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Two algorithms for calculation of theoretical seismograms for anisotropic media

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Two algorithms for the numerical modeling of the elastic wave propagation in non-homogeneous anisotropic media are proposed. A common feature of both algorithms is the reduction of the 3D problem of elastic wave propagation to a series of 1D problems, by means of the finite integral Fourier transform with respect to the horizontal coordinate x and y.

Algorithm I is based on the explicit finite difference method with the second order approximation with respect to time and with the fourth order approximation with respect to the spatial variable. Algorithm II is based on employing the Laguerre transformation with respect to the time coordinate and the finite difference approximation with respect to the spatial variable z. Both the proposed algorithms can be effectively implemented on massively-parallel computer architectures.

1. Introduction

Currently, there are a few algorithms for the numerical solution of the direct dynamic problem of the elastic wave propagation in an anisotropic medium. We mean the algorithms based on a combination of the finite integral transforms with the finite difference methods [3, 4]. One of the "bottlenecks" of the proposed methods is a low (as usual, the second) approximation order, resulting in certain constraints, when calculating elastic waves at large distances and a considerable difference in elastic parameters of the model.

In works [1, 5], the new algorithm for the numerical simulation of the elastic wave propagation is proposed. It is based on a combination of the Laguerre spectral transform with respect to time with a finite difference scheme. It is shown that the proposed approach has many advantages over the currently available algorithms. In this paper, an attempt is made to apply this new approach to the solution of the direct dynamic problem for an anisotropic medium.

2. Statement of the problem

In the Cartesian coordinate system, the problem is formulated as follows: it is necessary to find components of the elastic displacements vector which would satisfy the following system of equations:

$$
\begin{aligned}
\rho\frac{\partial^2 U}{\partial t^2} &= c_{11}\frac{\partial^2 U}{\partial x^2} + c_{12}\frac{\partial^2 V}{\partial x\partial y} + c_{13}\frac{\partial^2 W}{\partial x\partial z} + c_{66}\left(\frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 V}{\partial y\partial x}\right) + \\
&\quad \frac{\partial}{\partial z}\left(c_{55}\left(\frac{\partial W}{\partial x} + \frac{\partial U}{\partial z}\right)\right) + F_x\left(t,x,y,z\right), \\
\rho\frac{\partial^2 V}{\partial t^2} &= c_{66}\left(\frac{\partial^2 U}{\partial x\partial y} + \frac{\partial^2 V}{\partial x^2}\right) + c_{22}\frac{\partial^2 V}{\partial y^2} + c_{12}\frac{\partial^2 U}{\partial x\partial y} + c_{23}\frac{\partial^2 W}{\partial y\partial z} + \\
&\quad \frac{\partial}{\partial z}\left(c_{44}\left(\frac{\partial V}{\partial z} + \frac{\partial W}{\partial y}\right)\right) + F_y\left(t,x,y,z\right), \\
\rho\frac{\partial^2 W}{\partial t^2} &= c_{55}\left(\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 U}{\partial x\partial z}\right) + c_{44}\left(\frac{\partial^2 V}{\partial y\partial z} + \frac{\partial^2 W}{\partial y^2}\right) + \\
&\quad \frac{\partial}{\partial z}\left(c_{13}\frac{\partial U}{\partial x} + c_{23}\frac{\partial V}{\partial y} + c_{33}\frac{\partial W}{\partial z}\right) + F_z\left(t,x,y,z\right)
\end{aligned} \tag{1}
$$

with the initial conditions

$$
U|_{t=0} = \left.\frac{\partial U}{\partial t}\right|_{t=0} = 0, \quad V|_{t=0} = \left.\frac{\partial V}{\partial t}\right|_{t=0} = 0, \quad W|_{t=0} = \left.\frac{\partial W}{\partial t}\right|_{t=0} = 0 \tag{2}
$$

and the boundary conditions

$$
\begin{gathered}
c_{55}\left(\frac{\partial W}{\partial x} + \frac{\partial U}{\partial z}\right)\Big|_{z=0} = 0, \qquad c_{44}\left(\frac{\partial V}{\partial z} + \frac{\partial W}{\partial y}\right)\Big|_{z=0} = 0, \\
\left(c_{13}\frac{\partial U}{\partial x} + c_{23}\frac{\partial V}{\partial y} + c_{33}\frac{\partial W}{\partial z}\right)\Big|_{z=0} = 0.
\end{gathered} \tag{3}
$$

The coefficients $c_{ij}(z)$ and the density $\rho(z)$ (parameters of the anisotropic medium orthotropic type) are arbitrary piecewise continuous positive functions of the variable z, F_x, F_y, F_z are components of the force vector, describing the action of the source localized in time and space

$$
\vec{F}(t,x,y,z) = F_x\vec{i} + F_y\vec{j} + F_z\vec{k}. \tag{4}
$$

For example, for the source of the "vertical force" type

$$
\vec{F}(t,x,y,z) = \delta(x)\delta(y)\delta(z-d)f(t)\vec{k}.
$$

Here d is the depth of the source along the vertical axis.

3. Transformations in horizontal variables

To solve the problem, let us make use of the finite integral transformations of the form:

$$\left\{\begin{matrix} u(t,n,m,z) \\ v(t,n,m,z) \\ w(t,n,m,z) \end{matrix}\right\} = \int_0^a\int_0^a \left\{\begin{matrix} U(t,x,y,z)\sin(k_n x)\cos(k_m y) \\ V(t,x,y,z)\cos(k_n x)\sin(k_m y) \\ W(t,x,y,z)\cos(k_n x)\cos(k_m y) \end{matrix}\right\} dx\,dy, \tag{5}$$

where

$$\left\{\begin{matrix} U(t,x,y,z) \\ V(t,x,y,z) \\ W(t,x,y,z) \end{matrix}\right\} = \sum_{n=0}^{\infty}\sum_{n=0}^{\infty} p_n s_m \left\{\begin{matrix} u(t,n,m,z)\sin(k_n x)\cos(k_m y) \\ v(t,n,m,z)\cos(k_n x)\sin(k_m y) \\ w(t,n,m,z)\cos(k_n x)\cos(k_m y) \end{matrix}\right\} dx\,dy. \tag{6}$$

After the respective transformations of equation (1), boundary conditions (2), and initial conditions (3) we obtain the following problem:

$$\begin{aligned}
\rho\frac{\partial^2 u}{\partial t^2} &= \frac{\partial}{\partial z}\left(c_{55}\left(\frac{\partial u}{\partial z} - k_n w\right)\right) - k_n c_{13}\frac{\partial w}{\partial z} - \\
&\quad (c_{12}+c_{66})\,k_n k_m v - \left(c_{11}k_n^2 + c_{66}k_m^2\right)u + f_x(t,n,m,z), \\
\rho\frac{\partial^2 v}{\partial t^2} &= \frac{\partial}{\partial z}\left(c_{44}\left(\frac{\partial v}{\partial z} - k_m w\right)\right) - k_m c_{23}\frac{\partial w}{\partial z} - \\
&\quad (c_{12}+c_{66})\,k_n k_m u - \left(c_{66}k_n^2 + c_{22}k_m^2\right)u + f_y(t,n,m,z), \\
\rho\frac{\partial^2 w}{\partial t^2} &= \frac{\partial}{\partial z}\left(c_{33}\frac{\partial w}{\partial z} + c_{13}k_n u + c_{23}k_m v\right) + k_n c_{55}\frac{\partial u}{\partial z} + \\
&\quad k_m c_{44}\frac{\partial v}{\partial z} - \left(c_{55}k_n^2 + c_{44}k_m^2\right)u + f_z(t,n,m,z)
\end{aligned} \tag{7}$$

with the boundary and the initial conditions:

$$\begin{gathered}
c_{55}\left(\frac{\partial u}{\partial z} - k_n w\right)\Big|_{z=0} = 0, \qquad c_{44}\left(\frac{\partial v}{\partial z} - k_m w\right)\Big|_{z=0} = 0, \\
\left(c_{33}\frac{\partial w}{\partial z} + c_{13}k_n u + c_{23}k_m v\right)\Big|_{z=0} = 0, \\
\frac{\partial u}{\partial t}\Big|_{t=0} = u|_{t=0} = 0, \quad \frac{\partial v}{\partial t}\Big|_{t=0} = v|_{t=0} = 0, \quad \frac{\partial w}{\partial t}\Big|_{t=0} = w|_{t=0} = 0.
\end{gathered} \tag{8}$$

4. Algorithm I

After application of the finite integral Fourier transforms, the solution of the original problem (1)–(3) reduces to the solution of problems (7), (8) with respect to the coordinate t, z.

For the numerical solution of this problem, we make use of the finite difference method of the second order approximation with respect to time and the fourth order accuracy with respect to space.

Here we use the staggered grid scheme [2]. Let us introduce two types of uniform grid:

$$
\begin{aligned}
\omega &= \{z_i = i\Delta z;\ i = 0,\dots,K;\ t_{n,j} = 0,\dots,J,\ t_J = T\},\\
\omega^1 &= \left\{z_{i+1/2} = (i+1/2)\Delta z,\ i = 0,\dots,K-1;\ t_{n,j} = 0,\dots,J,\ t_J = T\right\}.
\end{aligned}
$$

The values of the components $u^j(t,n,m,z)$, $v^j(t,n,m,z)$ are grid functions given on the grid ω and the values of the components $w^j(t,n,m,z)$ on the grid ω^1. The difference scheme is the following:

$$
\begin{aligned}
\rho_i u_{\bar{t}t} &= -\frac{1}{24\Delta z}\left(\Theta^j_{i+3/2} - \Theta^j_{i-3/2}\right) + \frac{9}{8\Delta z}\left(\Theta^j_{i+1/2} - \Theta^j_{i-1/2}\right) - \\
&\quad c_i^{13} k_n \theta\left(w_i^j\right) - \left(c_i^{12} + c_i^{66}\right) k_n k_m v_i^j - \\
&\quad \left(k_n^2 c_i^{11} + k_m^2 c_i^{66}\right) u_i^j + f_x\left(i\Delta z, j\Delta t\right),\\
\rho_i v_{\bar{t}t} &= -\frac{1}{24\Delta z}\left(\Psi^j_{i+3/2} - \Psi^j_{i-3/2}\right) + \frac{9}{8\Delta z}\left(\Psi^j_{i+1/2} - \Psi^j_{i-1/2}\right) - \\
&\quad c_i^{23} k_n \theta\left(w_i^j\right) - \left(c_i^{12} + c_i^{66}\right) k_n k_m u_i^j - \\
&\quad \left(k_m^2 c_i^{22} + k_n^2 c_i^{66}\right) v_i^j + f_y\left(i\Delta z, j\Delta t\right),\\
\rho_{i+\frac12} w_{\bar{t}t} &= -\frac{1}{24\Delta z}\left(\Omega^j_{i+2} - \Omega^j_{i-1}\right) + \frac{9}{8\Delta z}\left(\Omega^j_{i+1} - \Omega^j_{i}\right) + \\
&\quad c^{44}_{i+1/2} k_m \theta\left(v^j_{i+1/2}\right) + c^{55}_{i+1/2} k_n \theta\left(u^j_{i+1/2}\right) - \\
&\quad \left(k_m^2 c^{44}_{i+1/2} + k_n^2 c^{55}_{i+1/2}\right) w_i^j + f_z\left(i\Delta z, j\Delta t\right),
\end{aligned}
\tag{9}
$$

where

$$
\begin{aligned}
\theta\left(\psi^j_{i+\frac12}\right) &= -\frac{1}{24\Delta z}\left(\psi^j_{i+2} - \psi^j_{i-1}\right) + \frac{9}{8\Delta z}\left(\psi^j_{i+1} - \psi^j_{i}\right),\\
\Theta^j_{i+1/2} &= c^{55}_{i+\frac12}\left(\theta\left(u^j_{i+1/2}\right) - k_n w^j_{i+1/2}\right),\\
\Psi^j_{i+1/2} &= c^{44}_{i+1/2}\left(\theta\left(v^j_{i+1/2}\right) - k_n w^j_{i+1/2}\right),\\
\Omega^j_i &= c_i^{33}\theta\left(w_i^j\right) + k_n c_i^{13} u_i^j + k_m c_i^{23} v_i^j,
\end{aligned}
$$

and

$$
\psi_{\bar{t}t} = \left(\psi_i^{j+1} - 2\psi_i^j + \psi_i^{j-1}\right) / (\Delta t)^2.
$$

5. Algorithm II

Now, to problem (7), (8) we apply the integral Laguerre transforms with respect to the temporal variable t of the form [1, 5]:

$$\begin{Bmatrix} u^j(n,m,z) \\ v^j(n,m,z) \\ w^j(n,m,z) \end{Bmatrix} = \int_0^\infty \begin{Bmatrix} u(n,m,z,t) \\ v(n,m,z,t) \\ w(n,m,z,t) \end{Bmatrix} (ht)^{-\alpha/2} l_j^\alpha(ht)\, d(ht) \tag{10}$$

with the respective inversion formulas

$$\begin{Bmatrix} u(n,m,z,t) \\ v(n,m,z,t) \\ w(n,m,z,t) \end{Bmatrix} = (ht)^{-\alpha/2} \sum_{j=0}^{\infty} \begin{Bmatrix} u^j(n,m,z) \\ v^j(n,m,z) \\ w^j(n,m,z) \end{Bmatrix} l_j^\alpha(ht), \tag{11}$$

where $l_j^\alpha(ht)$ are the orthonormal Laguerre function:

$$\int_0^\infty l_n^\alpha(ht) l_m^\alpha(ht)\, dt = \delta_{mn} \frac{(m+\alpha)!}{m!}.$$

The Laguerre function $l_j^\alpha(ht)$ is expressed by the classical Laguerre polynomials $L_j^\alpha(ht)$. We select the parameter α to be integer and positive, then

$$l_j^\alpha(ht) = (ht)^{-\frac{\alpha}{2}} e^{-\frac{ht}{2}} L_j^\alpha(ht).$$

The application of (10), (11) results in the system of equations (7), (8) of the following form

$$\begin{aligned} &\frac{\partial}{\partial z}\left(c_{55}\left(\frac{\partial u^j}{\partial z} - k_n w^j\right)\right) - k_n c_{13}\frac{\partial w^j}{\partial z} - (c_{12}+c_{66})k_n k_m v^j - \\ &\quad \left(c_{11}k_n^2 + c_{66}k_m^2 + \frac{h^2\rho}{4}\right)u^j \\ &\qquad = h^2\rho\sqrt{\frac{j!}{(j+\alpha)!}} \sum_{l+0}^{j-1} (j-l)\sqrt{\frac{(l+\alpha)!}{l!}} u^l + f_x^j, \\ &\frac{\partial}{\partial z}\left(c_{44}\left(\frac{\partial v^j}{\partial z} - k_m w^j\right)\right) - k_m c_{23}\frac{\partial w^j}{\partial z} - (c_{12}+c_{66})k_n k_m v^j - \\ &\quad \left(c_{66}k_n^2 + c_{22}k_m^2 + \frac{h^2\rho}{4}\right)v^j \\ &\qquad = h^2\rho\sqrt{\frac{j!}{(j+\alpha)!}} \sum_{l+0}^{j-1} (j-l)\sqrt{\frac{(l+\alpha)!}{l!}} v^l + f_y^j, \end{aligned}$$

$$\frac{\partial}{\partial z}\left(c_{33}\frac{\partial w^j}{\partial z} + c_{13}k_n u^j + c_{23}k_m v^j\right) + c_{55}k_n\left(\frac{\partial u^j}{\partial z}\right) + c_{44}k_n\left(\frac{\partial v^j}{\partial z}\right) -$$
$$\left(c_{55}k_n^2 + c_{44}k_m^2 + \frac{h^2\rho}{4}\right)w^j$$
$$= h^2\rho\sqrt{\frac{j!}{(j+\alpha)!}}\sum_{l+0}^{j-1}(j-l)\sqrt{\frac{(l+\alpha)!}{l!}}w^l + f_z^j.$$

After the application of the Fourier and the Laguerre transforms, the original problem (1)–(3) reduces to solving the boundary value problem for the system of ordinary differential equations, this system being dependent on two transformation parameters k_x and k_y. For the numerical solution of this system at fixed transformation parameters, a difference scheme of the fourth approximation order is applied. We use basically the same difference approximation scheme as in the first algorithm. In this case, the finite difference scheme can be written down in the vector form:

$$A_\Delta(n,m,i\Delta z)\vec{W}^j(n,m,i\Delta z) = \vec{F}\left(\sum_{l=0}^{j-1}\vec{W}^l(n,m,i\Delta z)\,\varphi_l\right).$$

In this formula, $\vec{W}^j(n,m,i\Delta z)$ stands for the sought for vector of displacement components defined on the difference grid, and $A_\Delta(n,m,i\Delta z)$ denotes the band matrix of coefficients. As a result, the original problem reduces to solution of the system of linear algebraic equations. It should be noted that the band matrix depends on the Fourier transform parameters k_n, k_m and is independent of the Laguerre transformation parameter j, and the right-hand side of this system of linear equations is defined from recurrent formulas of the parameter j.

Hence, at fixed parameters of the Fourier transform it is necessary to solve a system of linear equations with many right-hand sides. To do this the LU-decomposition method can be successfully used.

6. Some aspects of numerical implementation

Both algorithms proposed can be parallelized for multi-computers. Obviously, when implementing these algorithms on the multi-processor systems, the computation of the problem for the fixed parameters of the Fourier transform can be carried out on each processor, there being no necessity to perform the information exchange between the processes.

This is because of the fact that the summation operation by formula (6) can be done on completion of all the initiated processes of a separate program.

In the present work. we do not dwell on studying the stability of the proposed algorithms. We should only note that for Algorithm I, the stability criterion proposed in [2], is applicable.

The basic computing peculiarities connected with the numerical implementation of Algorithm II are considered in good detail in [1, 5].

For the numerical modeling, we have selected the model consisting of the transversely-isotropic layer with the horizontal axis of symmetry localized on the elastic half-space. The axis of symmetry is parallel to the axis x. The parameters of the anisotropic layer are the following:

- in the layer

$$\begin{aligned}
&c_{11} = 14,82 \cdot 10^{11}\ \mathrm{din/cm}^2, && c_{12} = 2,614 \cdot 10^{11}\ \mathrm{din/cm}^2,\\
&c_{13} = 6,703 \cdot 10^{11}\ \mathrm{din/cm}^2, && c_{22} = c_{11},\\
&c_{23} = 14,82 \cdot 10^{11}\ \mathrm{din/cm}^2, && c_{33} = 12,208 \cdot 10^{11}\ \mathrm{din/cm}^2,\\
&c_{44} = 1,305 \cdot 10^{11}\ \mathrm{din/cm}^2, && c_{55} = c_{44};\\
&c_{66} = 6,103 \cdot 10^{11}\ \mathrm{din/cm}^2, && \rho = 1\ \mathrm{g/cm}^3;
\end{aligned}$$

- in the half-space

$$\lambda + 2\mu = 6.25 \cdot 10^{11}\ \mathrm{din/cm}^2, \quad \mu = 2.25 \cdot 10^{11},\ \mathrm{din/cm}^2 \rho = 1\ \mathrm{g/cm}^3.$$

A source of the "explosive" type is localized at a depth of $d = \frac{\lambda}{3}$ from the free surface.

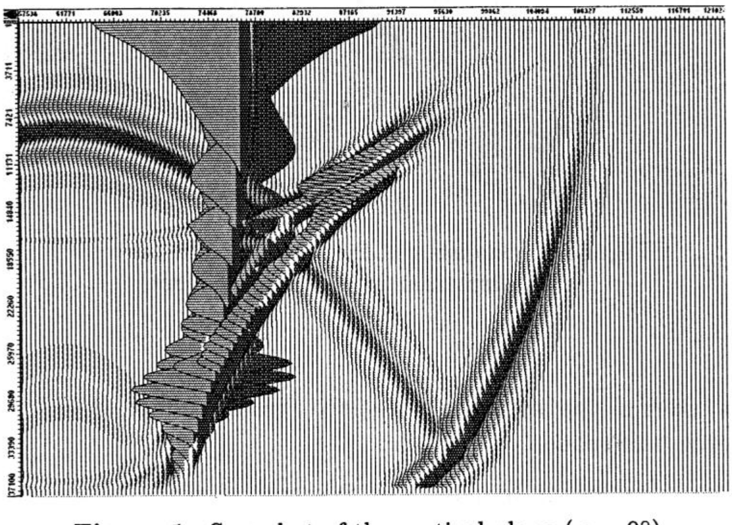

Figure 1. Snapshot of the vertical plane ($\varphi = 0°$) U_z displacement vector component

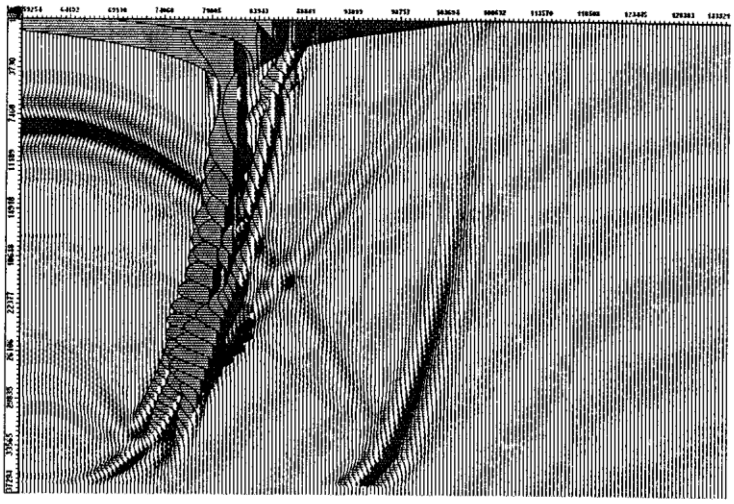

Figure 2. Snapshot of the vertical plane ($\varphi = 45°$)
U_z displacement vector component

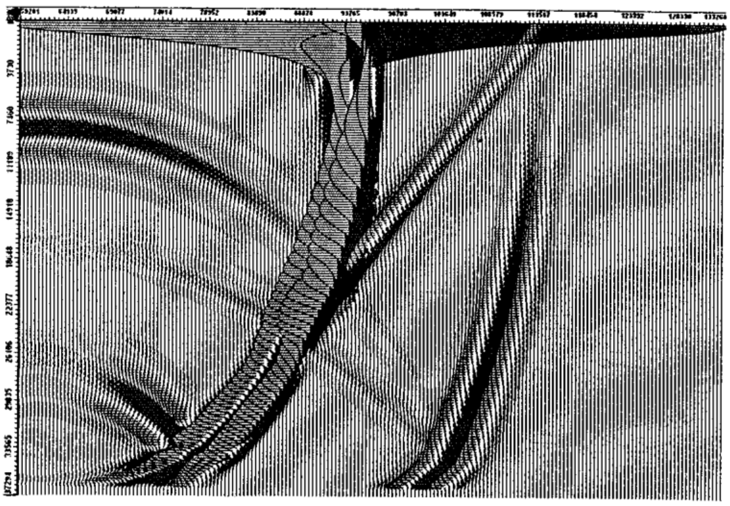

Figure 3. Snapshot of the vertical plane ($\varphi = 90°$)
U_z displacement vector component

Parameters of the isotropic half-space are selected close to parameters of an isotropic half-space.

Figures 1–3 present snapshots of the vertical plane of the wave field for the displacement vector component U_z for different angle ($\varphi = 0°$, $\varphi = 45°$, $\varphi = 90°$).

References

[1] Konyukh G.V., Mikhailenko B.G. Application of integral Laguerre transformation in solving dynamic seismic problems // Proc. of Institute of Computational Mathematics and Mathematical Geophysics. Mathematical Modeling in Geophysics. – 1988. – P. 106–123.

[2] Levander A.R. Fourth-order finite-difference P-SV seismograms // Geophysics. – 1988. – Vol. 53, № 11. – P. 1425–1436.

[3] Martynov V.N. Wave fields from concentrated sources in transversely-isotropic media // Izv. AN SSSR, Fizika Zemli. – 1986. – Vol. 11. – P. 19–26 (in Russian).

[4] Martynov V.N., Mikhailenko B.G. Calculation of complete theoretical seismograms for anisotropic media. Inverse modeling in exploration geophysics // Proceedings of the 6th International Mathematical Geophysics Seminar held at the Free University of Berlin. February, 3–6, 1988. – P. 323–348.

[5] Mikhailenko B.G. Spectral Laguerre method for the approximate solution of time dependent problems // Appl. Math. Lett. – 1999. – Vol. 12. – P. 105–110.