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Conservation laws for a time field (the eikonal equation solutions) in kinematic seismics (geometric optics)*

A.G. Megrabov

Abstract. A number of non-classical formulas of vector analysis are represented as differential identities that, on the one hand, relate the modulus to the direction of an arbitrary smooth vector field in a three- and two-dimensions. On the other hand, these formulas, in a sense, separate these characteristics. In particular, for an arbitrary smooth plane vector field $\boldsymbol{v}(x,y) = |\boldsymbol{v}|\boldsymbol{\tau}$ with the module $|\boldsymbol{v}|$ and a direction $\boldsymbol{\tau}$, the conservation law in two equivalent forms is presented: in terms of the direction $\boldsymbol{\tau}$ and in terms of the vector field $\boldsymbol{v}$. It is established that each of these forms is equivalent to a conservation law for vector lines of a field $\boldsymbol{v}$, expressed in terms of the curvature vector of vector lines L_τ of the field $\boldsymbol{v}$ and of the curvature vector of orthogonal to them curves. Application of these formulas to the solutions τ of the eikonal equation has allowed us to discover a number of the new identities relating the time field τ, the refractive index $n(x,y)$ and the ray slope (direction) angle α. In particular, differential conservation laws for the time field τ (the eikonal equation solutions) in the kinematic seismics (geometrical optics) were discovered. The geometric interpretation of these conservation laws from the point of view of the differential geometry in terms of curvature vectors of rays and fronts of waves corresponding to the time field τ is obtained.

Introduction

This paper is an extension of papers [1–5].

In [2–5], a number of non-classical formulas of the vector analysis in the form of differential identities of first, second and third orders are obtained which, on the one hand, *relate* the modulus $|\boldsymbol{v}|$ to the direction $\boldsymbol{\tau}$ of an arbitrary smooth vector field $\boldsymbol{v}$ in a three-dimensional (space, $\boldsymbol{v} = \boldsymbol{v}(x,y,z)$) and a two-dimensional (plane, $\boldsymbol{v} = \boldsymbol{v}(x,y)$) cases. The direction $\boldsymbol{\tau}$ of a vector $\boldsymbol{v}$, as usual [6, Ch. 1, § 2] is understood as unit vector $\boldsymbol{\tau} = \boldsymbol{v}/|\boldsymbol{v}|$, tangential to the vector line of vector field $\boldsymbol{v}$ so we have $\boldsymbol{v} = |\boldsymbol{v}|\boldsymbol{\tau}$. Thus, the quantities $\boldsymbol{v}$, $|\boldsymbol{v}|$, $\boldsymbol{\tau}$ can depend, in addition to the "spatial" variables x, y, z, also, on the "temporal" variable t, playing the role of a parameter in the general identities obtained. However, without necessity we will not explicitly indicate to a possible dependence on t assuming $\boldsymbol{v} = \boldsymbol{v}(x,y)$ and etc.

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On the other hand, the formulas obtained in a sense *separate* the modulus $|\boldsymbol{v}|$ and the direction $\boldsymbol{\tau}$ of the vector field $\boldsymbol{v} = |\boldsymbol{v}|\boldsymbol{\tau}$. Namely, the basic identity (see Section 1.2) to any smooth vector field $\boldsymbol{v}$ (force or non-force in the physical sense) explicitly assigns a vector field $\boldsymbol{Q} = \boldsymbol{Q}(\boldsymbol{v})$, representable in the form of the sum of the two vector fields $\boldsymbol{P}$ and $\boldsymbol{S}$ with the following properties. The field $\boldsymbol{P}$ is defined only by the modulus $|\boldsymbol{v}|$ of the field $\boldsymbol{v}$ and is potential in a two- and in a three-dimensional cases, while the field $\boldsymbol{S}$ is defined only by the direction $\boldsymbol{\tau}$ of the field $\boldsymbol{v}$ and is solenoidal in a two-dimensional case. In the case of a potential vector field $\boldsymbol{v} = \operatorname{grad} u$ with a potential, the obtained formulas relate the Laplacian of an arbitrary function $u(x, y, z)$ or $u(x, y)$, the modulus of its gradient and its direction, and the field $\boldsymbol{Q}$ is collinear with the initial field $\boldsymbol{v}$ [2–5]. A similar relation is also obtained and in the case of a solenoidal field $\boldsymbol{v} = \operatorname{rot} \boldsymbol{A}$ with the vector potential $\boldsymbol{A}$ [3–5].

In the given paper, we present the proof of the basic identity (Section 1.2) from which other identities follow as corollaries. In [2–5], this identity is presented without the proof.

In a two-dimensional case ($\boldsymbol{v} = \boldsymbol{v}(x, y)$) a differential conservation law for an arbitrary smooth plane vector field $\boldsymbol{v} = \boldsymbol{v}(x, y) = |\boldsymbol{v}|\boldsymbol{\tau}$ in two equivalent forms is obtained: in terms of the direction field $\boldsymbol{\tau} = \boldsymbol{\tau}(x, y)$ of the form

$$\operatorname{div} \boldsymbol{S}(\boldsymbol{\tau}) = 0, \tag{1}$$

where $\boldsymbol{S}(\boldsymbol{\tau}) = \boldsymbol{\tau} \operatorname{div} \boldsymbol{\tau} + \boldsymbol{\tau} \times \operatorname{rot} \boldsymbol{\tau}$ (other forms of the field $\boldsymbol{S}(\boldsymbol{\tau})$ are found as well), and in terms of the vector field $\boldsymbol{v}$ of the form

$$\operatorname{div} \left\{ \frac{\boldsymbol{v} \operatorname{div} \boldsymbol{v} + \boldsymbol{v} \times \operatorname{rot} \boldsymbol{v}}{|\boldsymbol{v}|^2} - \frac{1}{2} \operatorname{grad} \ln |\boldsymbol{v}|^2 \right\} = 0. \tag{2}$$

Based on the results obtained in [1], in the present paper we give the geometric interpretation of conservation laws (1) and (2) in terms of curvature vectors of the curves generated by the vector field $\boldsymbol{v}(x, y)$. Namely, both conservation law (1) for the field directions $\boldsymbol{\tau}$, and conservation law (2) for the vector field of $\boldsymbol{v} = |\boldsymbol{v}|\boldsymbol{\tau}$ are equivalent to the following conservation law for the vector lines L_τ of the field $\boldsymbol{v}$:

$$\operatorname{div} \boldsymbol{S}^* = 0, \tag{3}$$

where the vector field $\boldsymbol{S}^*$ is the sum of the vector field of the curvature vector of the vector line L_τ of the field $\boldsymbol{v}$ and the vector field of the curvature vector of the curves L_ν, orthogonal to the curves L_τ. The field $\boldsymbol{S}^*$ can also be expressed in terms of only one curvature vector of the vector lines L_τ of the field $\boldsymbol{v}$ or in other forms in terms of a tangential unit vector $\boldsymbol{\tau}$ of the curves L_τ or (and) its normal unit vector $\boldsymbol{\nu}$.

In [4, 5], the above-mentioned general identities are applied to the eikonal equation, which is the basic mathematical model of kinematic seismics (geometric optics). In the two-dimensional case, the eikonal equation for the time field $\tau = \tau(x, y)$ in an inhomogeneous isotropic medium with the refractive index $n(x, y)$ looks like [7, Ch. 1, § 1]:

$$|\operatorname{grad}\tau|^2 \stackrel{\text{def}}{=} \tau_x^2 + \tau_y^2 = n^2(x, y) \quad \Leftrightarrow \quad \Delta_1\tau \stackrel{\text{def}}{=} \frac{\tau_x^2 + \tau_y^2}{n^2(x, y)} = 1. \tag{4}$$

In [4, 5], this application leads to the following results (see Section 4). A number of vector and scalar differential identities of second and third orders relating the time field $\tau(x, y)$, the refractive index $n(x, y)$ and the ray slope (direction) angle $\alpha = \alpha(x, y)$, distinct from the classical ray equation and known identities from [8–10] are obtained.

In particular, in [4, 5] the existence in a kinematic seismic problem (geometric optics) of two conservation laws for the time field $\tau = \tau(x, y, t)$ (t is the parameter of a source defining its position) is detected. The first conservation law looks like:

$$\operatorname{div}\boldsymbol{T} = 0 \qquad (\text{at any } t = \text{const}), \tag{5}$$

where

$$\boldsymbol{T} = \operatorname{grad}\ln n - \frac{\Delta\tau}{n^2}\operatorname{grad}\boldsymbol{\tau},$$

is a special case of identity (2) for the case $\boldsymbol{v} = \operatorname{grad}\tau$, where $\boldsymbol{\tau}$ is a solution to the eikonal equation (4). This conservation law in its form is similar to the differential conservation law $\operatorname{div}\boldsymbol{v} = 0$ ($\boldsymbol{v}$ is velocity) for an incompressible fluid [11]. The second conservation law looks like:

$$\frac{\partial}{\partial t}\operatorname{div}\boldsymbol{Q} = 0 \qquad \Big(\Leftrightarrow \quad \operatorname{div}\frac{\partial\boldsymbol{Q}}{\partial t} = 0\Big), \tag{6}$$

where

$$\boldsymbol{Q} = \frac{\Delta\boldsymbol{\tau}}{n^2}\operatorname{grad}\tau, \qquad \operatorname{div}\boldsymbol{Q} = \Delta\ln n.$$

The role of the "time t" is thus played by the parameter t of a source. This conservation law means the existence in kinematic seismics (geometric optics) of a differential invariant of the wave propagation $I = \operatorname{div}\boldsymbol{Q} = \operatorname{div}\left\{\frac{\tau_{xx} + \tau_{yy}}{n^2}\operatorname{grad}\tau\right\}$, i.e., the quantity, expressed in terms of $\boldsymbol{\tau}$ and $n(x, y)$ and independent of the location of a wave source. To it, there corresponds an integral invariant.

In this paper, based on formula (3), the geometric interpretation of conservation laws (5) and (6) in terms of curvature vectors of rays and fronts of waves corresponding to the time field τ is given.

In addition, we present the vector fields $\boldsymbol{Q}(x,y)$ and $\boldsymbol{R}(x,y)$, found in [4, 5] in the explicit form, with following properties. A potential vector component of the vector $\boldsymbol{Q}$ is defined only by the refractive index $n(x,y)$ and coincides with the vector field $\operatorname{grad}\ln n$, and a rotational (solenoidal) component of the vector $\boldsymbol{Q}$ is defined only by the angular characteristic $\alpha(x,y)$ and coincides with $\operatorname{rot}(\alpha\boldsymbol{k})$. The potential and the rotational components of the vector $\boldsymbol{R}$ are equal to $(-\operatorname{grad}\alpha)$ and $\operatorname{rot}(\ln n\boldsymbol{k})$, respectively. There are also presented: formulas for $\Delta\ln n$, $\Delta\alpha$ in terms of divergence and a rotor of the same vector field ($\boldsymbol{Q}$ or $\boldsymbol{R}$); a non-classical form of the ray equation. In [4], the integral formulas, allowing the calculation of a number of functionals in the direct and the inverse kinematic problems, etc., are obtained.

The eikonal equation is of interest because the vector fields $\boldsymbol{Q}$, $\boldsymbol{R}$, $\boldsymbol{P}$, $\boldsymbol{S}$ and other quantities entering the differential identities obtained and constructed on a potential non-force vector field $\boldsymbol{v} = \operatorname{grad}\tau(x,y)$, where $\tau(x,y)$ is a solution to the eikonal equation, admit the physical and geometric interpretation: they are related to the refractive index, the geometrical divergence of rays, the ray transformation and the differential geometry (Section 3.4).

The symbols $(\boldsymbol{a}\cdot\boldsymbol{b})$ and $\boldsymbol{a}\times\boldsymbol{b}$ denote the scalar and the vector products of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, ∇ is the Hamiltonian operator ("a nabla"), $(\boldsymbol{v}\cdot\nabla)\boldsymbol{a}$ is a derivative of the vector $\boldsymbol{a}$ in the direction of the vector $\boldsymbol{v}$, $\Delta u = u_{xx} + u_{yy}$.

1. Differential identities and the conservation law for an arbitrary plane vector field $\boldsymbol{v}(x,y)$

1.1. Basic initial quantities. Let D be a domain in the plane x,y; $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ are unit vectors along the axes of the rectangular co-ordinates x,y,z; $\boldsymbol{v} = \boldsymbol{v}(x,y) = v_1\boldsymbol{i}+v_2\boldsymbol{j}$ is a vector field defined on D, $v_k = v_k(x,y)$ are scalar functions $(k = 1,2)$, $|\boldsymbol{v}|^2 = v_1^2 + v_2^2$; $\alpha = \alpha(x,y)$ is the angle of slope of the vector $(v_1\boldsymbol{i} + v_2\boldsymbol{j})$ to the axis Ox, so that $\cos\alpha = v_1/\sqrt{g}$, $\sin\alpha = v_2/\sqrt{g}$, where $g = v_1^2 + v_2^2$, i.e., $\alpha(x,y)$ is the polar angle of a point $(\xi = v_1,\ \eta = v_2)$ on the plane ξ,η:

$$\alpha \stackrel{\text{def}}{=} \arctan(v_2/v_1) + (2k+\delta)\pi, \qquad k\in\mathbb{Z}, \tag{7}$$

where $\delta = 0$ and $\delta = 1$ in quadrants I, IV and II, III of the plane ξ,η respectively. This means that $|\boldsymbol{v}|$, α are polar coordinates on the plane $\xi = v_1$, $\eta = v_2$. Thus, $\boldsymbol{v} = |\boldsymbol{v}|\boldsymbol{\tau}$, where

$$\boldsymbol{\tau} \stackrel{\text{def}}{=} \boldsymbol{v}/|\boldsymbol{v}| = \boldsymbol{\tau}(\alpha) \stackrel{\text{def}}{=} \cos\alpha\,\boldsymbol{i} + \sin\alpha\,\boldsymbol{j} \tag{8}$$

is a unit tangential vector to the vector line of the vector field $\boldsymbol{v}$ or the direction (unit vector) of the vector field $\boldsymbol{v}$ ($|\boldsymbol{\tau}| \equiv 1$).

1.2. The basic identity for an arbitrary plane vector field $\boldsymbol{v}(x, y)$

Theorem 1 (The basic identity). *For any plane vector field $\boldsymbol{v} = \boldsymbol{v}(x, y) = |\boldsymbol{v}|\boldsymbol{\tau}$ with the components $v_k(x, y) \in C^1(D)$ $(k = 1, 2)$, the modulus $|\boldsymbol{v}| \neq 0$ in D and direction $\boldsymbol{\tau}$, we have the identity*

$$\boldsymbol{Q} = \boldsymbol{Q}(\boldsymbol{v}) = \boldsymbol{P}(|\boldsymbol{v}|) + \boldsymbol{S}(\boldsymbol{\tau}), \tag{9}$$

where

$$\boldsymbol{Q}(\boldsymbol{v}) \overset{\text{def}}{=} \frac{\boldsymbol{v}\operatorname{div}\boldsymbol{v} + \boldsymbol{v} \times \operatorname{rot}\boldsymbol{v}}{|\boldsymbol{v}|^2}, \quad \boldsymbol{P}(|\boldsymbol{v}|) \overset{\text{def}}{=} \operatorname{grad}\ln|\boldsymbol{v}| = \frac{\operatorname{grad}|\boldsymbol{v}|^2}{2|\boldsymbol{v}|^2}, \tag{10}$$

$$\boldsymbol{S} = \boldsymbol{S}(\boldsymbol{\tau}) \overset{\text{def}}{=} \boldsymbol{\tau}\operatorname{div}\boldsymbol{\tau} + \boldsymbol{\tau} \times \operatorname{rot}\boldsymbol{\tau} = \boldsymbol{Q}(\boldsymbol{v}) - \boldsymbol{P}(|\boldsymbol{v}|). \tag{11}$$

For the vector field $\boldsymbol{S}(\boldsymbol{\tau})$, any of the following representations holds:

$$\boldsymbol{S}(\boldsymbol{\tau}) = \boldsymbol{\tau}\operatorname{div}\boldsymbol{\tau} - \boldsymbol{\tau}_s = -\{(\boldsymbol{\tau}\times\nabla)\times\boldsymbol{\tau} + (\boldsymbol{\tau}\cdot\nabla)\boldsymbol{\tau}\} = -((\boldsymbol{v}\times\nabla)\times\boldsymbol{v})/|\boldsymbol{v}|^2 \tag{12}$$

($\boldsymbol{\tau}_s = (\boldsymbol{\tau}\cdot\nabla)\boldsymbol{\tau} = \operatorname{rot}\boldsymbol{\tau}\times\boldsymbol{\tau}$ *is the derivative of the vector $\boldsymbol{\tau}$ in the direction $\boldsymbol{\tau}$*),

$$\boldsymbol{S}(\boldsymbol{\tau}) = \operatorname{rot}(\alpha\boldsymbol{k}) = \operatorname{grad}\alpha \times \boldsymbol{k}, \tag{13}$$

$$\boldsymbol{S}(\boldsymbol{\tau}) = \boldsymbol{\tau}\operatorname{div}\boldsymbol{\tau} - k\boldsymbol{\nu}, \tag{14}$$

where k is the curvature of the vector line of the field $\boldsymbol{v}$, $\boldsymbol{\nu}$ is its unit normal,

$$\begin{aligned}\boldsymbol{S}(\boldsymbol{\tau}) &= \boldsymbol{\tau}\operatorname{div}\boldsymbol{\tau} + \boldsymbol{\nu}\operatorname{div}\boldsymbol{\nu} = \boldsymbol{\tau}\times\operatorname{rot}\boldsymbol{\tau} + \boldsymbol{\nu}\times\operatorname{rot}\boldsymbol{\nu} \\ &= \boldsymbol{S}(\boldsymbol{\nu}) \overset{\text{def}}{=} \boldsymbol{\nu}\operatorname{div}\boldsymbol{\nu} + \boldsymbol{\nu}\times\operatorname{rot}\boldsymbol{\nu}.\end{aligned} \tag{15}$$

The vectors $\boldsymbol{\tau}$, $\boldsymbol{\nu}$ form the right-hand side system of the Frene unit vectors (the Frene basis) of the vector lines L_τ of the field $\boldsymbol{v}$ and are related by the Frene equations [12–14]

$$\frac{d\boldsymbol{\tau}}{ds} = k\boldsymbol{\nu}, \qquad \frac{d\boldsymbol{\nu}}{ds} = -k\boldsymbol{\tau}, \tag{16}$$

where s is a natural parameter of the curve L_τ, i.e., its length being calculated from its certain point.

Hence, under conditions of the theorem for any plane vector field $\boldsymbol{v}(x, y)$, we have the identity:

$$\boldsymbol{Q} \overset{\text{def}}{=} \frac{\boldsymbol{v}\operatorname{div}\boldsymbol{v} + \boldsymbol{v}\times\operatorname{rot}\boldsymbol{v}}{|\boldsymbol{v}|^2} = \operatorname{grad}\ln|\boldsymbol{v}| + \operatorname{rot}(\alpha\boldsymbol{k}) \quad \Rightarrow \tag{17}$$

$$\begin{aligned} \operatorname{div} \boldsymbol{v} &= (\{\operatorname{grad} \ln |\boldsymbol{v}| + \operatorname{rot}(\alpha \boldsymbol{k})\} \cdot \boldsymbol{v}), \\ \operatorname{rot} \boldsymbol{v} &= \{\operatorname{grad} \ln |\boldsymbol{v}| + \operatorname{rot}(\alpha \boldsymbol{k})\} \times \boldsymbol{v}. \end{aligned} \tag{18}$$

Identity (9) *or* (17) *can be written down in the equivalent form:*

$$\frac{\boldsymbol{v} \operatorname{div} \boldsymbol{v} + \boldsymbol{v} \times \operatorname{rot} \boldsymbol{v}}{|\boldsymbol{v}|^2} - \frac{1}{2} \operatorname{grad} \ln |\boldsymbol{v}|^2 = \boldsymbol{S}(\boldsymbol{\tau}). \tag{19}$$

Proof. Let us present the proof as in [3, 5] the given statement is given without it. From known formulas of the vector analysis [6, § 7]

$$\operatorname{div}(\varphi \boldsymbol{a}) = \varphi \operatorname{div} \boldsymbol{a} + (\operatorname{grad} \varphi \cdot \boldsymbol{a}),$$
$$\operatorname{rot}(\varphi \boldsymbol{a}) = \varphi \operatorname{rot} \boldsymbol{a} + \operatorname{grad} \varphi \times \boldsymbol{a},$$
$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b}(\boldsymbol{a} \cdot \boldsymbol{c}) - \boldsymbol{c}(\boldsymbol{a} \cdot \boldsymbol{b}),$$

where $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ are any vectors (vector functions), φ is a scalar function, for $\boldsymbol{v} = |\boldsymbol{v}|\boldsymbol{\tau}$, $\varphi = |\boldsymbol{v}|$, $\boldsymbol{a} = \boldsymbol{\tau}$ we have:

$$\frac{\boldsymbol{v} \operatorname{div} \boldsymbol{v}}{|\boldsymbol{v}|^2} = \boldsymbol{\tau} \operatorname{div} \boldsymbol{\tau} + (\operatorname{grad} \ln |\boldsymbol{v}| \cdot \boldsymbol{\tau})\boldsymbol{\tau},$$
$$\frac{\boldsymbol{v} \times \operatorname{rot} \boldsymbol{v}}{|\boldsymbol{v}|^2} = \boldsymbol{\tau} \times \operatorname{rot} \boldsymbol{\tau} - (\operatorname{grad} \ln |\boldsymbol{v}| \cdot \boldsymbol{\tau})\boldsymbol{\tau} + \frac{1}{2} \operatorname{grad} \ln |\boldsymbol{v}|^2.$$

Summarizing the last two equalities, we arrive at identity (9). From the known formula [6, § 17, the formula (10)]

$$\frac{1}{2} \operatorname{grad} |\boldsymbol{a}|^2 = (\boldsymbol{a} \cdot \nabla)\boldsymbol{a} + \boldsymbol{a} \times \operatorname{rot} \boldsymbol{a}$$

at $\boldsymbol{a} = \boldsymbol{\tau}$, $|\boldsymbol{\tau}| \equiv 1$ we have $\boldsymbol{\tau} \times \operatorname{rot} \boldsymbol{\tau} = -(\boldsymbol{\tau} \cdot \nabla)\boldsymbol{\tau} = -\boldsymbol{\tau}_s$ and from (11) we obtain representation (12), and taking into account the first of the Frene equations (16) – representation (14). Formula (13) is obtained in [3, 5] and is proved in [1]. From it and from (9)—(11), follows identity (17). Equalities (15) are proved in [1]. Formulas (18) are obtained by the scalar multiplication of equality (17) by the vector $\boldsymbol{v}$. The theorem is proved. □

Remark 1. As follows from the above proof, formulas (9)–(12), (14) also hold for the three-dimensional vector field $\boldsymbol{v} = \boldsymbol{v}(x, y, z)$. A three-dimensional analogy of Theorem 1 is obtained in [3, 5].

As is known [6], any smooth vector field can be represented as sum of a gradient of a certain scalar and a curl of a certain vector. Identity (17) gives such a representation for the vector field $\boldsymbol{Q}$ and, moreover, in an explicit form, directly in terms of the initial vector field $\boldsymbol{v}$. Hence:

Corollary 1 (The geometric sense of the basic identity (9) or (17)). *For any plane vector field* $\boldsymbol{v}(x,y)$ *under conditions of Theorem 1, there exists a vector field* $\boldsymbol{Q}(x,y)$*, defined in* (9) *and* (17)*, which is the sum of two vector fields with the following properties. The first field* $\boldsymbol{P} = \operatorname{grad}\ln|\boldsymbol{v}|$ *is defined only by the field modulus* $\boldsymbol{v}$ *and is potential, and the second field* $\boldsymbol{S} = \operatorname{rot}(\alpha\boldsymbol{k})$ *is defined only by direction of the field* $\boldsymbol{v}$ *(the angle* $\alpha(x,y)$*) and is solenoidal. Thus, the basic identity* (9) *or* (17)*, first, separates any smooth plane vector field* $\boldsymbol{v}$ *relative to its modulus and direction and, second, separates the vector field* $\boldsymbol{Q}(\boldsymbol{v})$ *to potential and rotational parts.*

1.3. A conservation law for an arbitrary smooth plane vector field $\boldsymbol{v(x,y)}$. The known identities of the vector analysis [6, § 17] $\operatorname{div}\operatorname{rot}\boldsymbol{a} = 0$ (for any vector field $\boldsymbol{a}$), $\operatorname{rot}\operatorname{grad}\varphi = 0$ (for any scalar field φ) and formulas (13), (17) imply

Corollary 2 (Conservation law for the field of the directions $\boldsymbol{\tau}$ of the vector field $\boldsymbol{v}$). *Under conditions of Theorem 1 we have* $(\boldsymbol{S}\cdot\operatorname{grad}\alpha) = 0$*, i. e., the vector lines of the vector field* $\boldsymbol{S}(\boldsymbol{\tau})$ *coincide with level lines of the scalar field of the angles* $\alpha(x,y)$*. If* $v_k(x,y) \in C^2(D)$ $(k = 1,2)$*, then we have identities*

$$\operatorname{div}\boldsymbol{S}(\boldsymbol{\tau}) = 0 \quad\Leftrightarrow\quad \operatorname{div}\{\boldsymbol{\tau}\operatorname{div}\boldsymbol{\tau} + \boldsymbol{\tau}\times\operatorname{rot}\boldsymbol{\tau}\} = 0, \tag{20}$$

$$\operatorname{rot}\boldsymbol{S}(\boldsymbol{\tau}) = -(\Delta\alpha)\boldsymbol{k} \quad\Rightarrow\quad \Delta\ln|\boldsymbol{v}| = \operatorname{div}\boldsymbol{Q},$$

$$(\Delta\alpha)\boldsymbol{k} = -\operatorname{rot}\boldsymbol{Q} \quad\Rightarrow\quad \Delta\operatorname{Ln}\{|\boldsymbol{v}|\,\mathrm{e}^{\pm i\alpha}\} = \operatorname{div}\boldsymbol{Q} \mp i(\operatorname{rot}\boldsymbol{Q}\cdot\boldsymbol{k}).$$

Identity (20) *represents a differential conservation law for a vector field of the directions* $\boldsymbol{\tau}$ *of the vector field* $\boldsymbol{v} = |\boldsymbol{v}|\boldsymbol{\tau}$*.*

From the identities $\operatorname{div}\boldsymbol{S}(\boldsymbol{\tau}) = 0$ and (19) follows

Theorem 2 (The differential conservation law for an arbitrary plane vector field $\boldsymbol{v}(x,y)$). *For any plane vector field* $\boldsymbol{v} = \boldsymbol{v}(x,y) = |\boldsymbol{v}|\boldsymbol{\tau}$ *with the components* $v_k(x,y) \subset C^2(D)$ $(k = 1,2)$*, modulus* $|\boldsymbol{v}| \neq 0$ *in* D *and direction* $\boldsymbol{\tau}$ $(\boldsymbol{\tau}| \equiv 1)$*, we have the identity*

$$\operatorname{div}\{\boldsymbol{Q}(\boldsymbol{v}) - \boldsymbol{P}(|\boldsymbol{v}|)\} = 0 \quad\Leftrightarrow\quad \operatorname{div}\boldsymbol{S}(\boldsymbol{\tau}) = 0$$

$$\Leftrightarrow\quad \operatorname{div}\left\{\frac{\boldsymbol{v}\operatorname{div}\boldsymbol{v} + \boldsymbol{v}\times\operatorname{rot}\boldsymbol{v}}{|\boldsymbol{v}|^2} - \frac{1}{2}\operatorname{grad}\ln|\boldsymbol{v}|^2\right\} = 0. \tag{21}$$

Obviously, according to identity (19)*, conservation laws* (20) *and* (21) *are equivalent.*

2. The conservation law for vector lines of a plane vector field $\boldsymbol{v}(x, y)$ and its equivalence to conservation laws (20), (21)

It is possible to geometrically interprete conservation laws (20), (21) as a certain property of the vector lines L_τ of the vector field $\boldsymbol{v}(x, y)$, related to the curvature vector of these curves L_τ. This property represents a conservation law for a set of the curves L_τ. For this purpose, let us apply results of [1, Theorem 2].

Let us consider a set $\{L_\tau\}$ of the vector lines L_τ of the vector field $\boldsymbol{v} = \boldsymbol{v}(x, y) = |\boldsymbol{v}|\boldsymbol{\tau}$ with the Frene basis $(\boldsymbol{\tau}, \boldsymbol{\nu})$ ($\boldsymbol{\tau}$ is the direction of the vector field $\boldsymbol{v}$ and at the same time — unit tangential vector of the curve L_τ, $\boldsymbol{\nu}$ being its unit normal), continuously filling some domain D on the plane with the rectangular coordinates x, y. Concerning the set $\{L_\tau\}$, everywhere below we will assume the following conditions to be satisfied:

(A) One and only one curve $L_\tau \in \{L_\tau\}$ passes at any point $(x, y) \in D$ so that curves L_τ do not intersect at any point $(x, y) \in D$.

(B) At any point (x, y) of any curve $L_\tau \in \{L_\tau\}$ there exists a Frene basis $(\boldsymbol{\tau}, \boldsymbol{\nu})$, the Frene unit vectors $\boldsymbol{\tau}$ and $\boldsymbol{\nu}$ being one-valued vector functions of the variables x, y in the domain D: $\boldsymbol{\tau} = \boldsymbol{\tau}(x, y)$, $\boldsymbol{\nu} = \boldsymbol{\nu}(x, y)$. Thus, in D, two mutually orthogonal vector fields of the unit vectors $\boldsymbol{\tau}$ and $\boldsymbol{\nu}$ are defined. We consider the unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$ along axes of the coordinates x, y and the unit vectors $\boldsymbol{\tau}$, $\boldsymbol{\nu}$ to form the right-hand side system of vectors.

(C) At any point $(x, y) \in D$, there exist quantities $\operatorname{div}\boldsymbol{\tau}$, $\operatorname{rot}\boldsymbol{\tau}$, $\operatorname{div}\boldsymbol{\nu}$, $\operatorname{rot}\boldsymbol{\nu}$, i.e., the vector fields $\boldsymbol{\tau}(x, y)$, $\boldsymbol{\nu}(x, y)$ are sufficiently smooth.

To the given set of the vector lines $\{L_\tau\}$ in D there corresponds a set $\{L_\nu\}$ of the curves L_ν, orthogonal to the curves L_τ. The tangential unit vector of the curve L_ν coincides with the normal unit vector $\boldsymbol{\nu}$ of the curve L_τ, and the normal unit vectors $\boldsymbol{\eta}$ to the curve L_ν coincides with a tangential unit vector $\boldsymbol{\tau}$ of the curve L_τ to within a sign. Sets of the curves $\{L_\tau\}$ and $\{L_\nu\}$ will be called *mutually orthogonal.* For a curve $L_\nu \in \{L_\nu\}$, the Frene equations look like $\dfrac{d\boldsymbol{\nu}}{ds_\nu} = k_\nu\boldsymbol{\eta}$, $\dfrac{d\boldsymbol{\eta}}{ds_\nu} = -k_\nu\boldsymbol{\nu}$, where s_ν is a natural parameter (variable length) of the curve L_ν, $\dfrac{d\boldsymbol{\nu}}{ds_\nu} = (\boldsymbol{\nu}\cdot\nabla)\boldsymbol{\nu}$ and $\dfrac{d\boldsymbol{\eta}}{ds_\nu} = (\boldsymbol{\nu}\cdot\nabla)\boldsymbol{\eta}$ are derivatives of the vectors $\boldsymbol{\nu}$ and $\boldsymbol{\eta}$ in the direction $\boldsymbol{\nu}$, k_ν and $k_\nu\boldsymbol{\eta}$ are the curvature and the curvature vector of the curve L_ν. The curves L_τ are vector lines of the vector field $\boldsymbol{v} = |\boldsymbol{v}|\boldsymbol{\tau}$ and of the vector field of its directions $\boldsymbol{\tau} = \boldsymbol{v}/|\boldsymbol{v}|$, and the curves L_ν are vector lines of the vector field of the normals $\boldsymbol{\nu}$ of the curves L_τ.

Theorem 2 [1] for the case, when a set of the curves $\{L_\tau\}$ is a set of vector lines of some vector field $\boldsymbol{v}(x, y) = |\boldsymbol{v}|\boldsymbol{\tau}$, can be formulated as follows:

Theorem 3 (A conservation law for vector lines of a plane vector field $\boldsymbol{v}(x,y)$). *Let $\boldsymbol{v} = \boldsymbol{v}(x,y) = |\boldsymbol{v}|\boldsymbol{\tau}$ be a plane vector field with components $v_k(x,y) \subset C^2(D)$ $(k = 1,2)$, the modulus $|\boldsymbol{v}|$ and the direction $\boldsymbol{\tau} = \boldsymbol{v}/|\boldsymbol{v}|$ $(|\boldsymbol{\tau}| \equiv 1)$. Let $\{L_\tau\}$ be a set of vector lines of the field $\boldsymbol{v}$ (or $\boldsymbol{\tau}$) with the Frene unit vectors $(\boldsymbol{\tau},\boldsymbol{\nu})$, satisfying in the domain D conditions* (A)–(C). *Then in the domain D we have the identity:*

$$\operatorname{div} \boldsymbol{S}^* = 0, \tag{22}$$

where

$$\boldsymbol{S}^* \overset{\text{def}}{=} \frac{d\boldsymbol{\tau}}{ds} + \frac{d\boldsymbol{\nu}}{ds_\nu} = (\boldsymbol{\tau}\cdot\nabla)\boldsymbol{\tau} + (\boldsymbol{\nu}\cdot\nabla)\boldsymbol{\nu} = k\boldsymbol{\nu} + k_\nu\boldsymbol{\eta}, \tag{23}$$

$\boldsymbol{\eta} = -\boldsymbol{\tau}$. *Identity* (22) *means that for such a set of vector lines $\{L_\tau\}$ there always exists a vector field $\boldsymbol{S}^* = k\boldsymbol{\nu} + k_\nu\boldsymbol{\eta}$, representing the sum of the curvature vector $k\boldsymbol{\nu}$ of the vector line $L_\tau \in \{L_\tau\}$ and the curvature vector $k_\nu\boldsymbol{\eta}$ of the orthogonal curve $L_\nu \in \{L_\nu\}$, which is solenoidal. This property can be interpreted as existence in the differential geometry of vector lines of an arbitrary plane vector field $\boldsymbol{v}$ of the conservation law for the vector field $\boldsymbol{S}^*$, having differential form* (23).

In addition, in the domain D the following identity takes place:

$$\boldsymbol{S}^* = -\boldsymbol{S}(\boldsymbol{\tau})$$

and, hence, for the vector field $\boldsymbol{S}^*$ of form (23) *we have any representation obtained from formulas* (11)–(15)*:*

$$\boldsymbol{S}^* = -\operatorname{rot}\{\alpha(x,y)\boldsymbol{k}\} = -\alpha_y\boldsymbol{i} + \alpha_x\boldsymbol{j} \quad \Rightarrow \quad \operatorname{rot} \boldsymbol{S}^* = (\Delta\alpha)\boldsymbol{k}, \tag{24}$$

$$\begin{aligned}\boldsymbol{S}^* &= -(\boldsymbol{\tau}\operatorname{div}\boldsymbol{\tau} + \boldsymbol{\tau}\times\operatorname{rot}\boldsymbol{\tau}) = -\boldsymbol{\tau}\operatorname{div}\boldsymbol{\tau} + k\boldsymbol{\nu} = -(\boldsymbol{\nu}\operatorname{div}\boldsymbol{\nu} + \boldsymbol{\nu}\times\operatorname{rot}\boldsymbol{\nu}),\\ &= -(\boldsymbol{\tau}\operatorname{div}\boldsymbol{\tau} + \boldsymbol{\nu}\operatorname{div}\boldsymbol{\nu}) = -(\boldsymbol{\tau}\times\operatorname{rot}\boldsymbol{\tau} + \boldsymbol{\nu}\times\operatorname{rot}\boldsymbol{\nu}).\end{aligned} \tag{25}$$

The vector field $\boldsymbol{S}^*$ of form (23)*, in addition, can be expressed only in terms of one curvature vector $k\boldsymbol{\nu} = \boldsymbol{\tau}_s = (\boldsymbol{\tau}\cdot\nabla)\boldsymbol{\tau}$ of the vector line L_τ of the field $\boldsymbol{v}$ by the formula*

$$\boldsymbol{S}^* = \boldsymbol{\tau}_s + \{(\boldsymbol{\tau}_s\cdot\nabla)\boldsymbol{\tau}_s - \boldsymbol{\tau}_s(\operatorname{grad}\ln|\boldsymbol{\tau}_s|\cdot\boldsymbol{\tau}_s)\}/|\boldsymbol{\tau}_s|^2. \tag{26}$$

Corollary 3. *Under conditions of Theorem 2, identity* (19) *can be noted as*

$$\frac{\boldsymbol{v}\operatorname{div}\boldsymbol{v} + \boldsymbol{v}\times\operatorname{rot}\boldsymbol{v}}{|\boldsymbol{v}|^2} - \frac{1}{2}\operatorname{grad}\ln|\boldsymbol{v}|^2 = -\boldsymbol{S}^*. \tag{27}$$

Hence, conservation law (22) *for vector lines of the vector field $\boldsymbol{v}(x,y)$, conservation law* (21) *for the vector field $\boldsymbol{v}(x,y)$ and conservation law* (20) *for the field of the directions $\boldsymbol{\tau}$ of the vector field $\boldsymbol{v}(x,y)$ are equivalent identities:*

$$\operatorname{div} \boldsymbol{S}^* = 0 \quad \Leftrightarrow \quad \operatorname{div}\Big\{\frac{\boldsymbol{v}\operatorname{div}\boldsymbol{v} + \boldsymbol{v}\times\operatorname{rot}\boldsymbol{v}}{|\boldsymbol{v}|^2} - \frac{1}{2}\operatorname{grad}\ln|\boldsymbol{v}|^2\Big\} = 0$$
$$\Leftrightarrow \quad \operatorname{div}\boldsymbol{S}(\boldsymbol{\tau}) = 0. \tag{28}$$

3. The cases of a plane potential and solenoidal field $\boldsymbol{v}(x,y)$. Identities for a scalar function $u(x,y)$ and conservation laws

For a plane potential field $\boldsymbol{v} = \operatorname{grad} u(x,y) = u_x\boldsymbol{i} + u_y\boldsymbol{j}$, and for a plane solenoidal field, representable without loss of generality as $\boldsymbol{v} = \operatorname{rot}\{u(x,y)\boldsymbol{k}\} = u_y\boldsymbol{i} - u_x\boldsymbol{j}$, the angle $\alpha = \alpha(x,y)$ is given by the formula (7) with $v_1 = u_x$, $v_2 = u_y$ and $v_1 = u_y$, $v_2 = -u_x$, respectively, and for the quantities $|\boldsymbol{v}|$, α_x, α_y, $\operatorname{grad}\alpha$, $\operatorname{rot}(\alpha\boldsymbol{k})$, $\boldsymbol{Q}$, $\boldsymbol{S}^*$, $\boldsymbol{S}$, we obtain the same formulas in terms of derivatives of the function $u(x,y)$. Thus, in both cases we have

$$|\boldsymbol{v}| = \sqrt{g}, \qquad g = |\operatorname{grad} u|^2 = |\operatorname{rot}(u\boldsymbol{k}))|^2 = u_x^2 + u_y^2, \tag{29}$$
$$\boldsymbol{Q} = \frac{\Delta u}{u_x^2+u_y^2}\operatorname{grad} u,$$

$$\boldsymbol{S}^* = -\{\boldsymbol{\tau}\operatorname{div}\boldsymbol{\tau} + \boldsymbol{\tau}\times\operatorname{rot}\boldsymbol{\tau}\} = -\operatorname{rot}(\alpha\boldsymbol{k}) = -(\alpha_y\boldsymbol{i} - \alpha_x\boldsymbol{j}) = \boldsymbol{P} - \boldsymbol{Q} \tag{30}$$
$$= -\Big\{\frac{\Delta u}{u_x^2+u_y^2}\operatorname{grad} u - \frac{1}{2}\operatorname{grad}\ln(u_x^2+u_y^2)\Big\} \tag{31}$$
$$= \frac{u_y u_{xy} - u_x u_{yy}}{u_x^2+u_y^2}\,\boldsymbol{i} + \frac{u_x u_{xy} - u_y u_{xx}}{u_x^2+u_y^2}\,\boldsymbol{j} = \frac{(\operatorname{grad} u\times\nabla)\times\operatorname{grad} u}{|\operatorname{grad} u|^2}, \tag{32}$$

$$\operatorname{div}\boldsymbol{S}^* = \operatorname{div}\boldsymbol{S} = 0, \qquad \operatorname{rot}\boldsymbol{S}^* = -\operatorname{rot}\boldsymbol{S} = (\Delta\alpha)\boldsymbol{k}. \tag{33}$$

From Theorem 1 in both cases follows:

Theorem 4. *For any scalar function $u(x,y) \in C^2(D)$ with the property $u_x^2 + u_y^2 \neq 0$ in D we have the identity*

$$\boldsymbol{Q} \overset{\text{def}}{=} \frac{\Delta u}{u_x^2+u_y^2}\operatorname{grad} u = \frac{\Delta u}{\sqrt{u_x^2+u_y^2}}\,\boldsymbol{\tau}$$
$$= \operatorname{grad}\Big\{\frac{1}{2}\ln(u_x^2+u_y^2)\Big\} + \operatorname{rot}\{\alpha(x,y)\boldsymbol{k}\} \quad \Leftrightarrow \tag{34}$$
$$\boldsymbol{R} \overset{\text{def}}{=} \frac{\Delta u}{u_x^2+u_y^2}\operatorname{rot}(u\boldsymbol{k}) = -\frac{\Delta u}{\sqrt{u_x^2+u_y^2}}\,\boldsymbol{\nu}$$
$$= -\operatorname{grad}\alpha(x,y) + \operatorname{rot}\Big\{\frac{1}{2}\ln(u_x^2+u_y^2)\boldsymbol{k}\Big\}. \tag{35}$$

Here $\alpha = \alpha(x, y)$ *is the slope angle of the vector* $\operatorname{grad} u$ *to an axis* Ox, *defined from formula* (7) *with* $v_1 = u_x$, $v_2 = u_y$, $\boldsymbol{\tau} = \operatorname{grad} u / |\operatorname{grad} u| = (u_x \boldsymbol{i} + u_y \boldsymbol{j}) / \sqrt{u_x^2 + u_y^2} = \cos\alpha\ \boldsymbol{i} + \sin\alpha\ \boldsymbol{j}$ *is the direction* (*the unit vector*) *of the vector field* $\operatorname{grad} u$ *or a unit tangent vector of a vector line of the field* $\operatorname{grad} u$, $\boldsymbol{\nu} = -\operatorname{rot}(u\boldsymbol{k}) / |\operatorname{rot}(u\boldsymbol{k})| = (-u_y \boldsymbol{i} + u_x \boldsymbol{j}) / \sqrt{u_x^2 + u_y^2} = -\sin\alpha\ \boldsymbol{i} + \cos\alpha\ \boldsymbol{j}$ *is the direction* (*the unit vector*) *of the normal to this line. Each of vector identities* (34), (35) *is equivalent to the system of scalar identities*

$$\begin{aligned} \big(\ln \sqrt{u_x^2 + u_y^2}\,\big)_x &= -\alpha_y + \frac{\Delta u}{u_x^2 + u_y^2}\, u_x, \\ \big(\ln \sqrt{u_x^2 + u_y^2}\,\big)_y &= \alpha_x + \frac{\Delta u}{u_x^2 + u_y^2}\, u_y. \end{aligned} \tag{36}$$

According to equalities (29), $\Delta u = \operatorname{div}(\sqrt{g}\,\boldsymbol{\tau})$, $\boldsymbol{\tau} = \boldsymbol{\tau}(\alpha)$, $\boldsymbol{\nu} = \boldsymbol{\nu}(\alpha)$ each of identities (34), (35) can be considered to be a relation only between the two scalar fields $g(x, y)$ and $\alpha(x, y)$ (between the modulus of the vector field $\operatorname{grad} u$ or $\operatorname{rot}(u\boldsymbol{k})$ and the angle characterizing its direction).

Corollary 4. *Identity* (34) *can be represented as*

$$\boldsymbol{T} \overset{\text{def}}{=} \frac{1}{2} \operatorname{grad} \ln(u_x^2 + u_y^2) - \frac{\Delta u}{u_x^2 + u_y^2} \operatorname{grad} u = \boldsymbol{S}^*, \tag{37}$$

where the vector field $\boldsymbol{S}^*$ *is expressed by any of formulas* (23), (24)–(26), (31), (32).

Other corollaries of Theorem 4 are considered in [2–5].

Corollary 5. *For any function* $u(x, y) \in C^3(D)$ $(|\operatorname{grad} u| \neq 0)$, *the identities of third order are valid:*

$$\Delta \ln \sqrt{u_x^2 + u_y^2} = \Big(u_x \frac{\Delta u}{g}\Big)_x + \Big(u_y \frac{\Delta u}{g}\Big)_y = \operatorname{div} \boldsymbol{Q}, \tag{38}$$

$$\Delta\alpha\, \boldsymbol{k} = -\Big\{\Big(u_y \frac{\Delta u}{g}\Big)_x - \Big(u_x \frac{\Delta u}{g}\Big)_y\Big\} \boldsymbol{k} = -\operatorname{rot} \boldsymbol{Q} \quad \Rightarrow \tag{39}$$

$$\Delta \operatorname{Ln}(u_x \pm i u_y) = \operatorname{div} \boldsymbol{Q} \mp i(\operatorname{rot} \boldsymbol{Q} \cdot \boldsymbol{k}),$$

as well as divergent identities

$$\operatorname{div} \boldsymbol{T} = 0 \quad \Leftrightarrow \quad \operatorname{div} \boldsymbol{S}^* = 0, \tag{40}$$

$$\operatorname{div} \boldsymbol{M} = 0, \tag{41}$$

where

$$\boldsymbol{T} \stackrel{\text{def}}{=} \frac{1}{2}\operatorname{grad}\ln(u_x^2+u_y^2) - \frac{\Delta u}{u_x^2+u_y^2}\operatorname{grad} u = \boldsymbol{P} - \boldsymbol{Q}, \tag{42}$$

$$\boldsymbol{M} \stackrel{\text{def}}{=} \operatorname{grad}\alpha(x,y) + \frac{\Delta u}{u_x^2+u_y^2}\operatorname{rot}\{u(x,y)\boldsymbol{k}\} = \operatorname{grad}\alpha + \boldsymbol{R}. \tag{43}$$

Corollary 6. *The equation of a vector line of the field* $\operatorname{grad} u(x,y)$ *can be represented as* $\boldsymbol{S}^* = (\boldsymbol{S}^*\cdot\boldsymbol{\tau})\boldsymbol{\tau} + \boldsymbol{\tau}_s$ *or as* $\operatorname{rot}(\alpha\boldsymbol{k}) = (\operatorname{rot}(\alpha\boldsymbol{k})\cdot\boldsymbol{\tau})\boldsymbol{\tau} - \boldsymbol{\tau}_s$*, i.e., in its classical equation* $\operatorname{grad}\ln\sqrt{g} = (\operatorname{grad}\ln\sqrt{g}\cdot\boldsymbol{\tau})\boldsymbol{\tau} + \boldsymbol{\tau}_s$*, the replacement* $\operatorname{grad}\ln\sqrt{g}$ *for* $\boldsymbol{S}^*$ *or for* $(-\operatorname{rot}(\alpha\boldsymbol{k}))$ *is admissible.*

4. Differential identities and conservation laws for a time field (for the eikonal equation solutions)

4.1. Differential identities for the eikonal equation. Let $c(x,y) = 1/n(x,y)$ be a propagation velocity of signals (waves) of any nature in the plane x,y with their kinematics satisfying the Fermat principle, $n = n(x,y)$ be a refractive index; and t be a parameter of the point source of the waves, determining its coordinates x,y. Assume D to be a domain in the plane x,y and $\tau = \tau(x,y,t)$—a solution to equation (4) for $n(x,y) \geq n_0 > 0$. The function $\tau(x,y,t)$ is the travel time (the time field) of a signal along the ray (along the geodesic of metric $ds^2 = n^2(x,y)(dx^2+dy^2)$), connecting the source with the parameter t and the point (x,y).

Let a source be located at the point M^* ($t = \text{const}$), then $\tau = \tau(x,y)$. The potential vector field $\boldsymbol{v} = \operatorname{grad}\tau(x,y) = |\boldsymbol{v}|\boldsymbol{\tau}$ has the modulus $|\boldsymbol{v}| = n(x,y)$ and the direction $\boldsymbol{\tau} = \cos\alpha\,\boldsymbol{i} + \sin\alpha\,\boldsymbol{j} = \operatorname{grad}\tau/n$, where $\alpha = \alpha(x,y)$ is the angle the ray (the vector $\operatorname{grad}\tau$ or $\boldsymbol{\tau}$) makes with the axis Ox at the point x,y;it is defined by formula (7) with $v_1 = \tau_x$, $v_2 = \tau_y$. The ray is the vector line L_τ of the field $\operatorname{grad}\tau$ with the unit tangent vector $\boldsymbol{\tau}$ and the unit normal $\boldsymbol{\nu} = -\sin\alpha\,\boldsymbol{i} + \cos\alpha\,\boldsymbol{j} = -\operatorname{rot}(\tau\boldsymbol{k})/n$. The vectors $\boldsymbol{\tau}$, $\boldsymbol{\nu}$ form the right-hand side system of the Frene unit vectors of a ray. The rays L_τ and fronts of the waves L_ν (lines of the level $\tau(x,y,t) = \text{const}$ at $t = \text{const}$) form mutually orthogonal sets of curves for any location of the source $t = \text{const}$. Thus, the unit vectors $\boldsymbol{\nu}$, $\boldsymbol{\eta} = -\boldsymbol{\tau}$ make up the right-hand system of the Frene unit vectors for a front curve L_ν ($\boldsymbol{\nu}$ is the tangential unit vector of the curve L_ν, $\boldsymbol{\eta}$ is the unit vector of its normal). The vectors $k\boldsymbol{\nu}$ and $k_\nu\boldsymbol{\eta}$, where k is the curvature of the ray L_τ, k_ν is the curvature of the front L_ν, represent curvature vectors of the ray and the front, respectively. The vector $\boldsymbol{S}^*$, defined by equality (23) as

$$\boldsymbol{S}^* = k\boldsymbol{\nu} + k_\nu\boldsymbol{\eta}, \tag{44}$$

in the case in question is the sum of the curvature vector of a ray L_τ and the curvature vector of a front L_ν. The vector $\boldsymbol{S}^*$ can be expressed by any of formulas (24)–(26), (31), (32) with $u = \tau$.

The classical ray equation has the form

$$\operatorname{grad} \ln n = (\operatorname{grad} \ln n \cdot \boldsymbol{\tau})\boldsymbol{\tau} + \boldsymbol{\tau}_s, \tag{45}$$

where $\boldsymbol{\tau}_s = (\boldsymbol{\tau}\cdot\nabla)\boldsymbol{\tau} = \operatorname{rot}\boldsymbol{\tau}\times\boldsymbol{\tau}$ is the derivative of the vector $\boldsymbol{\tau}$ along the ray (in the direction $\boldsymbol{\tau}$); $\boldsymbol{\tau} = d\boldsymbol{r}/ds$, $\boldsymbol{r} = x(s)\boldsymbol{i} + y(s)\boldsymbol{j}$; $x = x(s)$, $y = y(s)$ are parametric ray equations with the parameter s (the curve length).

At the point M^*, the value $\alpha(x, y)$ is not defined, and it is possible to show that $\lim_{(x,y)\to M^*} \tau(\Delta\tau/n^2) = 1$. From Theorem 4 with $u = \tau$ according to (4) follows

Theorem 5 (The basic identity for the eikonal equation). *Let $\tau(x, y) \in C^2(D)$ be the solution to equation* (4) *in some domain D with the parameter $n(x, y) \in C^1(D)$. Then the following identity holds:*

$$\boldsymbol{Q} \stackrel{\text{def}}{=} \frac{\Delta\tau}{n^2}\operatorname{grad}\tau = \frac{\Delta\tau}{n}\boldsymbol{\tau} = \operatorname{grad}\ln n + \operatorname{rot}(\alpha\boldsymbol{k}) \quad \Leftrightarrow \tag{46}$$

$$\boldsymbol{R} \stackrel{\text{def}}{=} \frac{\Delta\tau}{n^2}\operatorname{rot}(\tau\boldsymbol{k}) = -\frac{\Delta\tau}{n}\boldsymbol{\nu} = -\operatorname{grad}\alpha + \operatorname{rot}(\ln n\boldsymbol{k}), \tag{47}$$

which is equivalent to a system of the scalar identities

$$(\ln n)_x = -\alpha_y + (\Delta\tau/n^2)\,\tau_x, \qquad (\ln n)_y = \alpha_x + (\Delta\tau/n^2)\,\tau_y$$

or a system

$$\frac{\partial \ln n}{\partial \nu} = \frac{\partial\alpha}{\partial s}, \qquad \frac{\partial \ln n}{\partial s} = -\frac{\partial\alpha}{\partial\nu} + \frac{\Delta\tau}{n},$$

where $\partial/\partial s = (\tau_x/n)\,\partial/\partial x + (\tau_y/n)\,\partial/\partial y = (\boldsymbol{\tau}\cdot\nabla)$, $\partial/\partial\nu = -(\tau_y/n)\,\partial/\partial x + (\tau_x/n)\,\partial/\partial y = (\boldsymbol{\nu}\cdot\nabla)$ are operators of differentiation of a scalar function in the direction of the vector $\boldsymbol{\tau}$ and $\boldsymbol{\nu}$, respectively.

The basic identity (46) or (47) has the following physical and geometric meaning. To any smooth refractive index $n(x, y)$ and to any smooth time field $\tau(x, y)$ there correspond vector fields $\boldsymbol{Q}$ and $\boldsymbol{R}$, explicitly defined by (46), (47) with the following properties. A potential vector component of the vector $\boldsymbol{Q}$ is determined only by the index refractive $n(x, y)$ and coincides with the vector field $\operatorname{grad}\ln n$, while the rotational (solenoidal) component of the vector $\boldsymbol{Q}$ is determined only by the angular characteristic $\alpha(x, y)$ and coincides with $\operatorname{rot}(\alpha\boldsymbol{k})$. The potential and the rotational components of the vector $\boldsymbol{R}$ are equal to $(-\operatorname{grad}\alpha)$ and $\operatorname{rot}(\ln n\boldsymbol{k})$, respectively. As discussed below, the fields $\boldsymbol{Q}$ and $\boldsymbol{R}$ are related to a well-known medium characteristic such as the geometric divergence of rays. Since $\Delta\tau = \operatorname{div}\operatorname{grad}\tau$, $\operatorname{grad}\tau = n\boldsymbol{\tau}$, $\boldsymbol{\tau} = \boldsymbol{\tau}(\alpha)$, $\boldsymbol{\nu} = \boldsymbol{\nu}(\alpha)$, the basic identity (46) or (47) can also be treated as a relation only between the refraction index $n(x, y)$ and the ray angle slope $\alpha(x, y)$.

5. Conservation laws for a time field $\boldsymbol{\tau}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{t})$ (for the eikonal equation solutions)

From Theorem 5 follows

Corollary 7. *Identity* (46) *can be represented as*

$$\boldsymbol{T} = \boldsymbol{S}^*, \tag{48}$$

where

$$\boldsymbol{T} \overset{\text{def}}{=} \operatorname{grad}\ln n - \frac{\Delta\tau}{n^2}\operatorname{grad}\tau \tag{49}$$

$$= -\operatorname{rot}(\alpha\boldsymbol{k}) = -\alpha_y\boldsymbol{i} + \alpha_x\boldsymbol{j} \tag{50}$$

$$= -(\boldsymbol{\tau}\operatorname{div}\boldsymbol{\tau} + \boldsymbol{\tau}\times\operatorname{rot}\boldsymbol{\tau}) \tag{51}$$

$$= \{(\tau_y\tau_{xy} - \tau_x\tau_{yy})\boldsymbol{i} + (\tau_x\tau_{xy} - \tau_y\tau_{xx})\boldsymbol{j}\}/n^2, \tag{52}$$

$$\boldsymbol{S}^* = k\boldsymbol{\nu} + k_\nu\boldsymbol{\eta}, \tag{53}$$

which gives an explicit expression for the vector field $\boldsymbol{S}^*$ (and $\boldsymbol{T}$), whose vector lines coincide with the lines of the level of the function $\alpha(x,y)$. We also have $\boldsymbol{S}^* = (\boldsymbol{S}^*\cdot\boldsymbol{\tau})\boldsymbol{\tau} + \boldsymbol{\tau}_s$, so that ray equation (45) *can be represented as* $\operatorname{rot}(\alpha\boldsymbol{k}) = (\operatorname{rot}(\alpha\boldsymbol{k})\cdot\boldsymbol{\tau})\boldsymbol{\tau} - \boldsymbol{\tau}_s$, *i.e.,* $\operatorname{grad}\ln n$ *in* (45) *can be replaced by $\boldsymbol{S}^*$ or by* $(-\operatorname{rot}(\alpha\boldsymbol{k}))$.

Corollary 8. *For any refractive index $n(x,y)\in C^2(D)$ and any solution* (*the time field*) $\tau(x,y)\in C^3(D)$ *of equation* (4), *identities are valid:*

$$\operatorname{div}\boldsymbol{S} = 0, \quad \operatorname{rot}\boldsymbol{S}^* = (\Delta\alpha)\boldsymbol{k} \quad \Rightarrow \tag{54}$$

$$\Delta\ln n = \Big(\tau_x\frac{\Delta\tau}{n^2}\Big)_x + \Big(\tau_y\frac{\Delta\tau}{n^2}\Big)_y = \operatorname{div}\boldsymbol{Q} = -(\operatorname{rot}\boldsymbol{R}\cdot\boldsymbol{k}), \tag{55}$$

$$\Delta\alpha = -\Big\{\Big(\tau_y\frac{\Delta\tau}{n^2}\Big)_x - \Big(\tau_x\frac{\Delta\tau}{n^2}\Big)_y\Big\} = -(\operatorname{rot}\boldsymbol{Q}\cdot\boldsymbol{k}) = -\operatorname{div}\boldsymbol{R}, \tag{56}$$

$$\Delta\operatorname{Ln}(\tau_x \pm i\tau_y) = \operatorname{div}\boldsymbol{Q} \mp i(\operatorname{rot}\boldsymbol{Q}\cdot\boldsymbol{k}),$$

which give the explicit formulas for $\Delta\ln n$, $\Delta\alpha$ in terms of divergence and rotor of the same vector field $\boldsymbol{Q}$ or $\boldsymbol{R}$, defined in (46), (47).

Identities (55), (56) *mean that the field source intensity $\boldsymbol{Q}$ is determined only by the refractive index $n(x,y)$ and is equal to $\Delta\ln n$, while the field vortex intensity $\boldsymbol{Q}$ is determined only by the ray slope angle $\alpha(x,y)$ and is equal to $(-\Delta\alpha)$. For the field $\boldsymbol{R}$ the source and the vortex intensities are $(-\Delta\alpha)$ and $(-\Delta\ln n)$, respectively.*

In [4, 5], the following conservation laws for the non-force time field $\tau = \tau(x,y,t)$ in the two-dimensional kinematic seismics (geometric optics) are obtained. They follow from formulas (48)–(56).

Theorem 6 (Conservation laws for the eikonal equation solutions). *Let $n(x, y) \in C^2(D)$, $\tau(x, y, t) \in C^3(D)$ for any $t = const$ be a solution to the eikonal equation* (4). *Then in the domain D, the following conservation laws take place*:

(I) $\operatorname{div} \boldsymbol{T} = 0$ *at any $t = const$;*

(II) $\dfrac{\partial}{\partial t} \operatorname{div} \boldsymbol{Q} = 0 \quad \Leftrightarrow \quad \operatorname{div} \dfrac{\partial \boldsymbol{Q}}{\partial t} = 0 \quad \Leftrightarrow \quad \operatorname{div} \dfrac{\partial \boldsymbol{T}}{\partial t} = 0.$

Here the vector field $\boldsymbol{T}$ is determined by any of formulas (49)–(52),

$$\boldsymbol{Q} = \frac{\Delta \tau}{n^2} \operatorname{grad} \tau, \quad \operatorname{div} \boldsymbol{Q} = \Delta \ln n, \quad \operatorname{rot} \boldsymbol{Q} = -(\Delta \alpha) \boldsymbol{k} = -\operatorname{rot} \boldsymbol{T}.$$

The identity $\operatorname{div} \boldsymbol{T} = 0$ *means the existence of a time field conservation law in the two-dimensional kinematic seismics* (*geometric optics*) *with the differential form* $\operatorname{div} \boldsymbol{T} = 0$ *and the integral form* $\int_S (\boldsymbol{T} \cdot \boldsymbol{\eta})\, ds = 0$ (*for the flux of the vector field $\boldsymbol{T}$ through the boundary S of the domain D with the normal $\boldsymbol{\eta}$*). *The conservation law* (II) *means the existence of a differential wave-propagation invariant I with the following physical meaning. Although the time field function τ depends on the point source parameter t* (*the point source location in a medium*): $\tau = \tau(x, y, t)$, *the value* $I = \operatorname{div}\{(\Delta \tau / n^2) \operatorname{grad} \tau\} = \operatorname{div} \boldsymbol{Q} = -(\operatorname{rot} \boldsymbol{R} \cdot \boldsymbol{k}) = \Delta \ln n$ *is independent of t and of τ, i. e., it is invariant with respect to the source location. For the conservation law* (II) *the role of "time t" belongs to the point source parameter t, determining its location. To differential invariant* $I = \operatorname{div} \boldsymbol{Q}$, *there corresponds an integral invariant* $\int_S (\boldsymbol{Q} \cdot \boldsymbol{\eta})\, ds$, *equal to the flux of the vector $\boldsymbol{Q}$ through the boundary S of an arbitrary domain D with the normal $\boldsymbol{\eta}$, also independent of the source location.*

The following geometric interpretation of conservation laws (I) and (II) of Theorem 6 follows from Theorem 6 and consequences 3 or formulas (48)–(53).

Corollary 9. *The conservation law* (I) *of Theorem* 6 *for a time field $\tau(x, y, t)$ for any fixed location of a point source is equivalent to the following geometric property of the ray curves L_τ and the front curves L_ν orthogonal to them: the vector field $\boldsymbol{S}^*$ of the form of* (53), *which is the sum of curvature vectors of rays and fronts, is a solenoidal field:* $\operatorname{div} \boldsymbol{S}^* = 0$. *The conservation law* (II) *of Theorem* 6 *is equivalent to a conservation law* $\operatorname{div}(\partial \boldsymbol{S}^* / \partial t) = 0$ *for the vector field $\boldsymbol{S}^*$.*

5.1. Physical and a geometric meaning of the vector fields $\boldsymbol{P}$, $\boldsymbol{S}$, $\boldsymbol{Q}$, $\boldsymbol{R}$ and the quantities $\boldsymbol{\Delta_2 \tau}$, $\boldsymbol{\alpha_t}$. The relation with the geometric divergence of rays, the ray transformation, the differential geometry and the group analysis. In [4, 5] it is obtained

Theorem 7. *Let t be the point source parameter, $n(x,y) \in C^1(D)$, $\tau = \tau(x,y,t) \in C^3(D)$ be a solution of equation (4), $\alpha = \alpha(x,y,t)$; $\alpha_0 = \alpha_0(t,x,y)$ be the value of the direction angle α of the ray* $(\operatorname{grad}\tau)$ *at the source point with the parameter t, passing through the point (x,y); the value $\alpha_0(x,y,t)$ is a constant on any ray; $D = D(x,y,t)$ is the geometric divergence of rays with the vertex (source) at the point with parameter t; $J = J(x,y,t) \overset{\text{def}}{=} \partial\,(x,y)/\partial\,(\tau,\alpha_0)$ is the Jacobian of the (ray) transformation of the Cartesian coordinates x,y to the coordinates ray τ,α_0 (for a fixed t); we have $D^2 = n|J|$. Then the formulas are valid:*

$$\tau_t = \tau_t(x,y,t) \overset{\text{def}}{=} f(t,\alpha_0), \qquad (\operatorname{grad}\alpha_0 \cdot \boldsymbol{\tau}) = 0,$$

$$\alpha_t \overset{\text{def}}{=} \frac{\partial\alpha}{\partial t} = \frac{\tau_x\tau_{ty} - \tau_y\tau_{tx}}{n^2} = \frac{f_{\alpha_0}(t,\alpha_0)}{nD^2} = \frac{f_{\alpha_0}(t,\alpha_0)}{n^2 J},$$

$$(\operatorname{grad}\ln\alpha_t \cdot \boldsymbol{\tau}) = -(\operatorname{grad}\ln(nD^2)\cdot\boldsymbol{\tau}),$$

$$\Delta_2\tau \overset{\text{def}}{=} \frac{\Delta\tau}{n^2} = -\frac{(\operatorname{grad}\ln\alpha_t\cdot\boldsymbol{\tau})}{n} = \frac{(\operatorname{grad}\ln(nD^2)\cdot\boldsymbol{\tau})}{n},$$

$$\boldsymbol{Q} = (n\Delta_2\tau)\boldsymbol{\tau} = (\operatorname{grad}\ln(nD^2)\cdot\boldsymbol{\tau})\boldsymbol{\tau} = \frac{(\boldsymbol{R}_t\cdot\boldsymbol{\tau})}{\alpha_t}\boldsymbol{\tau},$$

$$\boldsymbol{R} = -(n\Delta_2\tau)\boldsymbol{\nu} = -(\operatorname{grad}\ln(nD^2)\cdot\boldsymbol{\tau})\boldsymbol{\nu} = -\frac{(\boldsymbol{Q}_t\cdot\boldsymbol{\nu})}{\alpha_t}\boldsymbol{\nu}.$$

Thus, the vector $\boldsymbol{Q}$ is a vector component of the vector $\operatorname{grad}\ln(nD^2)$ *in the tangential direction $\boldsymbol{\tau}$ of the ray; while the vector $\boldsymbol{R}$ is perpendicular to the vector $\boldsymbol{Q}$ and to the ray, and its value equals the modulus of the scalar component of the vector* $\operatorname{grad}\ln(nD^2)$ *in the direction $\boldsymbol{\tau}$.*

Thus, the vector fields $\boldsymbol{Q}$, $\boldsymbol{R}$, $\boldsymbol{P}$, $\boldsymbol{S}$, entering into identities (46)–(47) and constructed from the potential non-force vector field $\boldsymbol{v} = \operatorname{grad}\tau(x,y)$, where the potential $\tau(x,y)$ is a time field (the solution of eikonal equation (4)), have quite certain physical and (or) geometric meaning. Namely, the field $\boldsymbol{P}$ is a gradient of the scalar field $\ln n(x,y)$, where $n(x,y)$ is the refractive index of the medium, while $\boldsymbol{S} = \operatorname{rot}\{\alpha(x,y)\boldsymbol{k}\}$, where $\alpha(x,y)$ is the ray inclination angle. The physical and the geometric meaning of the fields $\boldsymbol{Q}$ and $\boldsymbol{R}$ (along with their interpretation in Subsection 4.1) is described in Theorem 7 and related to a well-known physical characteristic of the wave kinematics and the medium, namely, to the geometric divergence of rays $D(x,y,t)$.

According to the differential geometry [15, § 45, § 79–83; 16] and is explained in [2, 4, 5], expressions $\Delta_1\tau$ (in (4)), $\Delta_2\tau$ are the first and the second Beltramy differential parameters of the function $\tau(x,y)$ for a surface with the Riemannian metric $ds^2 = n^2(x,y)(dx^2 + dy^2)$, the quantity $K(x,y) = -(\Delta\ln n^2)/(2n^2)$ is its Gaussian curvature, the quantity

$\varkappa = 1/\rho_g = -\Delta_2\tau = -\Delta\tau/n^2$ is the geodesic curvature of the front curve $\tau(x,y) = \text{const}$ on this surface. We have $\boldsymbol{Q} = -\varkappa \operatorname{grad}\tau$, $K(x,y) = -$-$\operatorname{div}\boldsymbol{Q}/n^2$. Moreover, expressions $J^6 = -\alpha_t$, $J^7 = \Delta_1\tau$, $J^4 = \Delta_2\tau$, $J^{11} = K(x,y)$ are differential invariants of some Lie group, and the identities in Corollary 8 are the relations between them (for the relation of the expressions J^j to the group analysis, see [1, 17, 18]). The determination of integrals over a curve and over an area and the determination of functionals in the inverse problem are considered [4, 5].

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