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The variation approach to travel-time calculations on regular grids*

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Abstract. Application of the variation method to simulate tsunami wave rays and to estimate the wave travel times corresponding is studied. In some cases, the wave ray determined by this method, connecting two points of the water area, may not present the global extreme, which provides the shortest tsunami travel time. In this way, for the correct calculation of the wave travel time from one point to another a broken line passing through the nodes of the calculation grid, which is built by the method based on the Huygens principle must be taken as an initial approximation. The variation method described can be applied to optimize the grid method for the wave travel time determination.

One of the possible methods for the wave ray construction from a source point to a receiver is based on a variation approach. Briefly, the essence of the method is as follows. The initial approximation of the wave ray (usually a straight line or geodesic line connecting a source and a receiver) is split to short segments by intermediate points, which then are moving in the orthogonal to this line direction in order to minimize the tsunami travel time along this broken line. The positions of these intermediate points are varying until a minimum sum of the wave travel times along all the segments composing the wave ray is reached. Probably, the first who used such an approach to calculate the tsunami arrival time was Braddock [1]. Figure 1 taken from [1] shows the initial ray approximation being the arc of a big circle which is the shortest route between two points on the Globe.

Let us test this method on the known exact solutions for the wave ray obtained by the author [2, 3]. Let the coastline in a two-dimensional area be presented by the line $y = 0$, and the depth $D(x, y)$ linearly depends on the distance y to the shore

$$D(x, y) = 0.1\, y.$$

Let us consider two points (a source and a receiver) located just near the coastline 1 m off it. It is necessary to determine the trajectory of the wave ray along which the disturbance (wave) propagates from the source (point A) to the receiver (point B) in the shortest time. We take a segment of the

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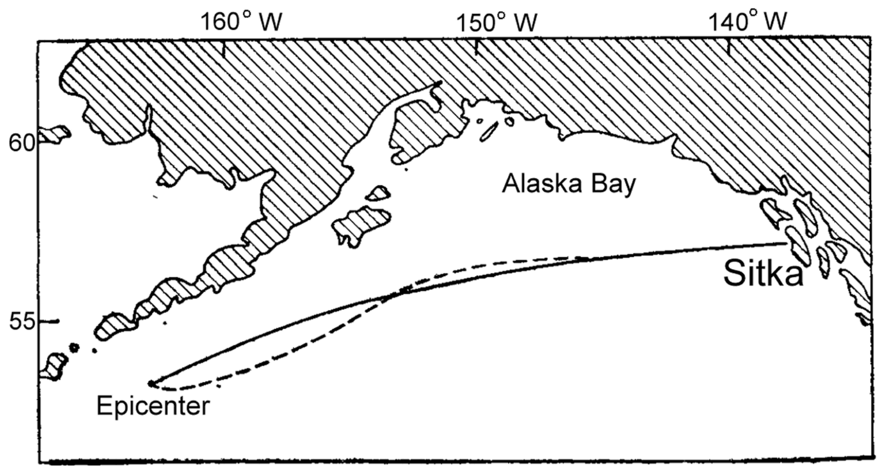


Figure 1. The wave-ray route (dashed line) connecting the tsunami source situated near Aleutian Islands and the Sitka Island built by method of variations [1]. The black line presents the geodetic trace being the shortest route on the spherical Globe

straight line AB as the initial approximation of the wave ray, along which 101 points (including points A and B) are posed within equal space intervals between them. The depth between the neighboring intermediate points is assumed to change linearly. Thus, according to [2], the approximate wave travel time along the segment connecting points F and H is expressed by the formula

$$T = \frac{2L}{\sqrt{gD_F} + \sqrt{gD_H}}, \tag{1}$$

where D_F and D_H are depth values at the points points F and H, L is the distance between them, and g is the acceleration of the gravity.

In order to find the wave ray trajectory, we will move one by one each of the intermediate points (starting from the closest to the point A) in parallel to the ordinate axis. In this case, the point can move by 1 m up (in the direction of increasing the ordinate) or down. After each variation, we find out whether the wave travel time along the new trajectory decreased as compared to the trajectory before this variation. If it decreases, then we fix a new travel time and this position of the point and proceed to vary the next one. After passing through all intermediate points, the variation procedure is repeated again. In this case, the direction of a passage can be inverse to the previous one (starting at the intermediate point closest to B). Here again, the value of the each point vertical shift is either plus or minus 1 m. This process continues until the wave travel time along the constructed ray stops decreasing after another pass along the entire ray. Figure 2 shows the comparison of the wave ray trajectory constructed by this method (painted red) with the exact solution, which is a segment of a cycloid [2] (painted black).

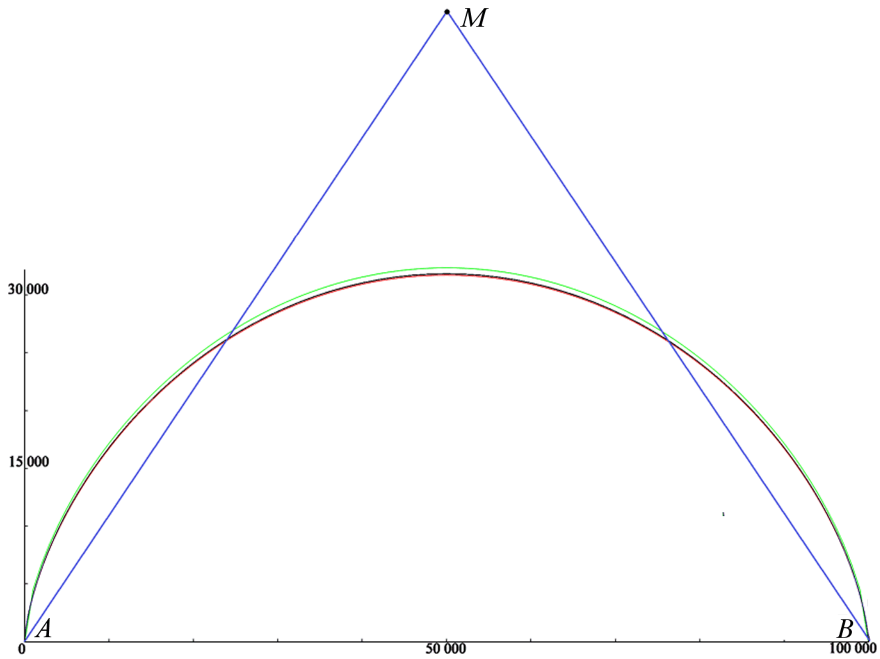


Figure 2. Comparison of exact solution with the wave-ray routes obtained by the method of variation: red — the final ray initiated by the straight segment AB; black — the exact solution (a segment of the cycloid); and green — the final wave ray trajectory initiated by the broken line AMB

In some cases, the wave ray presents only a local extreme route providing only a local minimum of propagation time along this trajectory. Publication [4] gives an example when due to the incorrectly chosen initial approximation, it is not possible using the described algorithm to build a wave ray that would provide a global minimum of the wave travel time between two points. Let us present this example. In the coastal area there is a shelf with a constant depth of D_0. The shelf at a distance y_0 off the shore is followed by the bottom slope. If we need to build a wave ray between the two points A and B situated on the shelf, then, taking a straight segment AB as the initial approximation of the ray, the algorithm described will result in the same segment of the straight line AB connecting these points. According to [2], for a sufficiently large distance between the source and the receiver, there are two solutions for this boundary value problem. The first wave ray will be a straight segment AB. And the second ray is represented by a smooth curve combined of a segment of the cycloid (above the sloping bottom) and two segments of the straight line going from the points A and B to the edge of the shelf (Figure 3).

Therefore, in order to avoid such a situation when determining a wave ray above the bottom slope, the broken line AMB shown in Figure 2 was also

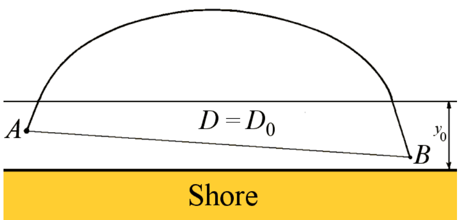


Figure 3. A non-unique solution of the edge problem for the wave ray in the case of the shelf of constant depth

taken as initial approximation of the wave ray. In both cases of initial ray approximation (segment AB and broken line AMB), the resulting broken lines providing a minimum tsunami travel time are quite close to an exact solution. The difference is limited by several tens of meters and is caused by the too large interval between positions of intermediate points. The wave travel time along all the three routes differs by only 1.5 seconds. The shapes of the obtained wave rays together with the exact solution are given in Figure 2. Here, the exact solution is located between the wave rays (red and green lines) obtained by the method described, when the initial approximation was the segment of the straight line between the source and the receiver (red line) and the broken line AMB (green line). The lower wave ray (colored red) practically merges with the arc of the cycloid, representing the exact solution.

Let us consider another test. The task remains the same. It is necessary to determine the wave ray between the two points A and B located 100,000 m away from each other, and one meter off the shoreline. This case differs from the previous test by the depth distribution in the coastal area. Here the bottom has the parabolic topography where the depth increases by the formula

$$D(x, y) = 10^{-6}y^2,$$

where y is the offshore distance. In this test, the initial approximation of the ray trajectory was a straight segment connecting both points. As is shown in [3], the exact solution for the wave ray trajectory with such a bottom topography will be the arc of a circle which is also presented in Figure 4.

Figure 4 shows, that the position of both lines is almost the same. At the same time, the difference between the calculated wave travel time and the exact solution is approximately 20 seconds. This can be explained by errors in estimating the wave travel time between the neighboring intermediate points above the parabolic bottom by the formula (1). Increasing the number of intermediate points (with decreasing the space intervals between them) will improve precision of the travel-time estimation.

The variation method presented can be used for the wave ray determination between two points of the water area, but it should be taken into account that this will be a local extreme, providing a minimum among the routs close to the initial approximation of the ray trajectory. In order to find the ray that give the global travel-time minimum it is necessary to con-

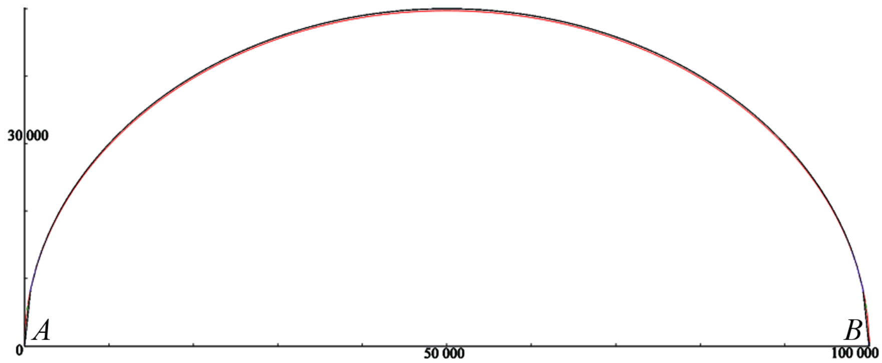


Figure 4. A comparison of the exact solution (painted black) with the wave ray constructed by the variation method (painted red) above the parabolic bottom topography

struct the approximate wave ray by some rough method (for example, the one described in [5]), then the method described here can clarify its position, ensuring that this will be the shortest (with respect to time) route for the tsunami propagation among all possible ones.

As an example, Figure 5 presents the wave rays that were built by the method based on the Huygens principle [5] which can help to roughly determine the wave rays in the grid computational domains. This figure presents the constructed wave rays, which connect the center of the tsunami source

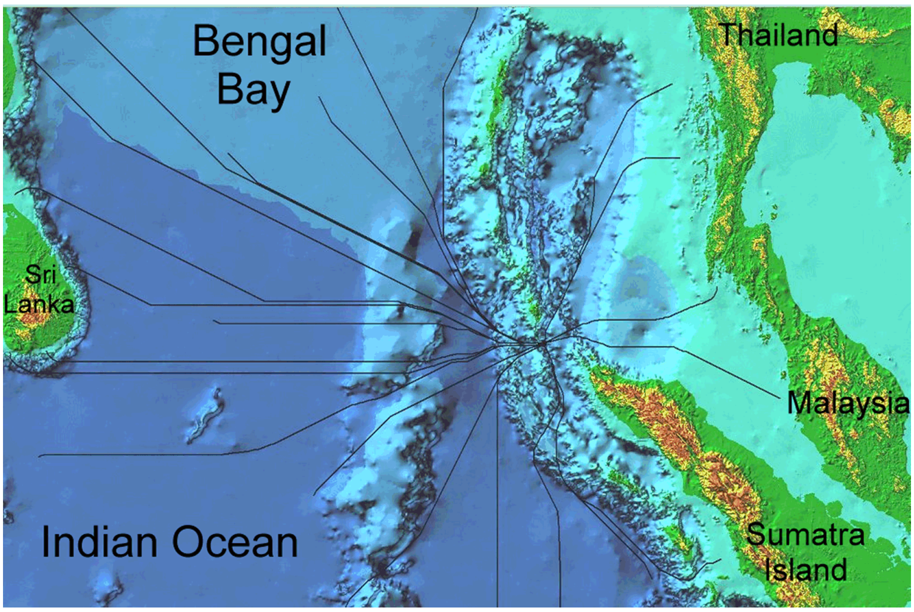


Figure 5. The bottom relief and tsunami wave rays in the Eastern Indian Ocean

of 26.12.2004 with some other points in the Indian Ocean. However, the wave rays constructed by this method are not smooth lines, but are the broken ones composed of segments whose spatial direction has only 16 possible variations [5].

These trajectories can be considered to be the first approximation of real wave rays. For a better accuracy, the trajectory refinement procedure based on the variation position of intermediate points can be proposed. After the end of calculation, each of the grid point through which the wave ray passes (the first approximation) moves along the grid lines in order to find a new position of the point at which the travel time along the changed broken line will be minimal. Such a procedure we do sequentially for each of the grid point through which the first approximation of the wave ray passes. As a result, the ray trajectory becomes smoother, since it no longer necessarily passes through the nodes of the computation grid.

We describe this optimization procedure using as an example the wave ray obtained by method [5]. Let the broken line $SABCD$ be the wave ray which provides minimum travel time between the grid-points S and D (Figure 6). In order to optimize the wave ray trajectory, we move the intermediate point A along the lines of the rectangular calculation grid in all the four directions (up, down, right, left) with a small step (approximately 1/10 of the mesh size). For each new position of the intermediate point A_1 we calculate the propagation time of the wave along the broken line. The depth along segments SA_1 and A_1B is assumed to change linearly. So the wave travel time along them is still determined by the formula (1). In order to

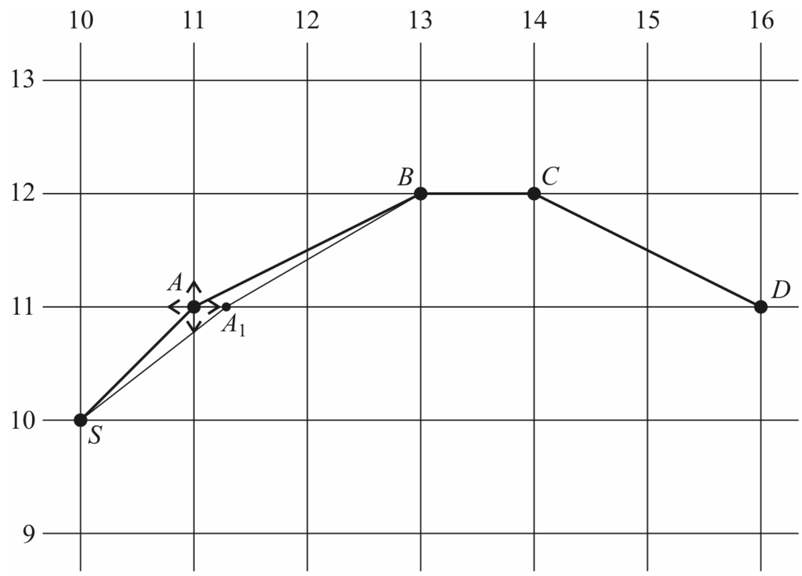


Figure 6. The scheme of the algorithm for optimizing the wave ray path

reach the better travel-time optimizing the intermediate point A_1 can be posed anywhere inside the mesh of computational grid. But this implies additional problems on realization and significant increment of the computer processing time. We are looking for the optimal location of the point A_1 at which the travel time along route SA_1B is minimized. Then we fix the point A_1 and repeat the procedure with the next segment of the ray A_1BC. Now we move the point B, and so on along the whole ray to the point D. Then we have to repeat the procedure in the same or in the reverse direction (from the point D to the point S). If a maximum accuracy is required, then such iterations (passes along the entire ray) must be carried out until the total propagation time cannot be reduced by shifting intermediate points. The practice shows that two passes along the wave ray are sufficient to achieve the accuracy desired. As a result, we get a wave ray that no longer necessarily passes through the nodes of the original calculation grid.

As an example, Figure 7 shows the wave ray above the parabolic bottom before carrying out the optimization procedure (left) and after it (right). Here, the depth increases proportional to the square of the distance from the left boundary of the computational domain. The shape of the "optimized" ray is well correlated with the exact solution, which has the appearance of an arc of a circle.

A similar procedure in the 70s of the last century was proposed and implemented by Braddock [1]. He used such an approach without initial

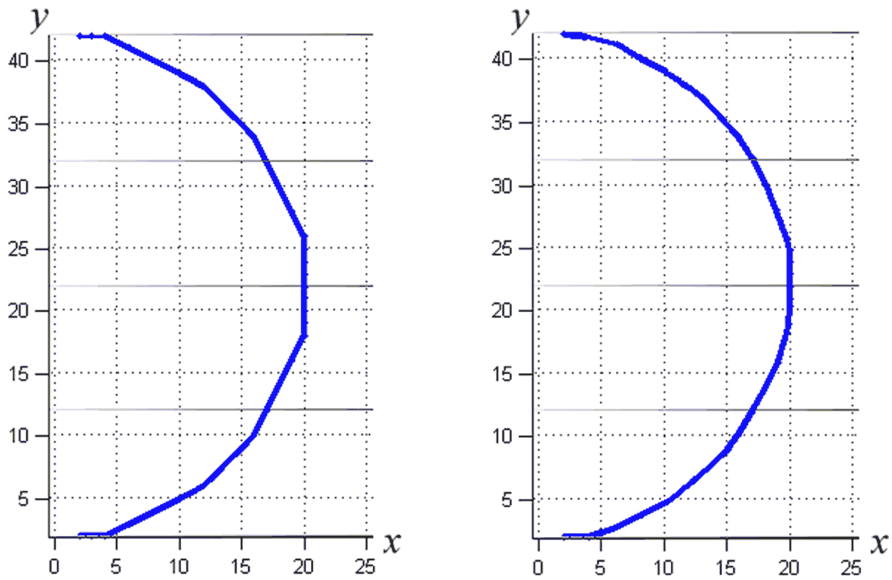


Figure 7. Comparison of the wave ray shapes above parabolic bottom before (left) and after optimization (right)

calculation grid, and as the first approximation of the wave ray trajectory, the arc of a large circle was used (the shortest distance between two points on the surface of the Globe in spherical coordinates). Then the ray route was split to segments whose edges position was moved in orthogonal to the ray direction. The main disadvantage of this approach is that a trajectory obtained gives a local travel time minimum, which is not always the fastest route for tsunami propagation.

This method makes it possible to improve the precision of the travel-time estimates in nodes of a computational grid. The variation approach described can be used in the course of calculation the tsunami arrival times to the grid points using methods based on the Huygens principle. The scheme of such a procedure is shown in Figure 8. The entire calculation stencil used for the arrival time estimation to the grid point A is shown in the bottom-left part of this drawing. It is necessary to find the wave arrival time at point A, using the tsunami arrival time at point C. As a result of the calculations, it was found out that the wave coming from the grid point B arrives at the grid point C in the earliest time as compared to the other neighboring to B points. Further, by varying the position of the point C_1, we are looking for such its position, which would minimize the wave travel time along the broken line BC_1A. This value added to the arrival time at the point B for finding the wave arrival time at the point A. And so on for each grid-point of the computational domain.

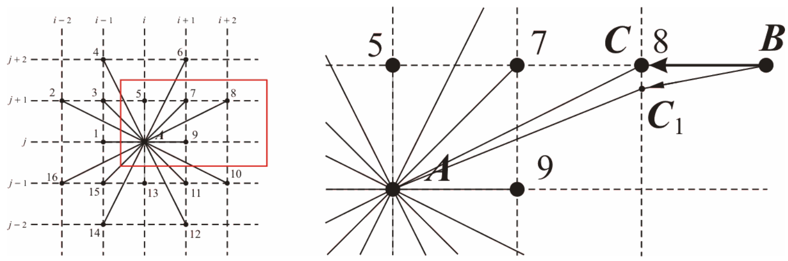
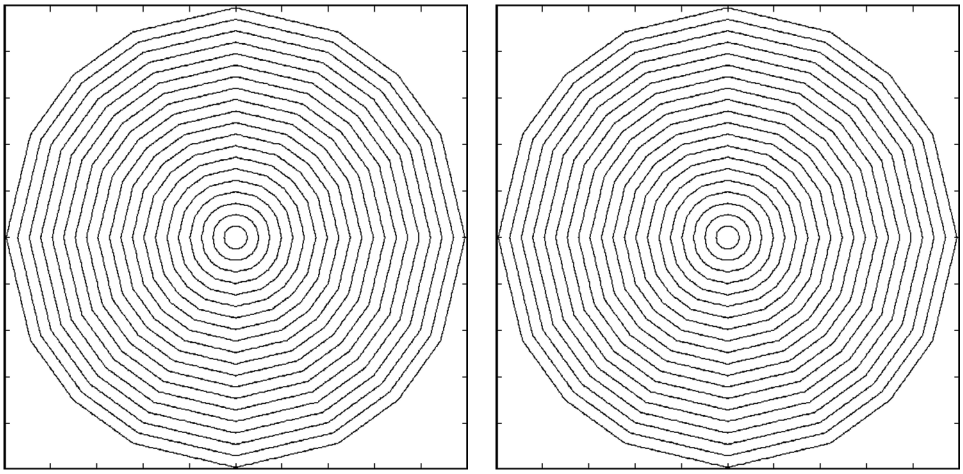


Figure 8. A scheme of the travel-time correction algorithm

The most rigorous test for tsunami gridded kinematic methods is the calculation of tsunami isochrones from a round (point) source in an area with a constant depth. A conventional algorithm with a sixteen-point template [5] gives as the result the tsunami isochrones have the shape of polygons instead of circles (Figure 9a). Here the size of the gridded computational domain is 1000×1000 nodes. The center of the rounded source is located in the central grid-point of the domain. The spatial grid step is equal to 1000 m in both directions. The depth is equal to 1000 m at all the grid points. Figure 9 shows the tsunami front positions in the range from 0 up to 5,000 s with the interval of 250 s.



a b

Figure 9. A tsunami isochrones as a result of testing the kinematic method for gridded data in the area of the uniform depth. The results of non-optimized method (a) and optimized one (b)

Using the described optimization procedure one can significantly correct a map of tsunami isochrones. The travel-time isolines, built according to the results of the calculation by the optimized method (Figure 9b), are much closer in the shape to the circles than the isochrones in the course of "non-optimized" calculation (Figure 9a). At the same time, the difference between the travel times obtained by the "optimized" and the "non-optimized" methods may at some points exceed one minute per each hour of tsunami propagation.

Conclusion

The variation method for determining the tsunami wave ray between two points of the water area can give a local extreme route which sometimes does not match the global extremal providing the shortest tsunami travel time along all possible trajectories. In order to build the global extreme wave trajectory by the variations method we need to take a rough solution obtained by a non-optimized method [5] as first approximation of a wave ray. The procedure of the trace variation can be implemented for the method based on the Huygens principle for improving precision of its travel-time estimates. The implementation of the procedure for travel-time optimization into the method [5] increases the processing time just slightly, because arithmetic calculation is the minor part of an algorithm.

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