Bull. Nov. Comp. Center, Num. Anal., 12 (2003), 57–71

The algorithm generator ALTROS*

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The generator of algorithms to calculate a set of Vandermonde and Hahkel algebraic structures elements is proposed.

1. Generator

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call ALTROS(Np, Npd, A, G, B, X, C, P, H, D, N)
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Remarks: The Jacobi matrix can be presented in the following form: $T = (c_i, b_i, a_i)$ or $S = (a_{i-1}, b_i, a_i)$ or $J = (1, b_i, g_i)$. The relations $DTD^{-1} = J$, $D = \mathrm{diag}(1, 1/c_2, \ldots, 1/(c_2 \cdots c_n))$, $g_i = c_{i+1}a_i$, $i = 1(1)n-1$, hold. The matrices T, S, J are presented by the arrays (C,B,A), (B,A), (B,G), respectively.

Arguments:

- N – the order n of the matrix H (and J, P);
- X – the nodes of the orthogonality array, X(i), $i = 1(1)n$; or the nodes of the Cauchy interpolation;
- C – the weights of the orthogonality array, C(i), $i = 1(1)n$, or subdiagonal of the Jacobi matrix T; or a unknown vector in SLAE; or the nodes of the Cauchy interpolation;
- G – the updiagonal of the Jacobi matrix J, G(i), $i = 1(1)n$; or the value function vector; or the denominator polynomials coefficients;
- A – the updiagonal of the Jacobi matrix S, A(i), $i = 1(1)n$; or the right-hand side vector in SLAE; or a function value from X in Cauchy interpolation;
- B – the diagonal of the Jacobi matrix, B(i), $i = 1(1)n$; or the function value from C in the Cauchy interpolation;
- H – the array of the moments, H(i), $i = 1(1)2n-1$;
- D – the diagonal matrix, D(i), $i = 1(1)n$; or the numerator polynomials coefficients;

*Supported by the Russian Foundation for Basic Research under Grant 01-07-90367.

P – $n \times n$-matrix: P(i,j), $i = 1(1)j$, are coefficients of polynomials of degree $j-1$; or P(i,j), $j = 1(1)n$, are elements of the i-th eigenvector of S; or factorization of the Vandermonde matrix (without identity diagonal); or the Levner matrix;

Np – a flag of the operation:

- Np = 1 – the construction of J:
  Npd = 1: input: H$(2n-1)$; output: A, B, D, P;
  Npd = 2: input: A, B (see comment VOSJP); output: A, B;
  Npd = 3: input: X, C; output: A, B;
- Np = 2 – the construction of P:
  Npd = 1: input: H$(2n-1)$; output: G, B, D, P;
  Npd = 2: J input: B, G; output: P;
  Npd = 3: x_i, c_i input: X, C; output: A, B, P;
- Np = 3 – the construction of x_i, c_i on S;
  input: A, B; output: X, C, P;
- Np = 4 – the solution to SLAE (A is input):
  Npd = 1: $Hc = a$ (input: H; output: C);
  Npd = 2: $Jc = a$ (input: B, G; output: C);
  Npd = 3: $VA = a$ (input: X; output: A);
  Npd = 4: $V^T A = a$ (input: X; output: A);
- Np = 5 – the matrix inversion:
  Npd = 1: V input: X; output: P;
  Npd = 2: J input: B, A; output: P;
- Np = 6 – the matrix factorization:
  Npd = 1: H^{-1} (input: H; output: B, G, D, P);
  Npd = 2: V^{-1} (input: X; output: P);
- Np = 7 – the interpolation:
  Npd = 1: Lagrange (input: X, G, H(1); output: H(2));
  Npd = 2: Newton (input: X, G, H(1); output: H(2));
  Npd = 3: Cauchy (input: the nodes X, C and the corresponding function values A, B; output: the coefficients of the polynomials of the denominator G and of the numerator D with the decrease of degrees (see (33), (34) for $q(x)$, $w(x)$));
  Npd = 4: Cauchy (input: the nodes X, C and the corresponding function values A, B, the node of interpolation H(1); output: the coefficients of the polynomials of the denominator G and of the numerator D with the decrease of degrees, the function value H(2) in the node of interpolation H(1));
- Np = 8 – the quadrature of the function $g(x)$ on [A(1), A(2)]:
  Npd = 1: Newton–Kotes (input: H, X, G; output: H(1));
  Npd = 2: Gauss (input: H, G; output: A(n));

Npd $= 3$: Radaux (input: H, G; output: A(n));
Npd $= 4$: Lobatto (input: H, G; output: A(n));

- Np $= 10$ – the calculation of eigenvalues of S by the Sturm method (input: A, B; output: X, P);
- Np $= 11$ – the calculation of Hamilton form (input: A, B, $0 \leq$ A$(n) = T \leq 1$; output: A, B, D);
- Np $= 12$ – the conjugate Sturm system:
  Npd $= 1$: the conjugate system and a complementary matrix (input: A, B; output: A, B, X, C of the new matrix; according to (32), $a = 1, b = 0$);
  Npd $= 2$: check of conditions (output: the strings).

Remarks: The subroutine ALTROS writes into the file ALTR the data to be computed and some control data as well:

Np $= 3$ and Np $= 10$:

- the string 'The error of Spur = ' mEp;
- the string 'It is not a Jacobi matrix' if $c_{i+1}a_i \leq 0$;
- the string 'The complex roots' if the matrix is not a Jacobi one;
- the string 'The multiple roots' if the matrix is not a Jacobi one;

Np $= 7$:

- the string 'N is too great';

Np $= 11$:

- the string 'The condition $a(i) < 0$ is violated';
- the string 'The absolute error $\mu =$ ' mEp, '$\mu =$ ' mEp;

Np $= 12$:

- the string 'The matrix is not a complementary one';
- the string 'The complementary matrix is obtained'.

2. Algorithms

The triple algebraic structure is a set of algebraic objects, connected with one-to-one relations. The Vandermonde and the Hankel structures, whose algorithms presented in ALTROS, are more often used.

The Vandermonde structure. In this structure [2, 4], the Vandermonde matrix V is the main object:

$$V = \begin{bmatrix} 1 & x_1 & \dots & x_1^{n-1} \\ 1 & x_2 & \dots & x_2^{n-1} \\ \vdots & \vdots & & \vdots \\ 1 & x_n & \dots & x_n^{n-1} \end{bmatrix}, \tag{1}$$

$V = \sum_{i=1}^{n} X^{i-1} e e_i^T$, where $e^T = (1, \dots, 1)$, $X = \operatorname{diag}(x_1, \dots, x_n)$.

Introduce the node polynomials, whose roots are $x_1, \dots, x_k$, $k = 1(1)n$:

$$\pi_k(x) = \prod_{i=1}^{k} (x - x_i) = \sum_{i=1}^{k} c_{k,k-i} x^i, \quad k = 1(1)n, \tag{2}$$

$$\varphi_j^{(k)}(x_j) = \frac{\pi_k(x)}{x - x_j},$$

$$\varphi_i(x) = \varphi_i^{(n)}(x) = \sum_{j=2}^{n-1} \psi_{i,n-j-1} x^j. \tag{3}$$

For example, the Vandermonde determinant is

$$|V| = \prod_{k=2}^{n} \pi_{k-1}(x_k).$$

If

$$\Psi = \begin{bmatrix} \psi_{1,n-1} & \psi_{2,n-1} & \dots & \psi_{n,n-1} \\ \vdots & \vdots & & \vdots \\ \psi_{1,0} & \psi_{2,0} & \dots & \psi_{n,0} \end{bmatrix}$$

and $\varphi = \operatorname{diag}(\varphi_1(x_1), \dots, \varphi_n(x_n))$, then

$$V^{-1} = \Psi \varphi^{-1}. \tag{4}$$

The last row of the matrix Ψ is $e = (1, \dots, 1)^T$, hence $e_n^T V^{-1} = e^T \varphi^{-1}$.

The polynomials

$$l_i(x) = \frac{\varphi_i(x)}{\varphi_i(x_i)}, \quad i = 1(1)n, \tag{5}$$

are called *the fundamental Lagrange polynomials*. They are defined by the property $l_i(x_j) = \delta_{ij}$, $i, j = 1(1)n$, providing their linear independence. The entries of the i-th column of the inverse Vandermonde matrix are coefficients of the polynomial $l_i(x)$.

The *interpolational polynomial* in the Lagrange form is the following:

$$P_{n-1}(x) = \sum_{i=1}^{n} f_i l_i(x),$$

where f_i is a number and $P_{n-1}(x_j) = f_j$, $j = 1(1)n$, i.e. the interpolational polynomial has the given values f_j at the nodes x_j (for example, the function values $f(x_j)$).

Let us introduce a vector

$$v(x) = \sum_{j=1}^{n} x^{j-1} e_j.$$

Then the form $P_{n-1}(x) = (f, (V^T)^{-1} v(x))$ or

$$P_{n-1}(x) = (V^{-1} f, v(x)) \tag{6}$$

is a coefficient representation of the interpolational polynomial.

The identities

$$x^k = \sum_{i=1}^{n} x_i^k l_i(x) + \delta_{kn} \pi_n(x), \quad k = 1(1)n,$$

and the equalities

$$f_j = \sum_{i=1}^{n} f_i l_i(x_j), \quad j = 1(1)n,$$

are hold.

The interpolational polynomial $P_{n-1}(x)$ can be presented in the Newton form, where the vector f in (4) is changed with the divided differences of order k:

$$\Delta[f_1, \dots, f_{k+1}] = \frac{\Delta[f_1, \dots, f_{k-1}, f_{k+1}] - \Delta[f_1, \dots, f_k]}{x_{k+1} - x_k}, \quad \Delta[f_1] = f_1 \tag{7}$$

(recurrence definition). Another representation

$$\Delta[f_1, \dots, f_{k+1}] = \sum_{i=1}^{k+1} \frac{f_i}{\varphi_i^{(k+1)}(x_i)}, \quad k = 1(1)n-1, \tag{8}$$

holds as well.

Theorem 1. *The triangle factorization of the Vandermonde matrix can be presented in the form* $V = TW$, *where* $T = (t_{ij})$, $W = (w_{ij})$, $i, j = 1(1)n$, *are the lower and the upper triangle matrices, respectively, and*

$$w_{ij} = q_{j-i}^{(i)} = \sum_{l=1}^{i} \frac{x_l^{j-1}}{\varphi_l^{(i)}(x_l)}, \quad t_{ji} = \pi_{i-1}(x_j).$$

$$w_{ij} = w_{i-1,j-1} + x_i w_{i,j-1}, \quad j \geq i, \ i = 2(1)n,$$

$$w_{1j} = x_1^{j-1}, \quad j = 1(1)n, \quad w_{ij} = 0, \quad j < i \ or \ i < 0.$$

In addition,

$$V^{-1} = W^{-1}T^{-1}, \tag{9}$$

and $T^{-1} = (t^*_{ij})$, $W^{-1} = (w^*_{ij})$, $i, j = 1(1)n$,

$$t^*_{jl} = \frac{1}{\varphi_l^{(j)}(x_l)}, \quad l = 1(1)j. \tag{10}$$

$$w^*_{ij} = c_{j-1,j-i}, \quad i = 1(1)j. \tag{11}$$

Let equation (8) for the interpolational polynomial be presented in the vector form. If $f = (f_1, \dots, f_n)^T$ and $Z = (z_1, \dots, z_n)^T$, $z_k = \Delta[f_1, \dots, f_k]$, then from (10) we obtain

$$Z = T^{-1}f, \quad P_{n-1}(x) = (W^{-5}Z, v(x)) = (Z, W^{-T}v(x)),$$

$$P_{n-1}(x) = \sum_{k=1}^{n} z_k \pi_{k-1}(x),$$

i.e., the Newton form of the interpolational polynomial. The recurrence definition (7) is suitable for the computation of z_k.

The interpolational polynomial has the coefficient form as well:

$$P_{n-1}(x) = \sum_{j=0}^{n-1} a_{j+1} x^j = (a, v(x)).$$

If $a = (a_1, a_2, \dots, a_n)^T$ and f is the vector of values of $P_{n-1}(x)$ at the nodes $x_1, \dots, x_n$, then

$$Va = f. \tag{12}$$

For solving system (12) with the Vandermonde matrix one can use representation (4) or (9) of the inverse matrix, for example.

The following lemma states a better method.

Lemma 1. *Let* $z_k = \Delta[f_1, \dots, f_k]$. *Then*

$$P_{n-1}(x) = \pi_j(x) P_{n-j-1}^{(j)}(x) + \sum_{k=1}^{j} z_k \pi_{k-1}(x), \quad j = 1(1)n,$$

where

$$P_{n-j-1}^{(j)}(x) = \sum_{k=0}^{n-j-1} a_{k+1}^{(j)} x^k = \Delta[f_1, \dots, f_j, P_{n-1}(x)]$$

and the coefficients $a_{k+1}^{(j)}$, $k = 0(1)n-j-1$, *are determined by the relations*

$$\sum_{k=0}^{n-j-1} a_{k+1}^{(j)} x_i^k = \Delta[f_1, \ldots, f_j, f_i], \quad i = j+1(1)n.$$

The identity

$$P_{n-j}^{(j-1)}(x) = P_{n-j}^{(j-1)}(x_j) + (x - x_j) P_{n-j-1}^{(j)}(x),$$

generates an efficient algorithm solving the system with a Vandermohde matrix. So, if the vector $Z = (z_1, \ldots, z_n)^T$ is obtained, then the coefficients of the polynomial $P_{n-1}(x)$ are defined by the algorithm [1], [4]:

$$\begin{aligned} a_1^{(n-1)} &= z_n, \\ a_0^{(j+1)} &= z_{j+1}, \\ a_{k+2}^{(j)} &= a_k^{(j+1)} - x_{j+1} a_{k+1}^{(j+1)}, \quad k = 0(1)n-j-2, \\ a_{n-j}^{(j)} &= a_{n-j-1}^{(j+1)}, \\ j &= n-2(-1)0 \end{aligned}$$

(the value $a_0^{(j+1)}$ is introduced as auxiliary one).

In another problem with a Vandermonde matrix we have

$$V^T C = M,$$

where $C = (c_1, \ldots, c_n)^T$ is an unknown weight vector, $M = (m_1, \ldots, m_n)^T$ is the moment vector. In this case the following lemma holds:

Lemma 2. *The relations*

$$\sum_{j=k+1}^{n} x_j^{i-1} c_j^{(k)} = m_{i+k}^{(k)}, \quad k = 0(1)n-1, \ i = 1(1)n-k,$$

take place, and the values $c_j^{(k)}$ *are determined by recursion* [1, 4]:

$$\begin{aligned} c_n^{(n-1)} &= m_n^{(n-1)}, \\ c_j^{(k-1)} &= c_j^{(k)}/(x_j - x_k), \quad j = n(-1)k+1, \\ c_k^{(k-1)} &= m_k^{(k-1)} - \sum_{j=k+1}^{n} c_j^{(k-1)}, \quad k = n-1(-1)1, \\ c_j &= c_j^{(0)}, \quad j = 1(1)n, \end{aligned}$$

when the values $m_k^{(k-1)}$,

$$m_l^{(0)} = m_l, \quad l = 1(1)n,$$
$$m_l^{(k)} = m_l^{(k-1)} - x_k m_{l-1}^{(k-1)}, \quad l = k+1(1)n, \quad k = 1(1)n-1,$$

are computed.

Henkel structure. The Hankel structure [4], [6], [7], [10], [11] is determined by the positive definite Hankel matrix H. It is characterized by a large set of relations. So,

$$H = R^T D R = V^T C V,$$

where $D = \mathrm{diag}(d_1, \dots, d_n)$, $R = (r_{ij})$, $i, j = 1(1)n$, is the upper identity triangular matrix,

$$d_i = \frac{|H_i|}{|H_{i-1}|}, \qquad r_{ij} = \frac{|H_i(i,j)|}{|H_i|},$$

V is a Vandermonde matrix (1), $C = \mathrm{diag}(c_1, \dots, c_n)$, $R_{\cdot,k+1} = JR_{\cdot,k}$, $k = 1(1)n-1$,

$$J = \begin{bmatrix} b_1 & g_1 & & \\ 1 & b_2 & \ddots & \\ & \ddots & \ddots & g_{n-6} \\ & & 1 & b_n \end{bmatrix}, \tag{13}$$

$$d_{k+1} = \prod_{i=1}^{k} g_i.$$

Another form of this representation is

$$HP = R^T D. \tag{14}$$

Here the k-th column of the matrix $P = (p_i^j)$, i, $j = 0(1)n-1$; $P = R^{-1}$, has the form of the coefficients of the orthogonal polynomial

$$P_{k-1}(x) = \sum_{i=0}^{k-1} p_i^{k-1} x^i, \quad k = 1(1)n,$$

with unit in a maximum term (in the Hankel structure $P_k(x)$ is not an interpolational, but orthogonal polynomial). The orthogonal relations are:

$$(VP)^T CVP = D \tag{15}$$

or

$$\sum_{i=1}^{n} c_i P_{l-1}(x_i) P_{k-1}(x_i) = \delta_{kl} d_k, \quad k, l = 1(1)n,$$

and

$$\begin{aligned} P_0(x) &= 1, \\ P_1(x) &= x - b_1, \\ P_k(x) &= (x - b_k)P_{k-1}(x) - g_{k-1}P_{k-2}(x), \quad k = 2(1)n, \end{aligned} \tag{16}$$

The roots of the polynomial $P_n(x)$, x_i, $i = 1(1)n$, are eigenvalues of the matrix J:

$$VPJ = XVP,$$

$X = \mathrm{diag}(x_1, \ldots, x_n)$, and its eigenvectors are expressed via the values of polynomials $P_k(x_j)$.

The polynomials defined in (16) have the Sturm property: the number of sign change $W(x)$ in the sequence $P_0(x), \ldots, P_n(x)$ decreases when x monotonically increases in such a manner, that $W(x_i - \varepsilon) = 1 + W(x_i + \varepsilon)$, $\varepsilon > 0$ is a small number. As polynomials (12) are computed with the total factor, we may take $P_n(x) = 1$, but not $P_0(x) = 1$. In such a system with a Jacobi matrix, the right-hand-side vector is e_k.

The solution to the system $TX = F$,

$$T = \begin{bmatrix} b_1 & a_1 & & \\ c_2 & b_2 & \ddots & \\ & \ddots & \ddots & a_{n-1} \\ & & c_n & b_n \end{bmatrix},$$

where $X = (x_1, \ldots, x_n)^T$, $F = (f_1, \ldots, f_n)^T$, is obtained in the following manner:

$$\begin{aligned} x_n &= v_n, \\ x_k &= u_k x_{k+1} + v_k, \quad k = n - 1(-1)1, \\ u_k &= -\frac{a_k}{b_k + c_k u_{k-1}}, \quad u_0 = 0, \\ v_k &= \frac{f_k - c_k v_{k-1}}{b_k + c_k u_{k-1}}, \quad v_0 = 0, \quad k = 1(1)n. \end{aligned}$$

If $|u_k| < \varepsilon$, then the next entry in the beginning of column is leading.

The eigenvalues x_i are the orthogonality nodes. The weights of orthogonality are defined by the relations

$$c_i = \frac{1}{d_n P_{n-1}(x_i)\varphi_i(x_i)}, \quad i = 1(1)n.$$

The entries of the matrix J are connected by the relations

$$\begin{aligned} b_j &= p_{j-2}^{j-1} - p_{j-1}^{j}, \quad j = 1(1)n, \\ g_j &= \frac{d_{j+1}}{d_j}, \quad j = 1(1)n - 1. \end{aligned}$$

In another problem, the values x_i, c_i, $i = 1(1)n$, are given, and it is required to find entries of the matrix S,

$$S = \begin{bmatrix} b_1 & a_0 & & \\ a_1 & b_2 & \ddots & \\ & \ddots & \ddots & a_{n-1} \\ & & a_{n-1} & b_n \end{bmatrix}, \tag{17}$$

which connect the polynomials $q_k(x)$, $k = 0(1)n-1$ ($a_0 = 0$, $a_n = 1$):

$$q_0 = 1, \quad a_k q_k(x) = (x - b_k)q_{k-1}(x) - a_{k-1}q_{k-2}(x), \quad k = 1(1)n. \tag{18}$$

The orthogonal conditions in this case have the form

$$\sum_{i=1}^{n} c_i q_k(x_i) q_l(x_i) = \delta_{kl}, \quad k = 1(1)n. \tag{19}$$

Then, from (18) and (19) we have

$$b_k = \sum_{i=1}^{n} c_i x_i q_{k-1}^2(x_i), \quad k = 1(1)n, \tag{20}$$

$$a_k = \left(\sum_{i=1}^{n} c_i((x_i - b_k)q_{k-1}(x_i) - a_{k-1}q_{k-1}(x_i))^2\right)^{1/2}, \qquad k = 1(1)n-1.$$

After the next calculation b_k, a_k, $k = 1(1)n-1$, we find the values $q_k(x_j)$, $j = 1(1)n$, $k = 1(1)n-1$,

$$q_k(x_j) = \frac{(x_j - b_k)q_{k-1}(x_j) - a_{k-1}q_{k-1}(x_j)}{a_k}. \tag{21}$$

Finally, b_n is computed by formula (20). The nodes x_i are the roots of the polynomial $q_n(x)$ from (21).

This algorithm [18] is stable for $n \le 50$.

Construction of the matrix J by Hankel matrices is also possible [3]. It follows from (14) and (16) that

$$\sum_{j=0}^{k} h_{k+j} p_j^k = d_{k+1}, \quad \sum_{j=0}^{k} h_{k+j-1} p_j^k = 0,$$

$$p_{-1}^k = p_{k+1}^k = 0, \quad p_k^k = 1,$$

$$p_j^k = p_{j-1}^{k-1} - b_k p_j^{k-1} - g_{k-1} p_j^{k-2}, \quad j = 0(1)k-1.$$

The coefficients of $P_0(x)$ and $P_1(x)$ are known first of all, and d_1, d_2, g_1 are known as well. Hence we find

$$b_k = \frac{\rho_{k-1}}{d_k} - \frac{\rho_{k-2}}{d_{k-1}},$$

where

$$\rho_l = \sum_{j=0}^{l} h_{l+j+1} p_j^l, \quad l = k-1, k.$$

After that the coefficients of the polynomials $P_k(x)$ and d_{k+1}, are computed. As $g_k = d_{k+1}/d_k$, then this recurrence continues.

Now let us consider [9] the generalized eigenvalue problem

$$S_H U = DUX, \tag{22}$$

for the Hamilton form S_H of the positive (nonnegative) definite Jacobi matrix, where

$$S_H = \begin{bmatrix} \beta_1 & \alpha_1 & & \\ \alpha_1 & \beta_2 & \ddots & \\ & \ddots & \ddots & \alpha_{n-1} \\ & & \alpha_{n-1} & \beta_n \end{bmatrix}, \tag{23}$$

with the conditions $\beta_i = \alpha_{i-1} + \alpha_i$. Here $D = \mathrm{diag}(m_1, \dots, m_n)$, $X = \mathrm{diag}(x_1, \dots, x_n)$, U is the corresponding fundamental matrix.

The transformation from (17) with $a_i < 0$ to (22), (23):

$$S_H = D^{1/2} S D^{1/2},$$

is a congruent one.

Let us denote $\theta_i = -a_i\sqrt{m_{i+1}/m_i} > 0$, $i = 1(1)n$,

$$b_1 \geq \varepsilon = \frac{\alpha_0}{m_1} \geq 0, \quad b_n \geq \mu = \frac{\alpha_n}{m_n} \geq 0,$$

then the relations

$$\theta_1 = b_1 - \varepsilon, \quad \theta_i = b_i - \frac{a_{i-1}^2}{\theta_{i-1}}, \quad i = 2(1)n-1, \quad 0 = b_n - \mu - \frac{a_{n-1}^2}{\theta_{n-1}}$$

are valid. Hence follows

$$\varepsilon = \frac{\mu g_{n-1} - g_n}{\mu h_{n-1} - h_n} \tag{24}$$

and

$$\mu = \frac{\varepsilon h_n - g_n}{\varepsilon h_{n-1} - g_{n-1}}, \tag{25}$$

$$g_2 = 1, \quad g_1 = b_1, \quad g_k = b_k g_{k-1} - a_{k-1}^2 g_{k-2}, \quad k = 2(1)n,$$
$$h_0 = 0, \quad h_1 = 1, \quad h_k = b_k h_{k-1} - a_{k-1}^2 h_{k-2}, \quad k = 2(1)n.$$

The choice of ε (or μ from relation (25)) can be arbitrary from the interval $[0, g_n/h_n]$ (or from the interval $[0, g_n/g_{n-1}]$). When the choice is made, we can find θ_i and then find m_i,

$$m_{i+1} = m_i \prod_{j=1}^{i} \left(\frac{\theta_j}{a_j}\right)^2,$$

also, $\alpha_0 = \varepsilon m_1$, $\alpha_n = \mu m_n$, $\alpha_i = -a_i\sqrt{m_i m_{i+1}}$, $i = 2(1)n-1$, and $\beta_i = \alpha_{i-1} + \alpha_i$, $i = 2(1)n$. The value m_1 must be known.

For the symmetric Jacobi matrix S (17) with the conditions $a_k > 0$, $k = 1(1)n-1$, with eigenvalues

$$x_n < x_{n-1} < \ldots < x_2 < x_1 \tag{46}$$

and the system of polynomials $q_k(x)$ (18), there is a problem of the conjugate Sturm system. For the sequence

$$q_1(x_i), \ldots, q_n(x_i) \tag{27}$$

with ordering of (26) there are $i-1$ sign changes.

Let us define [5, 8] the polynomials $q_i^y(y)$ by the relations

$$q_i^y(y_l) = \frac{\rho_i q_l(x_i)}{\rho_1 q_l(x_1)}, \quad i, l = 1(1)n, \tag{28}$$

where $\rho_i^2 = c_i$ are the orthogonality weights of the polynomials $q_k(x)$, $\rho_i > 0$.

Theorem 2. *The polynomials $q_i^y(x)$ of order $i-1$ are orthogonal ones with the weights $c_l^y = (\rho_l^y)^2$,*

$$c_l^y = c_1 q_l^2(x_1), \quad \sum_{l=1}^{n} c_l^y = 1, \tag{29}$$

at the nodes y_l, $l = 1(1)n$.

Lemma 3. *The polynomial $q_i^y(x)$, $i = 1(1)n$, has the degree $(i-1)$ if and only if the relations*

$$\sum_{i=1}^{n} \frac{\psi_{i,n-l}^y q_i(x_k)}{\varphi_i^y(y_i) q_i(x_1)} = 0, \quad l = k+1(1)n, \quad k = 3(1)n-1, \tag{30}$$

or

$$\frac{q_l(x_k)}{q_l(x_1)} = \sum_{i=1}^{k} \frac{q_i(x_k)}{q_i(x_1)} \prod_{j=1, j\neq i}^{k} \frac{y_l - y_j}{y_i - y_j}, \quad l = k+1(1)n, \ k = 1(1)n-1, \tag{31}$$

are valid.

Theorem 3. *Under the ordering of* (26), *the nodes* y_l, *where*

$$y_l = a\frac{\rho_2 q_l(x_2)}{\rho_1 q_l(x_1)} + b, \quad a > 0, \quad l = 1(1)n \tag{32}$$

are ordered similarly, i.e., $y_n < y_{n-1} < \ldots < y_2 < y_1$.

Theorem 4. *The fundamental matrix* O *of the matrix* S *is the left fundamental matrix of the matrix* S^c: $SO = OX$, $OS^c = YO$.

The polynomial $w(x) = -\pi_n(x)$ is connected with a nonsingular Hankel matrix H through its mutually distinct roots x_i, $i = 1(1)n$. Let $X = \{x_1 \ldots, x_n\}$. In addition, let us consider the sets Y, Z of the mutually distinct nodes y_i, z_i, $i = 1(1)n$, with the condition

$$w(y_i) \neq 0, \quad w(z_i) \neq 0, \quad i = 1(1)n.$$

Also, let us consider the polynomials

$$\pi_n^{(x)}(x), \quad \pi_n^{(y)}(x), \quad \pi_n^{(z)}(x), \quad \varphi_i^{(x)}(x), \quad \varphi_i^{(y)}(x), \quad \varphi_i^{(z)}(x),$$

so

$$\pi_n^{(x)}(x_j) = 0, \quad \pi_n^{(y)}(y_j) = 0, \quad \pi_n^{(z)}(z_j) = 0, \quad j = 1(1)n,$$

($\pi_n^{(x)}(x) = \pi_n(x)$, $\varphi_i^{(x)}(x) = \varphi_i(x)$, etc.).

The Vandermonde matrix $V(x) = V$, $V(y)$, $V(z)$ and $\Psi(x) = \Psi$, $\Psi(y)$, $\Psi(z)$, $\varphi(x) = \varphi$, $\varphi(y)$, $\varphi(z)$ correspond to the sets X, Y, Z.

For example, $\varphi(y) = \mathrm{diag}(\varphi_1^{(y)}(y_1), \ldots, \varphi_n^{(y)}(y_n))$, $V(y)\Psi(y) = \varphi(y)$. So, the Levner matrix $L = \Psi^T(y)H\Psi(z)$ has the form

$$L = \begin{bmatrix} \dfrac{s_1 - t_1}{y_1 - z_1} & \cdots & \dfrac{s_1 - t_n}{y_1 - z_n} \\ \cdots & & \cdots \\ \dfrac{s_n - t_1}{y_n - z_1} & \cdots & \dfrac{s_n - t_n}{y_n - z_n} \end{bmatrix},$$

where

$$s_j = \frac{q(y_j)}{w(y_j)}, \quad t_j = \frac{q(z_j)}{w(z_j)},$$

$q(x)$ is a polynomial of the degree $n - 1$. The polynomials $q(x)$, $w(x) = -\pi_n(x)$ are mutually distinct.

Let the sets (y_i, s_i), (z_i, t_i), $i = 1(1)n$, be given. In the Cauchy interpolation, the ration function $r(x) = q(x)/w(x)$, such that $r(y_i) = s_i$, $r(z_i) = t_i$, $i = 1(1)n$, can be calculated in the followisg form:

Theorem 5 (M. Fidler). *Let the Levner matrix* L *be nonsingular and* ξ *is a real number, such that*

$$w_\xi(x) = -\frac{1}{|L|}\begin{vmatrix} \frac{s_1-t_1}{y_1-z_1} & \dots & \frac{s_1-t_n}{y_1-z_n} & \varphi_1^{(y)}(x) \\ \dots & \dots & \dots & \dots \\ \frac{s_n-t_1}{y_n-z_1} & \dots & \frac{s_n-t_n}{y_n-z_n} & \varphi_n^{(y)}(x) \\ t_1-\xi & \dots & t_n-\xi & \pi_n^{(y)}(x) \end{vmatrix}, \quad (33)$$

$$q_\xi(x) = -\frac{1}{|L|}\begin{vmatrix} \frac{s_1-t_1}{y_1-z_1} & \dots & \frac{s_1-t_n}{y_1-z_n} & s_1\varphi_1^{(y)}(x) \\ \dots & \dots & \dots & \dots \\ \frac{s_n-t_1}{y_n-z_1} & \dots & \frac{s_n-t_n}{y_n-z_n} & s_n\varphi_s^{(y)}(x) \\ t_1-\xi & \dots & t_n-\xi & -\xi\pi_n^{(y)}(x) \end{vmatrix}, \quad (34)$$

and $w_\xi(y_i)$, $w_\xi(z_i)$ do not vanish. Then $q_\xi(x)/w_\xi(x)$ is the ration function $r(x)$ of the Caushy interpolation. In addition, the polynomials $q_\xi(x)$, $w_\xi(x)$ are mutually distinct ones, $w(x) = -\pi_n(x)$. The degree of the polynomial $q_\xi(x)$ is not greater than n.

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