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A subgrid model for the flow in the fractal porous media*

G.A. Kuz'min, O.N. Soboleva

We present a subgrid model for the flow of incompressible fluid through fractal porous rocks. Using the scaling hypothesis for the latter, we derive an expression for the effective permeability.

1. The problem of subgrid modeling

Rocks have a developed porous structure that influences the fluid flows, waves and other propagating fields. The choice of the mathematical model depends on the scales of the process under consideration and the scale of porosity. If the typical scale of porosity is small when compared to the scale of process, then continuous model may be suitable. In such models, characteristics of the media is described by the simple constant parameters such as thermoconduction or elasticity coefficient or another related parameters. After that the process is modeled by an equation with a set of effective parameters. In the opposite case, one deals with a model with parameters that vary in space and in time and the scales of variation are not small when compared to the scale of process. In that case, one is not able to use the constant parameters. That parameters fluctuate randomly. The mathematical model of such the media have to be formulated statistically.

It has been recognized that the porous structure may be approximated by the statistically scale invariant models. This has been led to the use of various models and the terminology of the physics of disordered phenomena. In particular, the percolation theory and the fractal models were recognized to be useful for modeling the flow and dispersion in the porous media (see, for example, the review in [1, 2]). We consider the scaling model of a random fractal fields. We derive an equation for the single phase flow in the fractal porous media, keeping in mind that the similar methods might be useful for the wave field evolution and for another related characteristics.

The formulation of the model is given in the next section. After that we consider the problem of subgrid modeling of the heat conduction in such a media. Physically, the problem is formulated as follows. Let the incompressible fluid steadily flow through a media with fluctuating permeability

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coefficient $\varepsilon(\boldsymbol{x})$. At low Reynolds number, the velocity is given by the Darcy law $\boldsymbol{v} = \varepsilon(\boldsymbol{x})\nabla p$, where p is the pressure. The incompressibility condition $\operatorname{div} \boldsymbol{v} = 0$ leads to the equation for p

$$\frac{\partial}{\partial x_j}\left[\varepsilon(\boldsymbol{x})\frac{\partial}{\partial x_j}p(\boldsymbol{x})\right] = 0. \tag{1}$$

We assume that the fluctuations of ε from extreme wide range of scales exists so that direct computation of the pressure field $p(\boldsymbol{x})$ from the equation is not possible. Is it possible to derive from 1 an equation that describe only the fluctuating pressure field of the largest scale? Similar problem arise in various regions of physics. In the theory of turbulent flow at very large Reynolds number, the main information is contained in the pulsations of the largest scale. Extensive efforts were spent to derive a mathematical model that contains only such large scale fluctuations. Similar model may be of interest in the problem under consideration.

Let us divide the fluctuating function $\varepsilon(\boldsymbol{x})$ into two components. The large scale component is obtained from $\varepsilon(\boldsymbol{x})$ via a spatial smoothing

$$\varepsilon_l(\boldsymbol{x}) = \int W(\boldsymbol{r}, l)\varepsilon(\boldsymbol{x} + \boldsymbol{r})\, d^D r. \tag{2}$$

The short wave (subgrid) component is $\varepsilon' = \varepsilon - \varepsilon_l$. In the above formula, $D = 1, 2, 3$ is the spatial dimension, $W(\boldsymbol{r}, l)$ is the filter function that tends to zero at $r > l$ and satisfies $\int W(\boldsymbol{r}, l) d^D r = 1$. Often one uses either the Gaussian filter $W(\boldsymbol{r}, l) = (1/\pi^{D/2})\exp(-r^2/l^2)$ or the Fourier filter. We use the Fourier filter that omits all the Fourier harmonics whose wavelength is shorter than some threshold value l.

The statistical distribution for p is determined by that for ε via equation (1). We define the large scale ongrid pressure field $p_l(\boldsymbol{x})$ as a solution of (1) where the large scale component ε_l is fixed, but that is averaged over the statistical distribution of ε', $p_l(\boldsymbol{x}) = \langle p(\boldsymbol{x})\rangle_l$. The complementary subgrid component $p' = p - p_l$ is not supposed to be interesting for us, but it can not generally be thrown out from the filtered equation

$$\frac{\partial}{\partial x_j}\left[\varepsilon_l(\boldsymbol{x})\frac{\partial}{\partial x_j}p_l(\boldsymbol{x}) + \left\langle \varepsilon'(\boldsymbol{x}), \frac{\partial}{\partial x_j}p'(\boldsymbol{x})\right\rangle_l\right] = 0 \tag{3}$$

because the second term may be essential. The choice of the form of the second term in (3) determines the subgrid model. We derive simple gradient form for this term using the scale and conformal invariance hypothesis for the fractal media.

2. The fractal porous media

Let us consider the statistics of the smoothed field (2). At $l \to 0$, $\varepsilon^l(\boldsymbol{x}) \to \varepsilon(\boldsymbol{x})$. The dimensionless field $\psi(\boldsymbol{x}, l, l') = \varepsilon^{l'}(\boldsymbol{x})/\varepsilon^l(\boldsymbol{x})$ is similar to the dimensionless fields of Kolmogorov [3] and will be supposed to have the scale symmetry. We suppose that all correlation functions of $\psi(\boldsymbol{x}, l, l') = \varepsilon^{l'}(\boldsymbol{x})/\varepsilon^{l}(\boldsymbol{x})$ are invariant to the scale transform: when the points $\boldsymbol{x}$ and the scales l are transformed as $\boldsymbol{x} \to K\boldsymbol{x}$, $l \to Kl$, where K is any numeric factor, the correlation functions of $\psi(\boldsymbol{x}, l, l')$ remain the same.

The field $\psi(\boldsymbol{x}, l, l')$ has too many arguments. A simpler field that has the same information is obtained by considering the limit $l' \to l$. From the definition of $\psi(\boldsymbol{x}, l, l')$, one concludes that the field

$$\varphi(\boldsymbol{x}, l) = \frac{\partial \ln \varepsilon^{l}(\boldsymbol{x})}{\partial \ln l} \tag{4}$$

has the correlation functions that are invariant to scale transformations. The relation (4) is considered as the equation for ε when the scale invariant field $\varphi(\boldsymbol{x}, l)$ is given. All essential information about the fractal porous media is contained in the statistical properties of $\varphi(\boldsymbol{x}, l)$. The media will be considered as known if the statistical distribution of $\varphi(\boldsymbol{x}, l)$ is given. Characterization of the media in terms of statistics of $\varphi(\boldsymbol{x}, l)$ has an obvious advantage. All statistical properties of the fractal density $\varepsilon(\boldsymbol{x}) = \varepsilon(\boldsymbol{x}, l \mid L \to 0)$ are determined by the field φ. Assuming some model for φ, we obtain full description of ε. In [4], the correlation functions for fractal were derived, assuming that φ is the scale and conformal symmetric field in some subrange.

In practice, the fluctuations may be observed in some finite range of scales $l_\eta < l < L$. Equation (4) have to be supplemented by the boundary condition on any end of the range (l_η, L). For definiteness, the boundary condition at $l = L$ will be assumed to be fixed $\varepsilon(\boldsymbol{x}, L) = \varepsilon_0 = \text{const}$. The solution to (4) is as follows:

$$\varepsilon(\boldsymbol{x}, l) = \varepsilon_0 \exp\Big[-\int_l^L \varphi(\boldsymbol{x}, l_1)\frac{dl_1}{l_1}\Big]. \tag{5}$$

3. Scale symmetry

We consider the scale and conformal pair correlation function of $\varphi(\boldsymbol{x}, l)$:

$$\Phi(\boldsymbol{x}, \boldsymbol{y}, l, l') = \langle \varphi(\boldsymbol{x}, l), \varphi(\boldsymbol{y}, l') \rangle.$$

The spatial homogeneity and isotropy implies

$$\Phi(\boldsymbol{x} - \boldsymbol{y}, l, l') = \Phi((\boldsymbol{x} - \boldsymbol{y})^2, l, l'),$$

where the same letter Φ, is used for the sake of simplicity in the right-hand side.

The scale symmetry says that the correlation remains unchanged if all spatial scales are extended in K times

$$\Phi((\boldsymbol{x}-\boldsymbol{y})^2, l, l') = \Phi\left(K^2(\boldsymbol{x}-\boldsymbol{y})^2, Kl, Kl'\right),$$

where K is a positive factor. This equation implies that Φ depends on two arguments rather than on three ones

$$\Phi((\boldsymbol{x}-\boldsymbol{y})^2, l, l') = \Phi\left(\frac{(\boldsymbol{x}-\boldsymbol{y})^2}{ll'}, \frac{l'}{l}\right). \tag{6}$$

The random field φ is assumed to be Gaussian distributed. Next, at $\boldsymbol{x} = \boldsymbol{y}$ it is assumed to be δ-correlated in the logarithm of scale

$$\Phi(l'/l) = \Phi_0 \delta(\ln l - \ln l'). \tag{7}$$

This supposition corresponds to the lognormal model [6].

4. The subgrid model

We use the renormalization group method to find a suitable model for the subgrid term. Subtracting (3) from (1), we obtain the equation for the subgrid pulsations

$$\Delta p' + \nabla\chi\nabla p_1 + \mu[\chi\nabla p_1 \nabla \ln \varepsilon_1 + \nabla p' \nabla \ln \varepsilon_1 + \chi \Delta p_1] + \\ \lambda\nabla(\chi\nabla p' - \langle\chi, \nabla p'\rangle_1) + \mu\lambda(\chi\nabla p' - \langle\chi, \nabla p'\rangle_1)\nabla \ln \varepsilon_1 = 0, \tag{8}$$

where $\chi(\boldsymbol{x}, \tau) = \int_{\ln l_1}^{\ln l} \varphi(\boldsymbol{x}, \tau) d\tau$, and the parameters μ, λ are introduced for the following reasons. Equation (8) is used to estimate the influence of the subgrid pulsations on the ongrid ones. If the solution to (8) was known, one might substitute this solution for p' in (3) obtaining the equation that contains the ongrid field p_1. In that case, we would have an exact model. Because this solution is not known, one might try to use the perturbation theory. Let us denote l_1, l to be the spatial scales of the ongrid and subgrid component. In our case $l \sim l_1$, but we shall suppose that $\mu \sim l_1/l \ll 1$ in order to derive a local gradient model. General model have to contain nonlocal functional dependencies of the fields p', p_1. Another useful parameter is $\lambda \sim \varepsilon'/\varepsilon_1$. The formal parameters μ, λ are used for ordering the perturbation expansion. The equation of the lowest order is as follows:

$$\Delta p' = -\nabla_m \chi \nabla_m p_1.$$

The solution $p_2 = -\Delta^{-1}(\nabla_m \chi \nabla_m p_1)$, where Δ^{-1} is the operator that is inverse to the Laplace operator. In the simplest model, one should retain the terms with lowest derivative of the large scale field p_1. One concludes that the simplest model for the gradient of the subgrid field is $\nabla_j p_2 = -\Delta^{-1}(\nabla_j \nabla_m \chi) \cdot \nabla_m p_1$. The similar solution was obtained [7] for the dielectric permeability of nonhomogeneous media. Substituting that solution into the filtered equation, one has the following expression for the subgrid term in (3)

$$-\frac{\Phi_0 \delta l}{lD}\nabla[\varepsilon_1(\boldsymbol{x})\nabla p_1(\boldsymbol{x})].$$

One sees that the constant ε_0 in (5) depends on the grid scale as the power

$$\frac{d\ln\varepsilon_0}{d\ln l} = -\frac{\Phi_0}{D}, \qquad \varepsilon_0 = \varepsilon_{00}(l/L)^{-\Phi_0/D}. \tag{9}$$

Thus, if one wishes to use the coarser grid when computing the flow through a fractal matter, one should multiply the effective permeability by a constant factor according to (9).

Equality (9) is the main result of the present paper. It contains two constants ε_0 and ε_{00}. The latter describes the mean flow through the media $\overline{v} = \varepsilon_{00}\nabla\overline{p}$. The former is the constant of the subgrid model. The fact that those constants are different, is the evidence that the subgrid fluctuations of the pressure are essential. On the other hand, one sees that the power in (9) is small at large D. This gives the hope that $1/D$ may be used as a parameter of expansion in a perturbation theory.

Formula (9) was obtained within a crude gradient model. Many questions have to be clarified. In order to incorporate more physics into the model, one have to use the next terms of the perturbation series that follows from (8). The numerical values will alter, but we hope that the scaling laws remain to be valid.

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