two-dimensional circulants for the given $N \in Q_4[d]$, formed by the description $\{N; d, d+1\}$, that is $E(N) = 1$. Thus combining this result with results of Theorem 6 [3] we obtain: in the sector $Q_4[d]$ the conditions $\gcd(N, d) = 1$ or $\gcd(N, d+1) = 1$ are the necessary and sufficient conditions for the existence of the optimal double-loop networks of the form $\{N; 1, s\}$.

4.2. The sectors $Q_3[d]$ and $Q_2[d]$

The right bounds of all sectors, namely, the values $q_1[d]$, $q_2[d]$, $q_3[d]$ and N_d, and also the points differing on unit from them, were considered earlier. The values $q_2[d]$ and N_d were investigated in [4, 6, 10, 13], where an existence of optimal loop networks $G(N; 1, s)$ for these N was proved. The existence of optimal loop networks for $q_1[d]$ and $q_3[d]$ follows from Theorem 6 [3]: in the first case $N = d(2d-1)$ and $\gcd(N, d-1) = 1$ take place, in the second case $N = d(2d+1)$ and $\gcd(N, d+1) = 1$ take place. Note that for $N = q_3[d]$ the optimal values for s must satisfy the following congruences: $ds \equiv d \pmod N$ and $ds \equiv (N-d) \pmod N$, where $\gcd(d, N) = d > 1$. As values of d and $N-d$ are divided on d, then d the solutions for these congruences exist, which may be easily found. For values $q_i[d] \pm 1$, $i \leq 3$, the existence of optimal loop networks was shown in [8]. Therefore we will consider the sectors $Q_2[d]$ and $Q_3[d]$ for the values $q_1[d] + 2 \leq N \leq q_3[d] - 2$, $d > 1$, excepting also from consideration the values $N = q_2[d] \pm 1$.

At first consider the point $q_2[d]$, i.e., $N = 2d^2$, which lies at the bound of $Q_2[d]$ and $Q_3[d]$ sectors. We will consider the odd d. For these N we give a new equivalence class of optimal descriptions for the loop networks which does not coincide with the earlier known from [10] and obtain analytical representation for s.

Lemma 2. *Let $N = 2(2m+1)^2$, $m > 0$. Then the equivalence class of optimal descriptions of the forms $\{N; m, 3m+1\}$, $\{N; m+1, 3m+2\}$ and $\{N; 1, s\}$, where $s = 4m^2$ under the odd m and $s = 4m^2 - 2$ under the even m, exists.*

Proof. For $N = 2d^2$ ($d > 1$ is an odd number) there are the optimal descriptions of the forms $\{N; d-1, d\}$ and $\{N; d, d+1\}$ which belong to one equivalence class. Transform them in the descriptions $\{N; d-1-(d-1)/2, d+(d-1)/2\}$ and $\{N; d-(d-1)/2, d+1+(d-1)/2\}$. Substituting d by $2m+1$ we obtain the descriptions of the forms $\{N; m, 3m+1\}$ and $\{N; m+1, 3m+2\}$. Define to what description of the form $\{N; 1, s\}$ these descriptions are equivalent.

1. Let $m = 2r+1$, $r > 0$. As long as $\gcd(m, N) = 1$, then from Lemma 1 it follows: an integer number $q \leq (N/2)$ exists for which $\gcd(q, N) = 1$ and $mq \equiv 1 \pmod{N}$. Take as q the number $((m-1)N/2+1)/m = 4m^2-3$. Then $s = (3m+1)(4m^2-3) - 3(m-1)N/2$. Hence under the odd m $s = 4m^2$.
2. Let $m = 2r$, $r > 0$. As long as $\gcd(m+1, N) = 1$, then from Lemma 1 it follows: an integer number $q \leq (N/2)$ exists for which $\gcd(q, N) = 1$ and $(m+1)q \equiv 1 \pmod{N}$. Take as q the number $(mN/2+1)/(m+1) = 4m^2+1$. Then $s = 3mN/2 - (3m+2)(4m^2+1)$. Thus if m is even $s = 4m^2-2$. □

Note that the optimal descriptions given in Lemma 2 relate to a equivalence class different from equivalence classes of descriptions $\{N; d-1, d\}$ and $\{N; d, d+1\}$ [3].

In the sectors $Q_2[d]$ and $Q_3[d]$ we will consider two infinite families of lines (values of N). Consider a family of the lines, parallel to a line $N = 2d^2 + d$ and lying at **odd** number of steps from it:

$$N = 2d^2 + d - 2k - 1, \quad 0 < k < d-1, \quad d > 1. \tag{5}$$

On the other hand consider a family of the lines, parallel to a line $N = 2d^2 - d$ and lying from it at odd number of steps, which are perpendicular to a family of lines (5):

$$N = 2d^2 - d + 2k_1 + 1, \quad 0 < k_1 < d-1, \quad d > 1. \tag{6}$$

The following connection takes place between the parameters d, k and k_1: $d = k + k_1 + 1$.

Every considered N belongs simultaneously to two lines of the families (5) and (6) and may be determinated by the assignment of any two parameters from d, k, k_1.

For N defined by (5) (or (6)), which has the optimal descriptions $\{N; d, d+1\}$, for $N > 2d^2 - 2$, and $\{N; d-1, d\}$, for $N \leq 2d^2 + 1$ [3], another equivalence class of the optimal descriptions exists. It is obtained analogously to the results of Lemma 2 in the following way:

1) for $2d^2 \leq N < 2d^2 + d$ (when $k \leq k_1$) the optimal description $\{N; d, d+1\}$ transforms in the form $\{N; d-k, d+1+k\}$, where k is obtained from (5) and equals to $k = d^2 - (N-d+1)/2$;

2) for $2d^2 - d < N \leq 2d^2$ (when $k_1 \leq k$) the optimal description $\{N; d-1, d\}$ transforms in the form $\{N; d-1-k_1, d+k_1\}$ or $\{N; k, 2d-k-1\}$.

Lemma 3. *Let N be defined by* (5) *and* $\gcd(N, s_1, s_2) = 1$, *where* (s_1, s_2) *is any from the following couples of numbers*

$$(k-1, 2d+k+2), \quad (k, 2d-k-1), \quad (d-k, d+1+k),$$
$$(d-k+1, 3d-k-2), \quad (d-1, d+1). \tag{7}$$

Then the optimal circulant $G(N; s_1, s_2)$ exists.

Proof. The realization of condition $\gcd(N, s_1, s_2) = 1$ denotes that the graph $G(N; s_1, s_2)$ is connected. Show that it is optimal. Let $s_1 = k-1$, $s_2 = 2d + k + 2$. By substituting the values s_1 and s_2 into congruences (1) we obtain they being its solutions. Thus these values define the optimal circulant. The proofs for remaining values are analogous. The values $s_1 = k$, $s_2 = 2d - k - 1$ are the solution to congruences (2), $s_1 = d - k$, $s_2 = d + k + 1$ are the solution to congruences (3), $s_1 = d - k + 1$, $s_2 = 3d - k - 2$ are the solution to congruences (4) and $s_1 = d - 1$, $s_2 = d + 1$ are a trivial solution to congruences (1). □

Theorem 3. *Let N be defined by* (5). *Then the optimal double-loop network $G(N; 1, s)$ exists, if N is relatively prime at least to one of the numbers: $k-1$, k, $d-1$, d, $d+1$, $d-k$, $d-k+1$, $d+k+1$, $2d-k-1$, $2d+k+2$, $3d-k-2$.*

Proof. If N is relatively prime to d, then according to Theorem 6 [3] the optimal loop network $G(N; 1, s)$ exists. Let N be relatively prime to any other number from (7), then according to Lemma 3 the optimal circulant $G(N; s_1, s_2)$ exist where (s_1, s_2) is one from (7). Then by Lemma 1 a number relatively prime to N exists that transfers (mod N) relatively prime with N number s_1 (or s_2) into 1. Thus in this case the optimal loop network $G(N; 1, s)$ also exists. □

Theorem 4. *Let N be defined by* (5). *Let* $\gcd(N, d) > 1$ *and* $\gcd(N, d-1) > 1$. *for $N \in Q_2[d]$, or* $\gcd(N, d+1) > 1$, *for $N \in Q_3[d]$. The optimal loop network $G(N; 1, s)$ does not exist: if* $\gcd(N, s_1, s_2) = 1$, *where (s_1, s_2) is at least one couple from* (7), *but* $\gcd(N, s_1) > 1$ *and* $\gcd(N, s_2) > 1$.

Proof. According to Theorem 7 [3] the optimal description of the form $\{N; 1, s\}$ does not exist which is equivalent to the description $\{N; d, d+1\}$ (or $\{N; d-1, d\}$). According to Lemma 3 a connected optimal circulant $G(N; s_1, s_2)$ exists, where (s_1, s_2) is at least one couple from (7). But by an equivalent transformation of the values (s_1, s_2) none of them are not transformed into 1 (see Lemma 1). Thus in the equivalence class of a description $\{N; s_1, s_2\}$ there is absent the description of the form $\{N; 1, s\}$. □

Based on the results of Theorems 3 and 6 [3] and considering under $s_1 = 1$ a possibility of integer-valued solutions of congruences (1)–(4) we obtain the necessary and sufficient conditions of existence of the optimal double-loop networks $G(N; 1, s)$.

Theorem 5. *Let N be defined by* (5). *Then the optimal double-loop network $G(N; 1, s)$ exists if and only if at least one of the following conditions is realized: 1)* $\gcd(N, d) = 1$; *2)* $\gcd(N, d-1) = 1$; *3)* $\gcd(N, d+1) = 1$; *4)* $\gcd(N, k) = 1$; *5)* $\gcd(N, 2d-k-1) = 1$; *6)* $\gcd(N, d-1) = 2$ *and* $\gcd(N, 2d-k-1)/k$ *and* $\gcd(N, 3d-k-2)/d+1-k$; *7)* $\gcd(N, d+1) = 2$ *and* $\gcd(N, k)/2d-k-1$ *and* $\gcd(N, d+1+k)/d-k$.

Now in the sectors $Q_2[d]$ and $Q_3[d]$ we will consider two infinite families of lines (values of N), parallel to a line $N = 2d^2 + d$ but lying from it at **even** number of the steps:

$$N = 2d^2 + d - 2k, \quad 0 < k < d, \quad d > 3. \tag{8}$$

Analogously consider a family of the lines, parallel to a line $N = 2d^2 - d$ and lying from it at even number of steps, which are perpendicular to a family of lines (8):

$$N = 2d^2 - d + 2k_1, \quad 0 < k_1 < d, \quad d = k + k_1 > 3.$$

The families of values of N defined by (5) and (8) cover in totality all values of N in the sectors $Q_2[d]$ and $Q_3[d]$.

Theorem 6. *Let N be defined by* (8). *Let* $\gcd(N, d) > 1$ *and* $\gcd(N, d-1) > 1$, *for $N \in Q_2[d]$, or* $\gcd(N, d+1) > 1$, *for $N \in Q_3[d]$. Then the optimal double-loop network $G(N; 1, s)$ does not exist.*

Proof. According to Theorem 7 [3] the optimal description of the form $\{N; 1, s\}$ does not exist which is equivalent to the description $\{N; d, d+1\}$ (or $\{N; d-1, d\}$). Therefore if the optimal description of the form $\{N; 1, s\}$ exists it must be in another equivalence class of optimal descriptions. But such class exists not for all values of N, namely for N from (8) it does not exist because under creation of optimal constructions in the sectors $Q_2[d]$ and $Q_3[d]$ (see Figure 6) they are not formed for N from (8). □

4.3. The sector $Q_1[d]$

In the sector $Q_1[d]$ an infinite line of values of N exists, for which there are two equivalence classes of optimal circulants. This line is parallel to a line $N = 2d^2 - 2d + 2$ and it is

$$N = 2d^2 - 2d + 5, \quad d > 5. \tag{9}$$

A number of N, lying at this line, has the second equivalence class of descriptions of optimal circulants different from class of description $\{N; d-1, d\}$. The graphs with N from (9) have 4 nodes at distance d from node 0 and it is possible the following optimal placement of nodes in rhombus at the plane (see Figure 7(5)). If a description of the form $\{N; 1, s\}$ exist for such optimal construction, then it must satisfy the following congruence: $(d + 1)s \equiv d - 2 \pmod{N}$. It means that a value of m exists for which $(d+1)s - mN = d - 2$ or $mN - (d+1)s = d - 2$. By solving the equations relatively s and taking into account that N is assigned by (9) we deduce: $s = 2dm - 4m + 1 + 3(3m-1)/(d+1)$, or $s = 2dm - 4m - 1 + 3(3m+1)/(d+1)$. From here s is an integer number, if $d = 3m - 2$ in the first case, or $d = 3m$ in the second case. So we receive

Lemma 4. *For N defined by* (9) *a second equivalence class of optimal descriptions exists having a description in the form of* $\{N; 1, s\}$ *for the following values of d and s: $s = 2(d^2 - 2d)/3 + 2$ under $d = 3m$ and $s = 2(d^2 - 4)/3 + 4$ under $d = 3m - 2$.*

The data of a catalogue analisys of optimal circulants [13] and their geometrical representation in the sector $Q_1[d]$ that can be seen in Figure 7, where the optimal constructions for equivalence class of the description $\{N; d-1, d\}$ are mentioned, indicate in behalf of the following:

Conjecture. All values of $N \in Q_1[d]$, except the values defined in Lemma 4, have one equivalence class of optimal circulants and, respectively, the conditions $\gcd(N, d) = 1$, or $\gcd(N, d-1) = 1$ are the necessary and sufficient conditions for the existence of the optimal double-loop networks $G(N; 1, s)$.

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