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Tsunami waves propagation along the waveguides*

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In this paper, the process of tsunami wave propagation along the waveguide with flat sloping sides is studied using the wave-ray method. When propagating along the waveguide, a single tsunami wave transforms into the wave train. The expression for the propagation velocities of the fastest and slowest signals is defined. The tsunami waves behavior above the ocean bottom ridges which have different model profiles is numerically investigated with the help of the finite difference method. Results of numerical experiments show that the highest waves are detected above the ridge with flat sloping sides.

Introduction

The qualitative theory of waveguides in a medium with a varying optical density (the velocity of signals or waves in it) was developed, for example, by L.M. Brekhovskikh [1]. Let us introduce, following this work, concept of the waveguide as applied to the tsunami problem. Let in a two-dimensional space the wave (disturbance) propagation velocity be constant and equal to v_0, and in the neighborhood of some straight line it monotonically decreases, reaching a minimum value just on this line. This line (the axis of the waveguide) with the neighborhood, where the wave propagation velocity is less, than v_0, is named "the waveguide". If a wave source is initially located on the waveguide, some part of the wave energy will be trapped by it, and the waves will propagate along the waveguide for a long distance (till its end). In [1], the concept of the waveguide with the total reflecting boundary is also given.

The waveguide with the linearly varying wave velocity is studied (it means that the wave propagation velocity linearly increases when moving away from an axis). The waveguides considered here look like the bottom ridges with flat sloping sides.

As a source of waves on the mean water surface, a segment of the plane wave is considered. The initial propagation direction of the wave is the same as the waveguide axis. Then the segment of the wave front, which was initially located above the waveguide is propagating along it. For waveguides of this type the quantitative theory of the wave propagating process is presented.

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1. Investigation of waveguides using wave rays

Wave rays, which are widely used for the wave kinematics research, can be determined as curves, orthogonally directed to the propagating wave front every time moment. In other words, they are the trajectories of the wave front line points. For example, in the area with constant depth all the rays have the form of straight lines.

Let us consider a rectangular water area with a sloping bottom. In the Cartesian coordinate system the depth is linearly increasing from zero on the axis Ox ($y = 0$) by the formula $H = y \operatorname{tg} \alpha$, where α is an inclination angle of the bottom slope (Figure 1). If from a point $(0, 2C_1)$ the wave ray starts in parallel with Ox axis direction, it will look like a segment of the cycloid of radius C_1, and its trajectory can be described by equations [2]

$$x = C_1(t - \sin t - \pi), \quad y = C_1(1 - \cos t), \quad t \in (\pi, 2\pi). \tag{1}$$

The ray will reach the axis Ox at the point $(x_2, 0)$, where $x_2 = \pi C_1$. If the plane of the bottom has the same declination, but a decrease the depth stops on the line $y = y_0$, the considered ray after arriving at the point M_1 at a distance y_0 from Ox axis, will be reflected from the line $y = y_0$. After reflection the wave ray will go to the point M_2 along the new cycloid with the same radius (see Figure 1). The condition of reflection of a wave ray on the line $y = y_0$, is caused by symmetry of the picture relative to the axis $y = y_0$ which is true for symmetric waveguides.

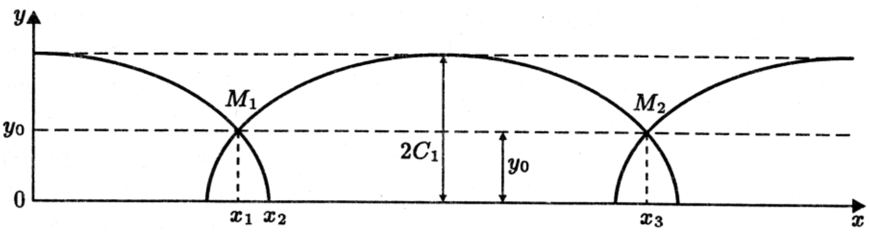

Figure 1. Trajectory of the wave ray above the bottom slope

Let us determine the length of a step of this truncated cycloid, and also, the wave travel time along it. As length of a step of a full cycloid is expressed as $L = 2C_1$, for the length of a step of a truncated cycloid we have (see Figure 1)

$$L_1 = x_3 - x_1 = L - 2(x_2 - x_1). \tag{2}$$

The value $(x_2 - x_1)$ can easily be defined from the equation of a cycloid (1):

$$L_1 = 2\pi C_1 - 2C_1\left(\arccos\left(1 - \frac{y_0}{C_1}\right) - \sin\left(\arccos\left(1 - \frac{y_0}{C_1}\right)\right)\right). \tag{3}$$

The wave travel-time along this cycloid will be written down as follows

$$T = \frac{2\sqrt{2C_1}}{\sqrt{g\,\mathrm{tg}\,\alpha}}\left(\pi - \arccos\left(1 - \frac{y_0}{C_1}\right)\right). \tag{4}$$

Here α is the inclination angle of the bottom plane to horizon, C_1 is a radius of a cycloid. If a maximum depth H_0 outside the waveguide and minimum depth on a ridge H_1 are given, then

$$C_1 = \frac{H_0}{2\,\mathrm{tg}\,\alpha}, \quad y_0 = \frac{H_1}{\mathrm{tg}\,\alpha}, \quad l = 2C_1 - y_0, \tag{5}$$

where l is a half-width of the waveguide. If it is twice wider ($l_1 = 2l$), but the depth variation remains the same, the declination of the bottom $\mathrm{tg}\,\beta$ will be reduced to half the former value and will be $\mathrm{tg}\,\beta = (\mathrm{tg}\,\alpha)/2$. Due to the above-said, the radius of a cycloid appropriate to "extreme" rays, which reach the edges of the waveguide, will be doubled, $C_2 = 2C_1$. The value y_2 will also increase ($y_2 = 2y_0$). The wave travel time from the axis up to the edges of the waveguide along the new cycloid will be written as

$$T_2 = \frac{2\sqrt{4C_1}}{\sqrt{(g\,\mathrm{tg}\,\alpha)/2}}\left(\pi - \arccos\left(1 - \frac{2y_0}{2C_1}\right)\right) = 2T_1. \tag{6}$$

The length of the cycloid step will be also twice increased. Therefore, the average velocity along the waveguide of the signals which propagate along the "extreme" rays does not depend on the waveguide width and is determined only by the difference of the depths H_0 and H_1.

Let us investigate the propagation velocity along the waveguide of different segments of the wave front depending on their initial distance from the axis. In Figure 2, the traces of the wave rays, which are start at the distances $2y_1$ and y_1 away from the axis of the waveguide and the wave ray, going along the axis, are shown.

Let l be a half-width of the waveguide, α – the angle of inclination of the bottom planes, H_0 – the depth outside the waveguide. The basic values we need for the calculation are defined as

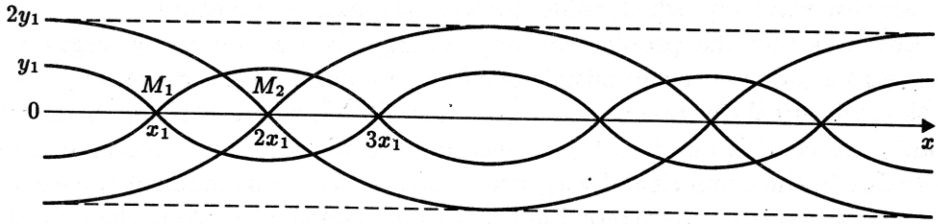

Figure 2. Wave rays in the symmetric waveguide

$$H_1 = H_0 - l \operatorname{tg}\alpha, \quad y_0 = \frac{H_1}{\operatorname{tg}\alpha}, \tag{7}$$

where H_1 is the depth above the ridge. The value C_1 (a radius of the cycloid) depends on the initial distance y_1 from the axis of the waveguide to the wave-ray starting point

$$C_1 = \frac{y_0 + y_1}{2}, \quad 0 \le y_1 \le l.$$

Using expressions (3) and (4) the ratio between the average wave velocity along the waveguide and the value y_1 can be defined

$$V = \frac{L}{T} = \frac{2\pi C_1 - \left(\arccos\left(1 - \frac{y_0}{C_1}\right) - \sqrt{\frac{2y_0}{C_1} - \left(\frac{y_0}{C_1}\right)^2}\right)}{\frac{2\sqrt{2C_1}}{\sqrt{g \operatorname{tg}\alpha}}\left(\pi - \arccos\left(1 - \frac{y_0}{C_1}\right)\right)}, \tag{8}$$

where the values C_1 and y_0 are determined by the ratio (5). The velocity of an appropriate wave motion along the waveguide depends on the value C_1 (or y_0) and it is as less, as closer the initial wave ray to the axis. Let us find a velocity limit V, when the parameter tends to zero. Let y_0 be equal to a small value ε. Then, keeping in formula (8) the terms of the least order by ε, we obtain

$$V = \sqrt{2gC_1 \operatorname{tg}\alpha}, \tag{9}$$

and due to the ratio

$$C_1 = \frac{H_1}{2 \operatorname{tg}\alpha}$$

the value corresponds to disturbances propagation velocity along the axis of the waveguide calculated by the Lagrange formula $V = \sqrt{gH_1}$, which is applied to the top of the ridge.

Thus, each waveguide (not only with flat bottom planes) has a specific dispersion, which is discovered in expansion of the initial plane wave along the waveguide during its propagation. Finally, the initial single wave will transform into a "wave train". This dispersion ability of each waveguide can be measured by the value $q = V_{\max}/V_{\min}$, where $V_{\max}$ is the running velocity of the fastest rays along the waveguide and $V_{\min}$ is the running velocity of the slowest wave ray, which coincides with the axis of waveguide.

Let us study the problem of wave energy focused by the waveguide. Above a flat bottom the amplitude of a circle tsunami wave is decreasing due to the cylindrical divergence proportional to $R^{-1/2}$, where R is the distance from a source. Coming into the waveguide, the wave does not decrease any more due to the cylindrical divergence. However from this moment the wave energy dissipates by the stretching the initially single wave along the axis of waveguide, which also causes the reduction of an amplitude. If the degree of the amplitude loss due to the wave dispersion along the waveguide is less than a possible amplitude loss due to the cylindrical stretching of a front, then in this case the waveguide has focusing properties. Mathematically, it can be written as follows:

$$(V_{\max} - V_{\min})T < \frac{2\pi}{K}R. \tag{10}$$

Here T is the propagation time of waveguide waves, K is the value indicating, which part of circle front of an incident wave is in the waveguide, R is the distance from the wave source up to the position of the front in the waveguide after the time period T. Thus, not every waveguide causes the focusing effect. The greatest focusing ability is intrinsic of the broad waveguides with a small difference in depths $(H_0 - H_1)$. A favorable situation for focusing arises in the case, when a tsunami waves source is localized above the waveguide.

Let us make some remarks about asymmetric waveguides, where the declination of a flat bottom is various for different sides from the ridge. Let the declination of one side be 2 times greater, than the other. The depth outside the waveguide is equal to H_0 and on the ridge (axis) – H_1. Let us consider the trajectories of "extreme" rays, which reach the edges of the waveguide. At the initial moment they are at the distances y_2 and $y_1 = 2y_2$ on both sides from the axis of the waveguide (Figure 3).

According to the previously-stated, the radius of cycloids, along which wave rays propagate above the most flat side, is twice greater, than the radius of the both considered rays, when they propagate above the other side of greater inclination. In Figure 3 it is seen, that both rays before they reach the cross-section $x = x_4$ have passed the identical-length paths which consist of two cycloid pieces each. Thus, at the cross-section $x = x_4$ this two rays arrive simultaneously. The cross-point M of these "extreme" rays is not located on the Ox axis of the waveguide.

In spite of the point M is a cross-point of wave rays (caustic) a noticeable increase of the wave amplitude in it is not detected, because the rays come to this point nonsimultaneously. Let us show this. As $y_1 = 2y_2$, then $x_3 = 2x_1$. The wave travel-time along the "large" cycloid is twice more, than along

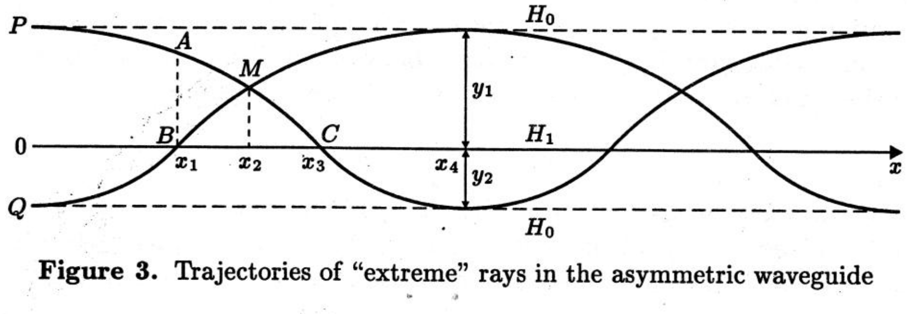

Figure 3. Trajectories of "extreme" rays in the asymmetric waveguide

"small" (time T_0). Taking into account the fact that the travel-time along the curve PC is equal to $2T_0$, travel-time along the curve PA will be less than T_0, because the length of PA is less than the length of AM. Furthermore, on PA the depths are greater. In addition, the wave travel-time along the curve AM will be less, than along BM, because the length of AM is less than the length of BM, and the curve AM is located at a greater depth. Thus, the wave reaches the point M along the curve PM much earlier, than along the curve QM. Let us estimate the propagation velocity of "extreme" rays along the waveguide. To the cross-section $x = x_4$ the "extreme" rays will come simultaneously, their travel-time being $3T_0$. Therefore the average wave velocity along the waveguide is expressed as x_1/T_0, that is equal to $V_{\max}$ for the symmetric waveguides with the same depths on the ridge and outside of the waveguide. Therefore the statement about independence of the wave propagation velocity along the waveguide of its width is correct also for asymmetric waveguides.

Now let us consider the ratio of tsunami amplitudes in the wave train that arises as a result of the above mentioned dispersion. Using relation (8), we have found some values of the propagation velocity V along the ridge for wave rays which are disposed different distances from the waveguide axis. The obtained values of the propagation velocity V show that the relation of V from y_1 is rather close to linear, but for the rays initially disposed closer to the waveguide edge, the rate of varying V depending on the distance y_1 is lower, than for the rays initially located closer to the waveguide axis. Thus, the dispersion of the waveguide impact on the slowly moving part of a "wave train" is somewhat stronger and, consequently, the leading waves from a dispersing signal have a slightly greater amplitude, than in the "tail" of the same "wave train".

Let us consider the case when at the entrance of the waveguide perpendicular to its axis there are not one, but two consecutive waves. Let between positive maxima of these waves the initial distance be equal to 2λ. Then the waves in the waveguide will be transformed into "wave trains". The "leading" part of the second wave can reach the "tail" of the first wave, thus causing amplification of the wave amplitude above the waveguide axis. It is possible to find the conditions to realize the given ability. Let the initial distance between wave maxima in the waveguide be equal to 2λ and maximum and minimum wave rays propagating velocities of the considered waveguide be equal to $V_{\max}$ and $V_{\min}$. The distance between the "tail" of the first wave and the "leading" part of the second wave will be expressed as

$$l = V_{\min}T + 2\lambda - V_{\max}T, \tag{11}$$

where T is the propagation time. Therefore, at the time T_0, when l tends to zero the dispersed signals from both waves will be imposed on each other, which will cause the amplification of oscillations near the waveguide axis at the distance $T_0 V_{\min}$ from its beginning. Thus, the maximum fluctuation of levels in the waveguide are watched on a distance $T_0 V_{\min}$ from the beginning of the waveguide. The time period T_0 is determined by the characteristics of the waveguide

$$T_0 = \frac{2\lambda}{V_{\max} - V_{\min}}. \tag{12}$$

If the initial wave signals, propagating in the waveguide, have negative parts, then the picture of interaction of such waveguide waves will be more complicated.

2. Numerical experiments

For the confirmation of a theoretical description of the behavior of tsunami waves in the waveguide of the indicated type, the numerical modeling of tsunami waves propagation above underwater ridge, having a model relief (which is shown in Figure 4) was carried out.

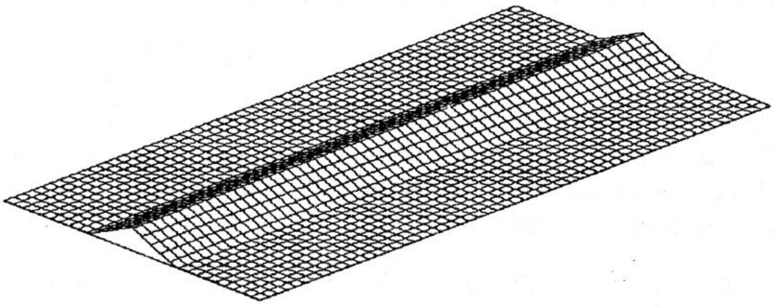

Figure 4. A bottom relief simulating underwater mountain ridge

The numerical algorithm based on variables splitting technique [3] was used for tsunami propagation modeling. Two-dimensional shallow water equations were used as the basic equations for the presented numerical method:

$$\begin{aligned} &H_t + (HU)_x + (HV)_y = 0, \\ &U_t + UU_x + VU_y + gD_x = 0, \\ &V_t + UV_x + VV_y + gD_y = 0, \end{aligned} \tag{13}$$

where $H = \eta + D$ is the total water depth; η is the vertical displacement of the water surface; U, V are velocities along X, Y directions; g is the acceleration of gravity. These equations neglect the wave dispersion effects, which can affect the tsunami wave only during it trans-oceanic propagation. Thus this model can be used for simulation of wave propagation along the waveguides with limited length (2–3 thousand km).

A finite-difference algorithm based on the splitting method has been developed. In order to solve shallow water equations, the splitting method reduces the numerical solving of the equations with two space variables to solving of two one-dimensional equations. This makes it possible to use the effective finite-difference scheme, developed for one-dimensional problem. Moreover, this method permits to set the boundary conditions for a finite-difference boundary value problem using the characteristic line method. The slitting method, which is used for shallow wave equations (13), comprises the consecutive numerical solution of two one-dimensional systems of equations:

$$\begin{cases} V_t + UV_x = 0, \\ U_t + UU_x + gH_x = gD_x, \\ H_t + (UH)_x = 0; \end{cases} \qquad \begin{cases} V_t + VV_x + gH_x = gD_x, \\ U_t + VU_x = 0, \\ H_t + (VH)_x = 0. \end{cases} \tag{14}$$

If we have two stable finite difference schemes, which approximate each of one-dimensional systems, the whole algorithm is stable and approximates the initial two-dimensional equations (13). All the eigenvalues of (14) are real and different and the system can be written down as

$$V'_t + \lambda_1 V'_x = 0, \quad P_t + \lambda_2 P_x = 0, \quad Q_t + \lambda_3 Q_x = 0, \tag{15}$$

where $\lambda_1 = U$, $\lambda_{2,3} = U \pm \sqrt{gH}$ are the eigenvalues, $V = V'$, $P = U + 2\sqrt{gH}$, $Q = U - 2\sqrt{gH}$ are the Riemannian invariants. The characteristic line method has been used to set the boundary conditions for this system. At the open sea boundary the conditions

$$V' = 0, \quad R = \pm 2\sqrt{gD}, \quad \text{where } R = U \pm 2\sqrt{gH} \ (\text{i.e. } R = P \text{ or } Q),$$

are used. At the land boundary the conditions of perfect reflection are used:

$$V = 0, \quad P = -Q.$$

For numerical solution of (15) the following finite difference scheme is used:

$$\frac{\vec{W}_i^{n+1} - \vec{W}_i^n}{\Delta t} + A\frac{\vec{W}_{i+1}^n - \vec{W}_{i-1}^n}{2\Delta x} - A\Delta t\frac{A(\vec{W}_{i+1}^n - \vec{W}_i^n) - A(\vec{W}_i^n - \vec{W}_{i-1}^n)}{2\Delta x^2}$$
$$= \frac{\vec{F}_{i+1} - \vec{F}_{i-1}}{2\Delta x} - A\Delta t\frac{\vec{F}_{i+1} - 2\vec{F}_i + \vec{F}_{i-1}}{2\Delta x^2}, \tag{16}$$

where

$$A = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \quad \vec{W} = \begin{pmatrix} V' \\ P \\ Q \end{pmatrix}, \quad \vec{F} = \begin{pmatrix} 0 \\ gD_x \\ gD_x \end{pmatrix}.$$

The criterion of stability of the scheme is $\Delta t \le \Delta x/\sqrt{gH}$.

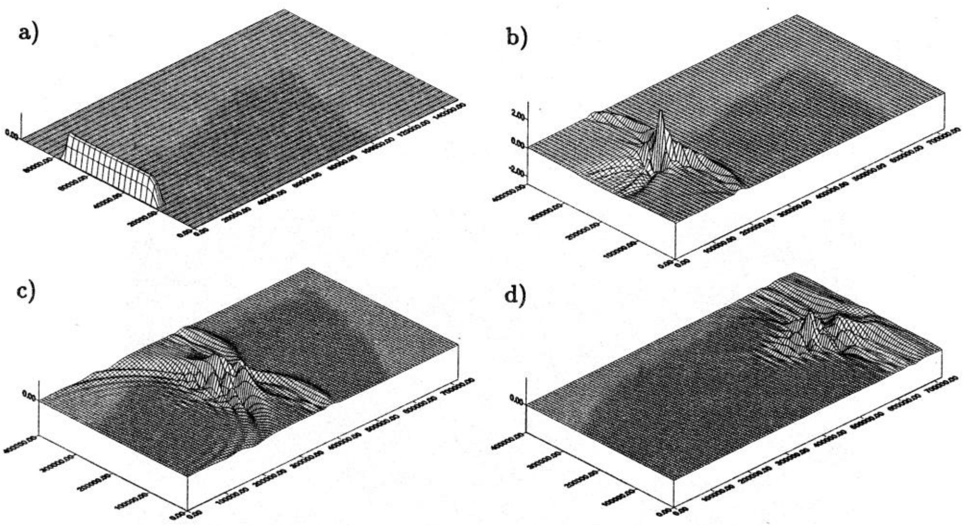

Figure 5. Water surface above the waveguide at various time moments

The computational area was 720 km × 400 km (720 × 400 grid points) and the width of the waveguide – 200 km. The depth on the axis of the waveguide was equal to 500 m, and outside the waveguide – 1000 m. The tsunami source has the form of a long narrow rectangle (20 × 200 km) with the profile, determined by the expression

$$\eta = 2 + 2\sin\left(\frac{i\pi}{10} - \frac{\pi}{2}\right), \quad i = 1, \ldots, 20. \tag{17}$$

Figure 5 shows the ocean surface at different moments of the wave propagation process along the waveguide. The source (Figure 5a) is forming the 2 meter high tsunami wave. As can be seen from the figures, the initial wave is expanded along the axis of the waveguide and transformed into the "wave train" with a narrow front, that completely confirms the theoretical results. The comparison of the waves amplitudes in the waveguide and above the flat bottom shows a significant amplitude growth above the axis (199 cm as compared to 85 cm without waveguide).

A number of different ridge profiles were used in numerical computations. The ridge profiles were determined by the formula

$$H(y) = H_1 + (H_0 - H_1)\frac{y}{y_0} - k\sin\Big(\frac{\pi y}{y_0}\Big), \tag{18}$$

where y is the distance from the axis, H_1 is the water depth on the axis, H_0 is the depth outside the waveguide and k is the parameter, which is varying from the value -150 up to $+150$. When $k = 0$, the sides are flat. The ridge profiles for $k = -100$, $k = +50$, $k = +150$ (from top to bottom of the figure) are shown in Figure 6.

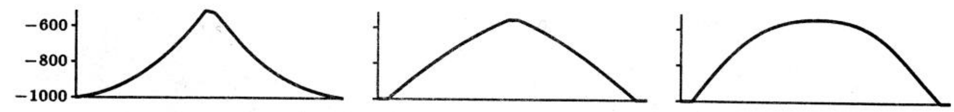

Figure 6. Profiles of the bottom ridge for different values of parameter k

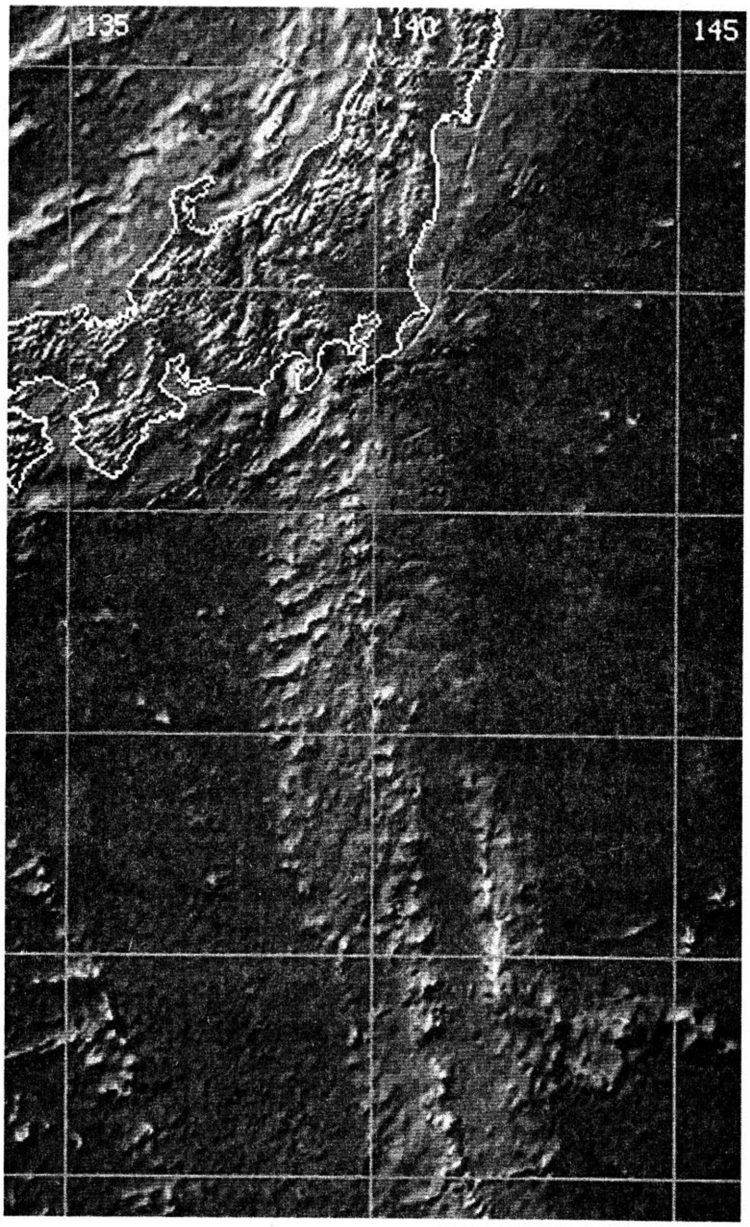

Figure 7. Bottom relief in numerical experiment of tsunami propagation along the Izu-Shoto island chain

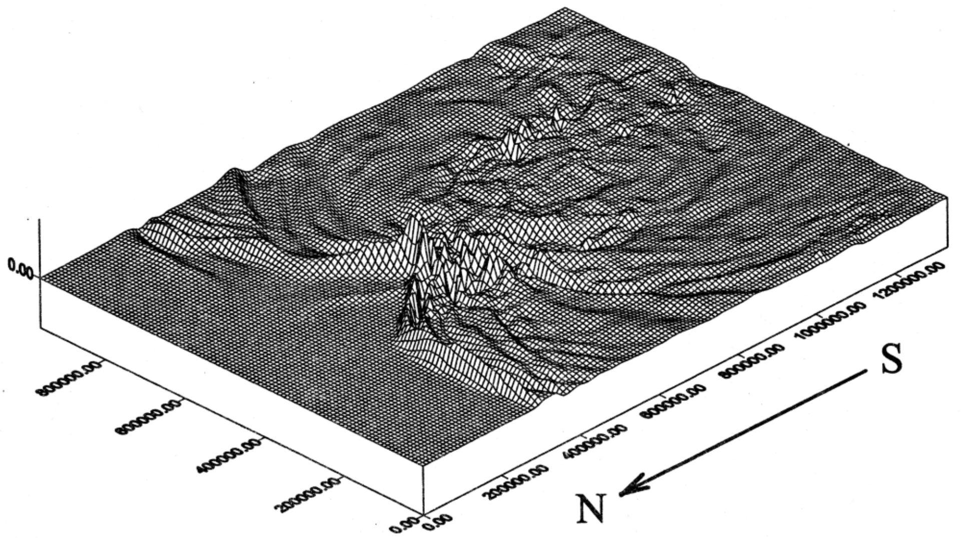

Figure 8. Water surface during tsunami wave propagation above the Izu-Shoto bottom ridge

The analysis of the results of numerical computations enables us to say, that the most effective waveguide is the one with flat sides (the case of $k = 0$). It produces the greatest wave amplitude during the tsunami propagation with the initial height of 2 meters.

Numerical modeling of tsunami propagation along the real Pacific waveguide was carried out with the model tsunami source, which produces the initial 2 meters high tsunami wave. The computational area includes the Izu-Shoto island chain southward of Japan (Figure 7).

In Figure 8, the ocean surface during the tsunami wave propagation along the Izu-Shoto island chain is presented. The results of computations show a significant tsunami amplitude growth near the axis of this waveguide.

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