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A numerical method for computing tsunami travel times to the rectangular grid nodes*

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Abstract. The new method for the tsunami travel times calculation to the nodes of rectangular grid was developed and tested. The algorithm is based on constructing the wave front segment inside the grid cell using tsunami arrival times to the cell angles. Testing the method proposed shows the better quality of the wave-front line approximation as compared to the existing method developed by the author earlier.

Introduction

A rapid and precise assessment of the tsunami wave travel times from a source to various points on the coast is important for the tsunami warning service. To solve this kinematic problem, there are several methods based on constructing wave rays [1–3] and the direct calculation of wave parameters and as the front location [4–6]. Some methods require a continuous function that would describe the depth of a two-dimensional area. For other methods, it is sufficient to know depth values at the nodes of a regular grid. The latter are more promising for the practical use, since the digital bathymetry of real water areas is represented by gridded depth sets of various resolutions. To use the methods of the first group in such cases the bilinear (or other) interpolation of depth at grid nodes to all other points of the region can be used. When implementing the grid methods, the kinematics of the wave front inside a grid cell is simulated assuming a linear depth change between the points there.

1. The tsunami kinematics basement

The tsunami wave belongs to the class of long waves, for whose description the system of differential equations of shallow water is used [7]. In the absence of external forces, except for gravity, these equations can be written down in the following form

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$$
\begin{aligned}
H_t + (uH)_x + (vH)_y &= 0, \\
u_t + uu_x + vu_y + gH_x &= gD_x, \\
v_t + uv_x + vv_y + gH_y &= gD_y,
\end{aligned}
\tag{1}
$$

where $H(x, y, t) = \eta(x, y, t) + D(x, y, t)$ is the full thickness of a water layer, $\eta(x, y, t)$ in the vertical water surface displacement (the wave height), $D(x, y)$ is the depth, u and v are the components of a water flow velocity vector, g is the acceleration of gravity. It follows from equations (1) that the wave front velocity does not depend on the tsunami parameters (length and height) and is expressed by the so-called Lagrange formula [7]

$$
c = \sqrt{gD}. \tag{2}
$$

It also follows from the same system of equations (1) that the wave front moves forward with the velocity indicated in the Lagrange formula (2) in the normal direction to this wave front line. It should be noted that due to the amplitude nonlinearity, the velocity of the crest of the wave slightly differs from the velocity of the front and is expressed as

$$
c = \sqrt{g(D + \eta)}. \tag{3}
$$

We will numerically calculate the kinematics of the tsunami wave front, so formula (2) will be used in the numerical algorithm. Let us assume that inside a grid cell, the depth changes according to the linear law. Let us find the travel time of the tsunami wave between two points, the distance between which is equal to L, and the depth linearly varies from the value H_1 to the value H_2. Let us introduce an auxiliary value, i.e. the bottom inclination angle $\operatorname{tg}\alpha = (H_2 - H_1)/L$. Then the travel time along the straight route with the length L is determined in the form

$$
\begin{aligned}
T &= \int_0^L \frac{dl}{\sqrt{g(H_1 + l \operatorname{tg}\alpha)}} = \frac{1}{\sqrt{g \operatorname{tg}\alpha}} \int_0^L \left(l + \frac{H_1}{\operatorname{tg}\alpha} \right)^{-1/2} d\left(l + \frac{H_1}{\operatorname{tg}\alpha} \right) \\
&= \frac{2}{\sqrt{g \operatorname{tg}\alpha}} \left(l + \frac{H_1}{tg\alpha} \right)^{1/2} \Bigg|_0^L = \frac{2}{\sqrt{g \operatorname{tg}\alpha}} \cdot \frac{\sqrt{H_2} - \sqrt{H_1}}{\sqrt{\operatorname{tg}\alpha}} \\
&= \frac{2}{\sqrt{g} \operatorname{tg}\alpha} \cdot \frac{H_2 - H_1}{\sqrt{H_2} + \sqrt{H_1}} = \frac{2L}{\sqrt{gH_2} + \sqrt{gH_1}}.
\end{aligned}
\tag{4}
$$

Therefore, the travel time of the tsunami wave between two points inside the grid cell is equal to the distance between them divided by the arithmetic mean of the tsunami velocities at these nodes. It should be noted that over an uneven bottom, the optimal route along which the disturbance propagates in the shortest time will not always be a straight line. However, for short distances, the difference in the travel time between the movement along the optimal trajectory and the line segment can be neglected.

2. Description of the method

The tsunami wave front moves in the direction of the external normal to the line of this front, which can be defined as the separation border between the points of the water area where the disturbance has already arrived by this moment and all other points of the area [8]. Publication [9] proposes a numerical method of orthogonal advance of the wave front for the simulation of the tsunami kinematics. But its implementation requires information about the depth values at any location in the area, which can be obtained using the bilinear interpolation of the grid bathymetry.

In the method, proposed the direction of the outer normal to the wave front line is geometrically determined based on the wave arrival times at the nodes of a regular grid, where the depth values are known. The essence of the method is to calculate the wave arrival time at the considered node of the grid from a segment of the wave front, which can be correctly built based on the grid travel-time data calculated at that moment. Now we describe the method as it is. At the initial moment, at all nodes of the calculation grid, we set the time values $t(i,j) = -1$, $i = 1, \ldots, i_{\max}$, $j = 1, \ldots, j_{\max}$. At the grid nodes located inside a tsunami source, we equate to zero the values of $t(i,j)$. After that, we begin to look through the points of the area where $t(i,j) = -1$ and calculate the wave arrival times. It is clear that the arrival time can be found only at the nodes having neighbors where the travel times are non-negative. As an example, Figure 1 shows the grid nodes located inside the round-shaped tsunami source (where $t(i,j) = 0$) which border with the nodes where the wave arrival times are unknown.

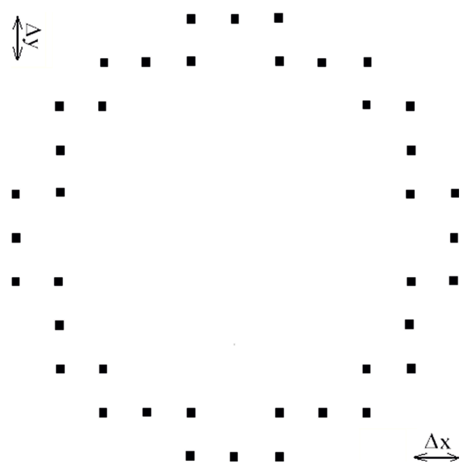


Figure 1. The location of the rectangular grid nodes representing the initial circular wave front. The length of the spatial steps along the abscissa and the ordinate axes are Δx and Δy, respectively

There are not many versions for the relative position of those nodes of the calculation grid, where the tsunami arrival time has been found bordering those where the arrival time has not been found yet. Only three configurations of the relative location of such nodes are possible without taking into account their orientation on the plane. They are shown in Figure 2.

In this figure, the grid nodes, where the tsunami arrival times are already known, are shown with squares, and the grid nodes, where the times to be

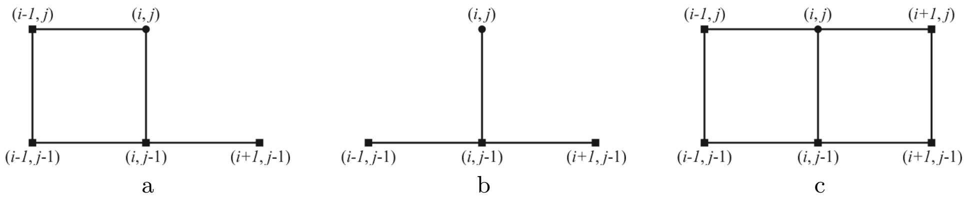


Figure 2. The possible configurations for the computation of the wave arrival time to the grid node (i, j)

determined, are drawn as circles. The node configurations shown in Figure 2 may be oriented at other angles. In the configuration shown in Figure 2a the grid node with indices $(i+1, j-1)$ may not exist. The algorithm for calculating the tsunami travel time at the computation grid node located as in Figure 2a will be described in this study. Figure 3 shows the scheme of this calculation, based on simple geometric constructions.

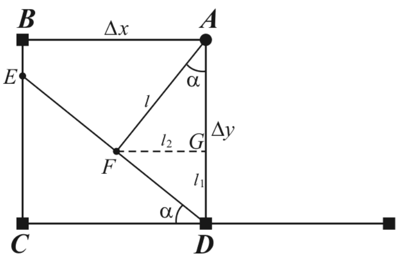


Figure 3. The scheme for calculating the arrival time at the point A based on the known arrival times of the wave at the points B, C and D. At the angles of the cell, depths are given

Let us consider the calculation grid cell, where it is necessary to determine the wave arrival time T_A at the point A when the tsunami arrival times T_B, T_C and T_D at the other three grid points of the cell B, C and D are known (see Figure 3). The depth values H_A, H_B, H_C and H_D are given for all angles of the cell. Let for the certainty $T_B > T_D$. In this case, the intersection point of the wave front segment corresponding to the time T_D and the segment BC (together with the angle α) can be found using formula (4). Considering that the velocity of the intersection point of the front with the section BC is greater than the velocity of the wave front by $1/\cos\alpha$ times, the value of $\cos\alpha$ is calculated by the formula

$$\cos\alpha = \frac{(T_B - T_C)(\sqrt{gH_C} + \sqrt{gH_B})}{2\Delta y}. \tag{5}$$

Further, it is easy to find the distance l from the point A to the wave front segment DE:

$$l = \Delta y \cos\alpha. \tag{6}$$

Now, to calculate the time when the tsunami reaches the point A, it is sufficient to find the depth at the point F determined by its coordinates. These coordinates X_F and Y_F are linked to the coordinates X_D and Y_D of the point D by the formulas

$$X_F = X_D - l_2 = X_D - l \sin\alpha, \quad Y_F = Y_D + l_1 = Y_D + l \cos\alpha.$$

The depth at the point F is calculated by the bilinear interpolation using the coordinates of the point F inside the grid cell and the depth values at the angles of the considered cell

$$H_F = \frac{H_C(X_D - X_F) + H_D X_F}{\Delta x} \cdot \frac{Y_B - Y_F}{\Delta y} + \frac{H_B(X_D - X_F) + H_A X_F}{\Delta x} \cdot \frac{Y_F}{\Delta y}. \quad (7)$$

As a result, the required time for the wave to enter the point A will be expressed in the form

$$T_A = T_D + \frac{2l}{\sqrt{gH_A} + \sqrt{gH_F}},$$

where l and the depth H_F are calculated from formulas (6) and (7) based on the known depth values at all the angles of the cell and the tsunami arrival time values at three of them (the points B, C and D). Similarly, geometric constructions can help to calculate the wave travel times to grid nodes for other node configurations with known and unknown arrival times (see Figure 2b, c).

3. Testing the method

First, we test the method proposed to find sequential positions of the wave front, which initially has a slope of 45 degrees to the abscissa axis (as well as to the ordinate axis). The depth at all the nodes of the 2000×1000 km computational area is constant and is equal to 1000 m. The length of the spatial steps in both directions is the same and equal to 1000 m. The initial wave front was a line segment connecting the upper left angle of the computational domain with the middle of its lower border (Figure 4).

Now we will test the described algorithm on the known exact solution for the wave front above the parabolic bottom topography [10]. Consider the following task: in a gridded area with a size 1000×1000 nodes, the depth increases from the lower border to the upper by the formula

$$H_{ij} = 0.001(j + 100)^2, \quad i = 1, \ldots, 1000, \; j = 1, \ldots, 1000.$$

The length of the grid step in both directions is the same and is equal to 100 m. The initial position of the wave front coincides with the left border of the computational domain. Then we start a repeated search for all the nodes of the computation grid with the calculation of the tsunami arrival times to the right from the left border.

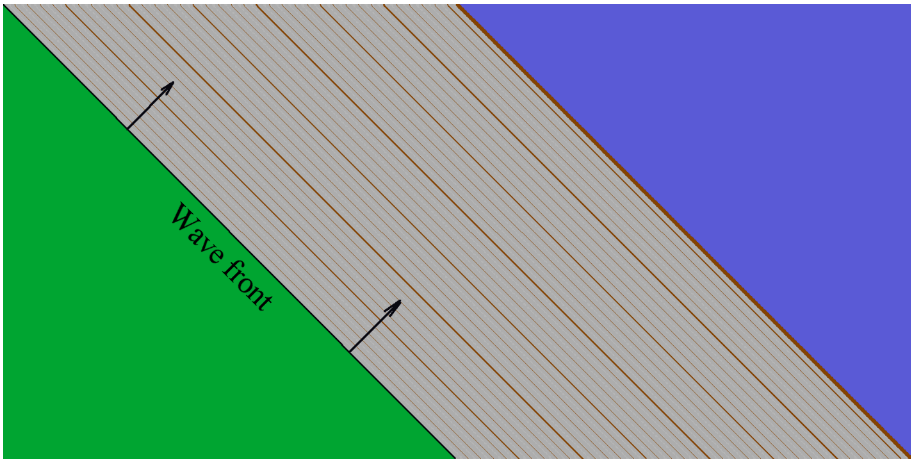


Figure 4. Sequential positions of the wave front when calculating the kinematics of the oblique front in an area with a constant depth. The zone of initial disturbance where $t(i,j)=0$ is colored by green

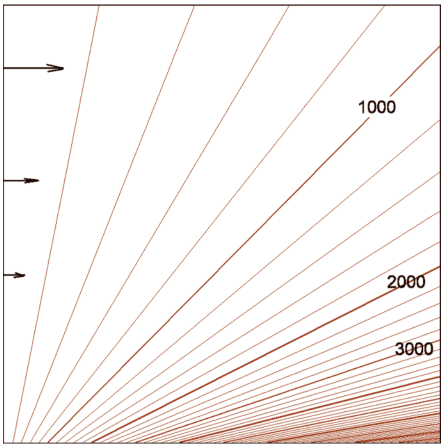
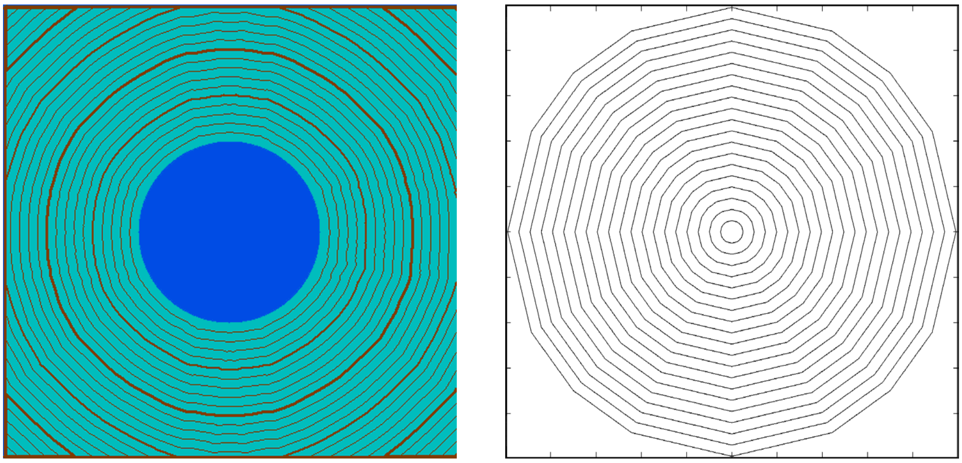


Figure 5. Tsunami wave front positions in the area with the parabolic bottom topography

Figure 5 visualizes the wave front positions every 200 seconds, built based on the results of the tsunami kinematics calculation by the method described. The figure shows that the wave front remains rectilinear throughout the tsunami propagation. This corresponds to the exact solution for such a model depth distribution in the area [10].

Another test for checking the correctness of the method is the simulation of the initially circular wave front kinematics in an area with a constant depth. Theoretically, in this case, the wave front line will have the shape of a circle at any time. To test the method proposed, a tsunami source is located at the center of the 1000×1000 nodes computation area, bounded by a circle with a radius of 5000 m. The grid step length is equal to 100 m in both directions, and the depth in the entire area is constant and is equal to 1000 m. Based on the three possible configurations (see Figures 2) multiple searches for all the grid nodes are carried out and the tsunami arrival times there are calculated. For the correct functioning of the algorithm, the direction of retrieval of the nodes changes to the opposite after every time cycle.



a b

Figure 6. The isolines of the calculated wave arrival times from a round source to the nodes of the calculation grid when using the method proposed (a) and as a result of the calculation using the algorithm based on the Huygens principle (b)

Figure 6a visualizes the time isolines of the wave arrival times to the grid nodes obtained as a result of the calculation using the method proposed. The figure shows that the shape of the calculated wavefront is close to a circle at any time. This once again confirms the correctness of the method proposed. For the comparison, in Figure 6b, the results of testing on the same task of the Huygens method [11] are presented in the same form. It can be seen that using the method described in this paper, the resulting isochrones are closer to the exact solution than the wave front positions obtained by the grid method [11].

Conclusion

A new method for finding the arrival times at the nodes of a rectangular calculation grid has been developed. It is based on the advancement of the wave front segments in the direction of the outer normal to them. Testing the method on several test tasks has shown its validity and effectiveness.

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