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Exchanges in circulant networks: algorithms and lower bounds*

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In this paper we consider the routing, broadcast and gossiping problems in circulant networks. The circulant graphs are studied extensively as reliable interconnection networks for the multiprocessor systems. The optimal circulants have the minimum diameter and the minimum average distance and, respectively, maximum of the reliability and connectivity. Review of the earlier published results in Russia and also new results are presented in the paper. The distributed routing algorithm for the optimal circulants proposed here has constant complexity independent of the number of processors in the system and is adapted to failures of processors and links. We consider gossiping in the store-and-forward, full-duplex and shouting model for the case when communicating nodes can exchange up to a fixed number p of packets at each round of gossiping (p-gossiping). A general method for evaluation of the lower bounds for p-gossiping in circulant graphs is established. The efficient parallel decentralized broadcast and gossiping algorithms for two-dimensional optimal circulants are proposed.

1. Introduction

In most parallel algorithms the data exchanges take place between the processors of computing systems and affect on the efficiency of parallel programs execution. The basic patterns of interprocessor communications are routing, broadcast and gossiping. A significant variety of parallel algorithms including sorting, matrix operations, linear system solution uses such patterns of communications as broadcast and gossiping. One-to-all broadcast is an operation where a single processor (source) must send identical data in message (packet) to all other processors of the system. All-to-all broadcast, also known as gossiping or the total exchange, is a generalization of one-to-all broadcast in which all processors simultaneously initiate a broadcast. The broadcast and gossiping are widely studied for the Cayley graphs under different communication models [15, 4, 16, 17]. We consider the store-and-forward, full-duplex and shouting model. In such a model the protocol consists of a sequence of rounds (steps) and during each round each node can send (and receive) messages from all its neighbors. Here we research the

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case where at each round communicating nodes can exchange up to a fixed number p of packets. The gossiping in the Cayley graphs under $p = 1$ has been considered in [17]. We will consider the case of $p > 1$. Let us call this as p-gossiping. In [17] the authors indicated the lower bound of p-gossiping time (number of rounds) equal to $\lceil (N-1)/\delta p \rceil$, N is a number of nodes, δ is a minimum degree of a graph. Analogous problem has been considered in [16] for the toroidal mesh under the limited size of the buffers. A number of papers investigates the problem of estimation of the minimum p-gossiping time for other graphs and communication models. We will consider the problem of estimating of the lower bounds of p-gossiping time in circulant graphs which are the symmetric Cayley graphs. We introduce the general method for evaluation of the lower bounds of p-gossiping for circulants.

The efficiency of algorithms for the basic communication schemes directly depends on the topology of links between processors. Namely, the efficiency is determined by the diameter of topology (the maximum of all over the minimum distances between each pair of processors). Therefore, consideration of optimal graphs as a model of parallel systems topology allows realization of communication algorithms in a minimum time (for the given number of processors) without excessive copies of messages. In this paper efficient algorithms are presented for routing, broadcast and p-gossiping in the optimal two-dimensional circulant networks. Circulant graphs are studied extensively [1–10, 13], see also the survey of Bermond et al. [14], including more than 70 references. The circulant graphs are used in the design and realization of local area networks and architectures of parallel computing systems: Illiac IV, MPP, CRAY T3D.

2. Graph definitions

The circulant graph is defined as the graph $G(N; s_1, s_2, \ldots, s_n)$ with N nodes, labeled with integer modulo N, having $i \pm s_1$, $i \pm s_2$, ..., $i \pm s_n$ (mod N) nodes adjacent to each node i. The numbers s_i are named generators (jump sizes). Degree $\delta = 2n$ (n is a dimension) of a node in an undirected graph G is the number of edges incident to it. Note, some toroidal meshes are the circulants. Synthesis of the optimal circulants is a certain problem of graph theoretical optimization representing the generalization (in the class of circulants) of the (d, k)-graph problem [11]. This problem is in the search for a graph with the minimum diameter and the minimum average distance among all circulants having N nodes and dimension n (let this set be $C(N, n)$). The diameter of G is defined by $D = \max_{ij} d_{ij}$, where d_{ij} is the length of the shortest path from the node i to the node j. The average distance of G is defined as $\overline{D} = \sum_{ij} d_{ij}/N^2$. Let for any graph $G \in C(N, n)$, $K_{n,m}$ denotes the number of nodes to be attained from node

0 by using at most m jump sizes and let $K^*_{n,m}$ be the upper bound for $K_{n,m}$. Let $L_{n,m} = K_{n,m} - K_{n,m-1}$, $L^*_{n,m}$ be the upper bound for $L_{n,m}$ on set $C(N,n)$. The values of $K^*_{n,m}$, $L^*_{n,m}$ for any n, m were determined in [9, 12, 3]. For $n = 2$ $K^*_{2,m} = 2m^2 + 2m + 1$, $L^*_{2,m} = 4m$. For $n = 3$ $K^*_{3,m} = (4m^3 + 6m^2 + 8m)/3 + 1$, $L^*_{3,m} = 4m^2 + 2$.

The graphs achieving the upper bounds are called the optimal graphs, namely, a graph $G \in C(N,n)$ is optimal, if $L_{n,m} = L^*_{n,m}$ for any $0 \leq m \leq D^* - 1$ and $L_{n,D^*} = N - K^*_{n,D^*-1}$, where an optimal diameter D^* is given from the correlation $K^*_{n,D^*-1} < N \leq K^*_{n,D^*}$. As it is shown in [9] the optimal graph has a minimum D and $\overline{D}$ and a maximum of the reliability and connectivity among all the graphs from $C(N,n)$.

As a model of interprocessor networks of parallel systems the optimal two-dimensional circulant graphs with analytically obtained description $G(N; s, s+1)$, where $s = \lfloor(\sqrt{2N-1} - 1)/2\rfloor$, will be considered. These graphs were given in [2, 3, 6]. The optimal graphs $G(N; s, s+1)$ have the following properties:

(a) to exist for any natural $N > 4$;

(b) to have a minimum D and a minimum $\overline{D}$, equal to the lower bounds, and a maximum of reliability and connectivity among all two-dimensional circulants with N nodes;

(c) to have the description $G(N; s, s+1)$ for given s for all $2s^2 - 1 \leq N \leq 2s^2 + 4s + 3$.

The geometrical representation of the considered graph is in Figure 1a. The integers in squares are the numbers of nodes. Two nodes i and j are

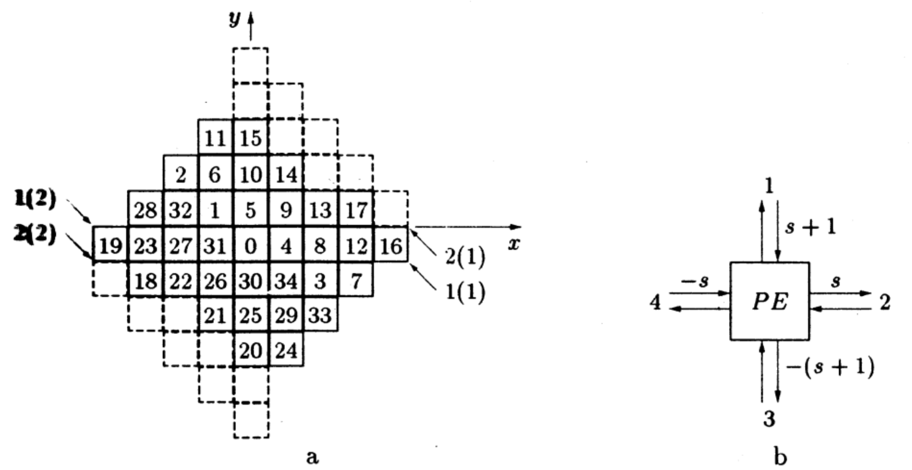

Figure 1. (a) Geometrical representation of the optimal circulant graph G(35;4,5); (b) Numeration of output (input) poles

connected by the edge, if $(i-j) \pmod N \in \{s, s+1, -s, -(s+1)\}$. For simplicity not all edges are shown. Under such a representation all the nodes of the graph lie inside a rhombus with the diagonal $2D^*$. The number D^* is the diameter of considered graph and $D^* = \lceil(\sqrt{2N-1}-1)/2\rceil$ [2]. Figure 1a illustrates the graph $G(35; 4, 5)$. When N increases from $2s^2-1$ to $2s^2+4s+3$ the rhombus is filled out sequentially the new nodes (dotted line squares) in the order indicated by arrows. Let nodes of a graph be associated to the processors and the edges to links between the adjacent processors. The output (input) poles of every processor (node) are numerated by numbers $1, 2, 3, 4$ (Figure 1b), and correspond to the generators $s+1$, s, $-(s+1)$, $-s$.

3. Routing algorithm

The relative address in circulants assigns the whole set of the shortest paths from the node i into the node j and is n-dimensional vector $A_{ij} = (x_0^{ij}, x_1^{ij}, \ldots, x_{n-1}^{ij})$, whose each component indicates the number of edges with the corresponding generator and the direction of movement along the given generator on the shortest path.

The main problem in circulants in organization of a routing algorithm is determination of the shortest path between two nodes according to their given relative addresses with respect to the third node (without loss of generality the node with number 0 will be assumed to be the third node). The following results permit us to solve the given problem efficiently using the parameters of the graph description [4].

Proposition 1. *For the circulant $G(N; s.s+1)$ for all $i, j \in \{0, 1, \ldots, N-1\}$, if $|x^{0j}-x^{0i}|+|y^{0j}-y^{0i}| < D^*$, then $A_{ij} = A_{0j} - A_{0i}$ else $A_{ij} = A_{0k}$, where $k = ((x^{0j}-x^{0i})s + (y^{0j}-y^{0i})(s+1)) \pmod N$.*

Proposition 2. *For the circulant $G(N; s.s+1)$ let $k \in \{0, 1, \ldots, N-1\}$ be the number of a node, $\Delta_k = k \pmod s$. Then $A_{0k} = (x^{0k}, y^{0k})$, where under $k < i^*$*

$$x^{0k} = [k/s] - \Delta_k - (s+1), \quad y^{0k} = \Delta_k + s \quad \textit{when } [k/s] - \Delta_k \geq s+1, \tag{1}$$

$$x^{0k} = [k/s] - \Delta_k + (s+1), \quad y^{0k} = \Delta_k - s \quad \textit{when } [k/s] - \Delta_k < \Delta_k - s, \tag{2}$$

$$x^{0k} = [k/s] - \Delta_k, \quad y^{0k} = \Delta_k \quad \textit{when } \Delta_k - s \leq [k/s] - \Delta_k < s+1; \tag{3}$$

under $k \geq i^*$ $A_{0k} = -A_{0N-k}$, A_{0N-k} is computed by formulas (1)–(3),

$$i^* = \begin{cases} N - s^2 + 1 & \textit{when } 2s^2 - 1 \leq N \leq 2s^2 + 1, \\ s^2 + s + 1 & \textit{when } 2s^2 + 1 \leq N \leq 2s^2 + 2s + 1, \\ N - s^2 - 1 & \textit{when } 2s^2 + 2s + 1 \leq N \leq 2s^2 + 3s + 2, \\ s^2 + 2s + 2 & \textit{when } 2s^2 + 3s + 2 \leq N \leq 2s^2 + 4s + 3. \end{cases}$$

The proof of Propositions 1–2 and the routing algorithm based on relative addresses are presented in [4]. The analogous results for routing were shown in [13].

The distributed algorithm realizes the pair interactions of processors and is based on the dynamic adaptive decentralized search of the shortest paths. The proposed distributed algorithm is executed in each transit processor belonging to the shortest path between the two given nodes. Let it be necessary to transmit the data from the source i to the destination j along the shortest path under the possible failures of processors and links. The data transmission begins with formation of a message in the source. The relative address A_{ij} of destination is initially computed in the source (Propositions 1 and 2) and defines all shortest paths between the source and destination. The relative address A_{kj} is modified in each transit node k thus that the current relative address assigns the whole set of the shortest paths from this transit node to destination. The choice of a transit node originates according to distinct from zero coordinates of A_{kj} and the current state of adjacent nodes and links. Namely, when the message arrives in the processor the adjacent processor belonging to the shortest path with a smaller message loading is chosen as transit. In this case the modification of A_{kj} consists in decrease of the corresponding coordinate on unit. If all directions belonging to the set of the shortest paths up to the rejected destination, then the choice produces among remaining directions according to degree of adjacent processors loading. In this case if a distance separating transit node from destination is larger then a diameter D^*, then the relative address is computed anew (Proposition 1). The end of the routing algorithm is the equality to zero all coordinates of relative address. The proposed dynamic adaptive routing algorithm possesses the constant complexity independent of N and is compared to the complexity of routing for hypercubes.

4. Lower bounds

We determine now the lower bounds of the number of rounds for completion of the gossip for any circulant graph with arbitrary degree under considered communication model. In the accepted communication model the protocols for both broadcast and gossiping consist of a sequence of steps (rounds) and at during each step, each processor can send messages to and receive different messages from all its neighbours. Each processor can send (and receive) up to a fixed number p of messages via each link at each step. We denote $T_b(p, G)$ and $T_g(p, G)$, respectively, the broadcast and the gossiping time, that are numbers of steps to complete the protocol in the network G under above conditions and $T_b^*(p, G)$ and $T_g^*(p, G)$ the minimum broadcast and the minimum gossiping time.

Proposition 3. *The minimum number $T_g^*(p, G)$ of rounds to complete the p-gossiping process in the circulant graph $G \in C(N, n)$ is not less than $k + (N - K_{n,k}^*)/\delta p$, where k is given from conditions $L_{n,k}^* \leq \delta p < L_{n,k+1}^*$.*

Proof. Note, the same view of gossiping process from any node of a graph follows from the properties of the circulants. For any graph $G \in C(N, n)$ we have $L_{n,i} \leq L_{n,i}^*$. So, no more than one packet $(L_{n,1}^*/\delta)$ can be transmitted using a given link at the first step of gossiping, no more than $L_{n,2}^*/\delta$ packets at the second step, and so on; and at step k and further at most p packets can be transmitted, where k is a minimum number that satisfies the inequalities: $L_{n,k}^* \leq \delta p < L_{n,k+1}^*$. For completion of the algorithm, a total of $(N-1)/\delta$ packets must pass through the link. The minimum number of steps (denoted by t) must satisfy the following inequality:

$$\sum_{i=1}^{k} L_{n,i}^* + (t-k)\delta p \geq N - 1.$$

Taking into account $K_{n,k}^* = \sum_{i=0}^{k} L_{n,i}^* = 1 + \sum_{i=1}^{k} L_{n,i}^*$, one sees that $T_g^*(p, G) \geq (N - K_{n,k}^*)/\delta p + k$. □

Note. The given lower bound $T_g^*(p, G)$ improves the lower bound given in [17] for the Cayley graphs, where optimal gossip time is evaluated by $\lceil (N-1)/\delta p \rceil$ rounds, δ is minimum degree of a graph.

Proposition 4. *When one allows to exchange up to a fixed number p of packets at each round, the minimum number of rounds $T_g^*(p, G)$ to complete the gossiping process in the circulant graphs with degree 4 is not less than $(N-1)/4p + (p-1)/2$.*

Proof. Under $\delta = 4$ from $L_{n,k}^* \leq \delta p < L_{n,k+1}^*$ and the expressions for $K_{2,m}^*$ and $L_{2,m}^*$ it follows that $k = p$ and

$$T_g^*(p, G) \geq \frac{N-1}{4p} + p - \frac{2p^2 + 2p}{4p} = \frac{N-1}{4p} + \frac{p-1}{2}.$$ □

Proposition 5. *The optimal number p needed to reach the minimum number of rounds to execute the gossiping process in the circulant graphs with degree 4 is not less than*

$$\frac{2D+1}{2} \pm \sqrt{\frac{(2D+1)^2}{4} - \frac{N-1}{2}}.$$

Proof. In order to reach the minimum number of rounds, t must be equal to the diameter. So we search for optimal value of p from the equality $D = (N-1)/4p + (p-1)/2$. This corresponds to the following optimal number of packets:

$$p^* \geq D + \frac{1}{2} \pm \sqrt{\frac{(2D+1)^2}{4} - \frac{N-1}{2}}. \qquad \square$$

Note. For the optimal circulants with $N = 2D^2 + 2D + 1$ we have $p^* = D$.

Proposition 6. *When one allows to exchange up to a fixed number p of packets at each round, the minimum number of rounds $T_g^*(p, G)$ to complete the gossiping process in the circulant graphs with degree 6 is not less than $(N-1)/6p + k[1 - (2k^2 + 3k + 4)/9p)]$, where $k = [\sqrt{(3p-1)/2}]$.*

Proof. The value of k must satisfy the inequalities $4k^2 + 2 \leq 6p < 4k^2 + 8k + 6$. Solving the quadratic equation $2k^2 + 4k + 3 - 3p = 0$, we will obtain $k_1 = \sqrt{(3p-1)/2} - 1$. The expression for k follows from the inequalities $\sqrt{(3p-1)/2} - 1 < k \leq \sqrt{(3p-1)/2}$. $\square$

5. Parallel decentralized broadcast and gossiping algorithms

Here we consider the broadcast problem for two-dimensional optimal circulants and then consider two different optimal gossiping algorithm for them.

The statement of the problem is the following: to organize transmission of messages from any processor to all the others (broadcast) and from every processor of the system to all others (gossiping) in the minimum time and without transmission of excess copies of messages (no duplication). The latter requirement is induced by the necessity to minimize the loading on a network under transmission.

5.1. Broadcast algorithm

A solution of this problem (cf. [4]) is in 1) construction of the regular minimum spanning trees with the root in a source and 2) transmitting copies of messages only along the directed edges of these trees. The form of these trees is identical from any node-source in consequence of the regularity of connections. With the increase of N the features of the considered optimal graphs make possible to conserve a view of the constructed tree only by adding the necessary edges to the last layer. In Figures 2 and 3 two of possible minimum spanning trees for graphs $G(N; 4, 5)$, $31 \leq N \leq 51$ are shown. A node-source is situated in the center, the arrows at the edges indicate the directions of transmission copies of messages between the adjacent processors. The spanning tree for $N = 31$ (the minimal number of processors under which the description $(N; 4, 5)$ is optimal) is denoted by arrows and circles. The arrows with the digits indicate an order in which new edges of a spanning tree appear under increase N from $2s^2 - 1$ to $2s^2 + 4s + 3$. For

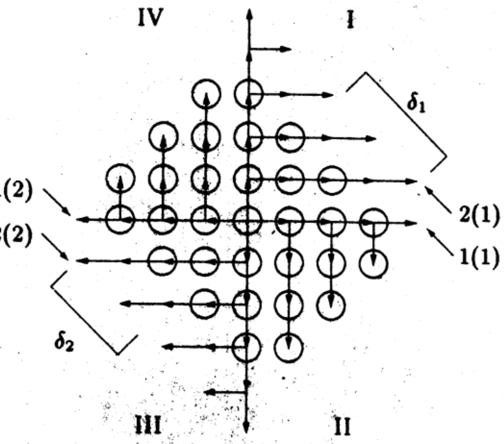

Figure 2. Minimum spanning tree for graph $G(N; 4, 5)$

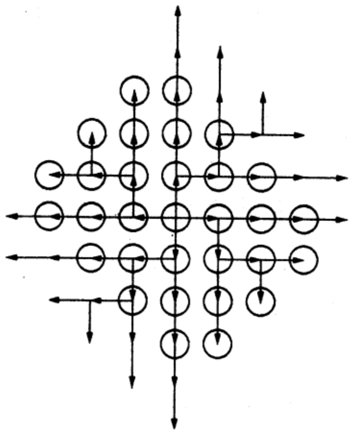

Figure 3. A variant of minimum spanning tree

a more uniform filling in of the quadrants new nodes appear in turn in the opposite quadrants. There will be considered the algorithms of broadcast and gossiping for the tree represented in Figure 2.

Consider the parallel decentralized algorithm of broadcast which is also node-invariant and invariant to the number N of processors in the system. The structure of a broadcast message is $msg := \{T, R, U, V\}$, where: T is the text of the message; $R \in \{0, 1, \ldots, D^*\}$ is the counter of the distance between the processor, executing the broadcast algorithm, and the processor-source; $U \in \{0, 1\}$, $U + 1$ is the number of output poles on which the processor executing the broadcast algorithm transmits copies of a message; $V \in \{0, 1, \ldots, D^* - 1\}$ determines the endpoint of the algorithm work when $R = D^* - 1$ (it is necessary to send a copy of a message only for $V > 0$).

When it is necessary to transmit copies of the message to neighbours the broadcast algorithm forms the parameters R, U, V, determines the numbers of output poles and then sets the message to the output queue.

Changing N in the range of optimality two intervals

$$2s^2 - 1 \leq N \leq 2s^2 + 2s + 1, \tag{4}$$

$$2s^2 + 2s + 2 \leq N \leq 2s^2 + 4s + 3 \tag{5}$$

will be distinguished (where $s > 1$ is any positive integer).

Let $\delta = N - (2s^2 - 1)$ for interval (4) and $\delta = N - (2s^2 + 2s + 1)$ for interval (5). Compute $\delta_1 (\delta_2)$ as a number of spanning tree edges situated in the last layer from the node-source respectively in I (III) quadrant (see Figure 2): $\delta_1 = \lceil \delta/2 \rceil$, $\delta_2 = \delta - \delta_1$. Using these values the broadcast algorithm in the node-source forms a parameter V. The proposed broadcast algorithm

Program 1. The broadcast algorithm in the processor–source

```
begin
  R := 0;
  U := 1;
  if inequality (4) and δ1(δ2) > 0, then V1(3) := δ1(δ2) − 1;
  else V1(3) := δ1(δ2);
  send msg from input 0 to output 1(3);
  if inequality (5), then V2(4) := 0;
  else
    if δ1(δ2) > 0, then V2(4) := s;
    else V2(4) := s − 1;
  send msg from input 0 to output 2(4);
end
```

Program 2. The broadcast algorithm in the transit processor

```
begin
  send msg from input i to output 0;
  R := R^m + 1;
  if (R = D*) or ((R = D* − 1) and (V^m = 0)), then stop;
  else (V := V^m);
  if (U^m ≠ 0), then
    begin
      U := 0;
      send msg from input i to output (i + 2) (mod 4) + 1;
      U := U^m;
      if (R ≠ D* − 1) and (V^m ≤ R) and (V^m ≠ 0), then V := 0;
      if (R = D* − 1), then stop;
    end;
  send msg from input i to output (i + 1) (mod 4) + 1;
  stop;
end
```

is shown in Programs 1 and 2. The algorithm begins its work in the source (or transit) processor after receiving a message from input 0 (or 1, 2, 3, 4). In Program 2. the parameters R^m, U^m, V^m of the incoming message from the input i transform in parameters R, U, V for message in output queue.

Using the value V allows us to avoid of the excess transmission of message copies at distance $R = D^* - 1$ steps from a source. If a message with $R = D^*$ arrives in the processor, then it is to be received, because it entered the first time, after that the processor terminates the broadcast algorithm. Taking into consideration the above-said one can come to the following conclusion.

Proposition 7. *For the circulant $G(N; s, s+1)$ with any $N > 4$ using the proposed broadcast algorithm*

$$T_b(1, G) = T_b^*(1, G) = D^*.$$

5.2. Gossiping algorithms

The gossiping algorithms are based at the proposed broadcast. They are considered as a superposition of broadcast algorithms executed for all processors of the system. An exit to gossiping is realized by each processor when a message appears. As a result every processor transmits its data to all others without excess copies and receives data from all the processors.

The first presented gossiping algorithm [4] introduces the ordering in the incoming messages processing. As we know the numbers of the output poles for incoming messages are defined by the broadcast algorithm. A sequence of acceptance from the input poles and of the incoming messages processing for gossiping will be assigned according to the following rules which follow from the accepted communication model (the number p packets in this case is bounded by a diameter of the graph):

1) the first to treat message arrived at input 0;
2) $i := 1$;
3) to process $F_R(i)$ messages, arrived at input i;
4) $i := i + 1$, if $i > 4$, then $i := 1$; go to 3).

The function $F_R(i)$ sets a number of the messages which have passed the distance R from the source to a current processor and arrived at input i. The values of $F_R(i)$ are defined in the following way:

- If $R \in \{1, 2, \ldots, D^* - 1\}$ and $i \in \{1, 2, 3, 4\}$, then $F_R(i) = R$.
- If $R = D^*$ and $\delta_2(\delta_1) = s + 1$, then

$$F_R(2)\,(F_R(4)) = \delta_2(\delta_1) - 1,$$
$$F_R(1)\,(F_R(3)) = \begin{cases} s, & \text{if } (4), \\ 1, & \text{if } (5). \end{cases}$$

- If $R = D^*$ and $\delta_2(\delta_1) < s + 1$, then

$$F_R(2)\,(F_R(4)) = \delta_2(\delta_1),$$
$$F_R(1)\,(F_R(3)) = \begin{cases} s - 1, & \text{if } (4), \\ 0, & \text{if } (5). \end{cases}$$

The function $F_R(i)$ is defined, if we consider a sink tree or a tree of paths which is generated by flowing of the messages to an arbitrary processor of the system (it corresponds to the central circle of the rhombus) from the other processors. This tree is represented in Figure 4 and is formed as a superposition of paths incoming to the central processor from each processors along the relevant to it tree represented in Figure 2. The arrows

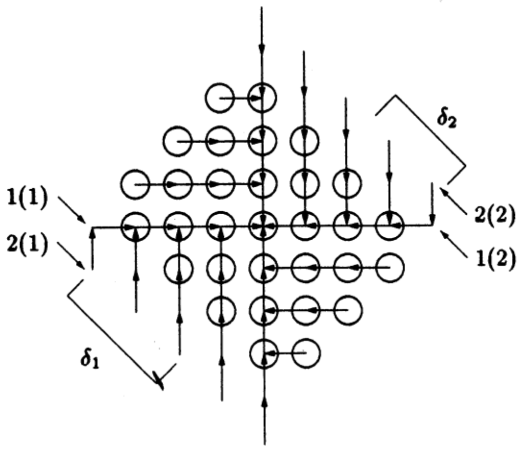

Figure 4. The sink tree for graph $G(N; 4, 5)$

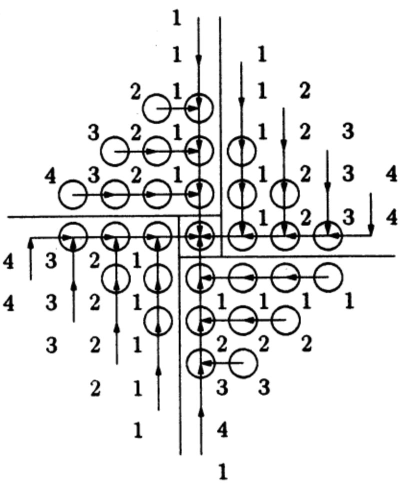

Figure 5. The order of messages arrival situated at equal distance from source

with the numbers assign a sequence of appearance new edges under change of the number of processors in the system. In this case all the processors of the system are divided on four domains corresponding to the poles on which messages from these domains arrive to the central processor. The numbers in Figure 5 indicate the order of messages arrival lying at equal distance from the accepting processor. The gossiping algorithm completes the work after processing of $F_R(i)$ messages incoming at the input i for $i = 4$ and $R = D^*$.

Note that

$$\sum_{i=1}^{4} \sum_{R=1}^{D^*} F_R(i) = N - 1.$$

Based on the analysis of the gossiping algorithm the following result was proved.

Proposition 8. *For the circulant $G(N; s, s+1)$ with any $N > 4$ under proposed gossiping algorithm*

$$T_g(D, G) = T_g^*(D, G) = D^*.$$

Consider the gossiping for the optimal circulants with $N = 2D^2 + 2D + 1$ for any fixed p. Let the broadcast tree be a tree in Figure 2. But in contrast to the given above gossiping algorithm at each round of this gossip the communicating nodes can exchange no more than p packets.

The numbers of the output poles for incoming messages will be defined by the proposed broadcast algorithm. A sequence of acceptance from the input poles and of the incoming messages processing for p-gossiping will be assigned according to the following rules which follow from the accepted communication model:

1) the first to treat message arrived at input 0;
2) $i := 1$;
3) to process all packets up to p, arrived at input i;
4) $i := i + 1$, if $i > 4$, then $i := 1$; go to 3).

In this case we can state the following results:

Proposition 9. *For the circulant $G(N; D, D+1)$ with $N = 2D^2 + 2D + 1$ minimum gossiping time of the proposed gossiping algorithm is*

$$T_g^*(p, G) = \left\lceil \frac{D(D+1)}{2p} - \frac{p-1}{2} \right\rceil + p - 1.$$

Proof. For every node of circulant $G(N; D, D+1)$ we will consider the sink tree. This tree is represented in Figure 4. In this case all the processors of the system are divided on four domains corresponding to the poles on which messages from these domains arrive to the central processor. Consider a domain corresponding to one of the poles. The number of nodes of the domain is $\frac{N-1}{4} = \frac{D(D+1)}{2}$. After $p-1$ steps of the gossiping algorithm the states of the nodes in the considered domain are shown in Figure 6.

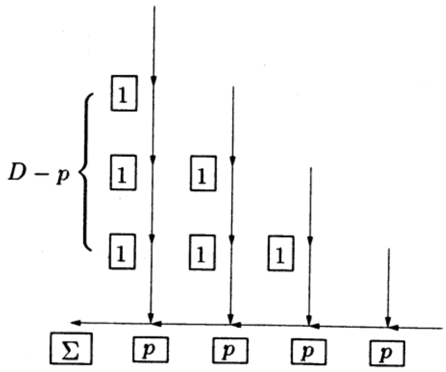

Figure 6

When all the remaining packets will come in the central node (at most p packets with one link during a round), the gossiping algorithm will be finished. The necessary minimum number of rounds x must satisfy the inequality:

$$xp \geq \sum_{i=p}^{D} i = \frac{(D+p)(D+1-p)}{2}.$$

This is simplified to $x = \lceil (D+p)(D+1-p)/2p \rceil$, and

$$T_g^*(p, G) = \left\lceil \frac{D(D+1)}{2p} - \frac{p-1}{2} \right\rceil + p - 1.$$

Thus we gave the optimal algorithm of p-gossiping for two-dimensional circulants and its time is coincided with the given lower bound. □

6. Conclusion

In this paper the basic patterns of the interprocessor exchanges including routing, broadcast and gossiping for two-dimensional circulant networks are proposed. The distributed dynamic adaptive routing algorithm for the optimal circulants considered here has constant complexity independent of the number of processors in system and is adapted to the failures (or loading degree) of processors and links. A new method of evaluation of lower bounds for p-gossiping time in arbitrary circulant graphs is established. This method may be extended to the other classes of the homogeneous Cayley graphs. The parallel decentralized node-invariant broadcast and p-gossiping algorithms for optimal two-dimensional circulants are proposed and the bounds of execution times of the protocols depending on the diameter of the networks are obtained. That implies the existence of the minimum broadcast and gossiping time (and the minimum message loading) for optimal two-dimensional circulant networks under execution of the proposed protocols.

References

[1] Du D.-Z., Hsu D.F., Li Q. and Xu Y, *A combinatorial problem related to distributed loop networks*, Networks, **20**, 1990, 173–180.

[2] E.A. Monakhova, *On analytical representation of optimal two-dimensional D_n-structures of homogeneous computer systems*, Vychislitelnye sistemy, **90**, Novosibirsk, 1981, 81–91 (in Russian).

[3] F.T. Boesch, J.-F. Wang, *Reliable circulant networks with minimum transmission delay*, IEEE Trans. Circuits Syst., CAS-32, 1985, 1286–1291.

[4] E.A. Monakhova, *Algorithms of interprocessor exchanges and reconfiguration of interconnection networks in computer systems with programmable structure*, Vychislitelnye sistemy, **94**, Novosibirsk, 1982, 81–102 (in Russian).

[5] D.F. Hsu, X.D. Jia, *Extremal problems in the combinatorial construction of distributed loop networks*, SIAM J. Discrete Math., **7**, 1994, 57–71.

[6] J.-C. Bermond, G. Illiades, C. Peyrat, *An optimization problem in distributed loop computer networks*, Third International Conference on Combinatorial Math., New York, June 1985, Ann. New York Acad. Sci., **555**, 1989, 45–55.

[7] J.-C. Bermond, D. Tzvieli, *Minimal diameter double-loop networks: Dense optimal families*, Networks, **21**, 1991, 1–9.

[8] J. Zerovnik, T. Pisanski, *Computing the diameter in multiple-loop networks*, J. Algorithms, **14**, 1993, 226–243.

[9] V.V. Korneev, *Macrostructure of homogeneous computer systems*, Vychislitelnye sistemy, **60**, Novosibirsk, 1974, 17–34 (in Russian).

[10] V.A. Vorobiev, *Simplest structures of homogeneous computer systems*, Vychislitelnye sistemy, **60**, Novosibirsk, 1974, 35–49 (in Russian).

[11] G. Memmi, Y. Raillard, *Some new results about the (d, k) graph problem*, IEEE Trans. Comput., **31**, 1982, 784–791.

[12] C.K. Wong, T.W. Maddocks, *A generalized Pascal's triangle*, Fibonacci Quart., **13**, 1975, 134–136.

[13] R. Beivide, E. Herrada, J.L. Balcazar, A. Arruabarrena, *Optimal distance networks of low degree for parallel computers*, IEEE Trans. Computers, **40**, No. 10, 1991, 1109–1124.

[14] J.-C. Bermond, F. Comellas, D.F. Hsu, *Distributed loop computer networks: a survey*, J. Parallel Distributed Comput., **24**, 1995, 2–10.

[15] S.M. Hedetniemi, S.T. Hedetniemi, A.L. Liestman, *A survey of gossiping and broadcasting in communication networks*, Networks, **18**, 1988, 319–349.

[16] J.-C. Konig, P.S. Rao, D. Trystram, *Analysis of gossiping and broadcasting algorithms in torus with restricted bufferization capabilities*, Techical Report IMAG, Grenoble, 1994.

[17] J.-C. Bermond, T. Kodate, S. Perennes, *Gossiping in Cayley graphs by packets*, Combinatorics and Computer Science, (CCS'95), LNCS 1120, 1996, 301–315.