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Global attractors for the Lorenz model on the sphere*

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The Lorenz model of baroclinic flows on two-dimensional sphere is considered in this article. The existence of a global attractors of the model is established.

Introduction

Meteorologists consider simultaneous weather states for large enough time, to infer "climate" of the atmosphere. The qualitative theory of differential equations offers strong tools that we are able to state precisely the meaning of "climate" [3, 4, 5, 8, 9]. The introduction of global attractors in atmosphere dynamics and their name are motivated in the hope that attractors would exist for the climate dynamics equations and that they would be able to describe the climatic regimes. Global attractors are compact sets that exponentially attract all orbits of a dissipative dynamical system.

In this article we address the case, which is very important for atmosphere dynamics of the Lorenz model equation on the sphere S^2. The existence of global attractors of this model is established.

1. Lorenz model equations on two-dimensional sphere

Consider the spherical coordinate system (λ, φ) on S^2, in which $\lambda \in (0, 2\pi)$, $\varphi \in (-\pi/2, \pi/2)$ and $\mu = \sin\varphi$. We can then write the Lorenz model equation for baroclinic atmosphere on S^2

$$\frac{\partial \Delta\psi}{\partial t} + J(\psi, \Delta\psi + 2\mu) + J(\tau, \Delta\tau) = k(2\Delta\tau - \Delta\psi) + \nu\Delta^2\psi, \tag{1.1}$$

$$\frac{\partial \Delta\tau}{\partial t} + J(\tau, \Delta\psi + 2\mu) - \nabla(2\mu_0\nabla\chi) = -k(2\Delta\tau - \Delta\psi) - k_1\Delta\tau + \nu\Delta^2\tau, \tag{1.2}$$

$$\frac{\partial \theta}{\partial t} + J(\psi, \theta) - \sigma\Delta\chi = h\theta_* - h_1\theta + \nu\Delta^2\theta, \tag{1.3}$$

$$\Delta\theta = \nabla(2\mu_0\nabla\tau), \tag{1.4}$$

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where

$$J(\psi,\chi)=\psi_\lambda\chi_\mu-\psi_\mu\chi_\lambda,$$
$$\nabla\chi=\Big(\frac{1}{\sqrt{1-\mu^2}}\chi_\lambda,\sqrt{1-\mu^2}\chi_\mu\Big),$$
$$\Delta\chi=((1-\mu^2)\chi_\mu)_\mu)+(1-\mu^2)^{-1}\chi_{\lambda\lambda},$$

where $\mu=\sin\varphi$, χ – velocity potential, φ – stream function, θ – potential temperature.

This system has the following conservation law:

$$\frac{\partial(A+K)}{\partial t}=0,$$

where

$A=\frac{1}{\sigma}\int_{S^2}\theta^2 ds$ – available potential energy,
$K=\int_{S^2}[(\nabla\psi)^2+(\nabla\tau)^2]ds$ – kinetic energy.

2. Functional spaces on S^2

Let $L^p(S)$, $(p\in[1,\infty))$ be the Banach space with the norm

$$\|\psi\|_{L^p(S)}=\Big(\int_S|\psi|^p dS\Big)^{1/p}.$$

We assume that $t\in(0,T)$, $Q=(0,T)\times S^2$ and X is the function space with the norm $\|\psi\|_X$. Let X be the function spaces H^s, H_0^s or $L_p(S)$. The space H_0^s here is defined by

$$H_0^s=\Big\{\psi\in H^s:\ \int_{S^2}\psi dS^2=0\Big\}.$$

The stream function formulation of atmosphere dynamics equations is considered in this article.

Given $T>0$, denote by $L_p(0,T;X)$ the Banach space of all measured functions $g(t)$, $g:(0,T)\to X$ on interval $(0,T)$ with values in X and the norm $(\int_0^T\|g\|_X{}^p)^{1/p}=\|g\|_{L_p(0;T;X)}<\infty$. We assume that $s\in R_+$, and $-\Delta$ – the Laplace–Beltrami operator (LB), $Y_\alpha=P_{n_\alpha}^{m_\alpha}(\sin\varphi)e^{im_\alpha\lambda}$ – eigenfunctions of the Laplace–Beltrami operator, where $P_{n_\alpha}^{m_\alpha}$ – associated polynomials, $\lambda_\alpha=n_\alpha(n_\alpha+1)$ – eigenvalues (multiplicity $2n_\alpha+1$), $0\le\lambda\le2\pi$, $|\varphi|\le\pi/2$.

The operator $-\Delta$ is unbounded, self-adjoint linear operator with compact inverse and with domaine $D(-\Delta)=H_0^2$, using the spectrum of $-\Delta$, we can define its power $\delta^s\equiv(-\Delta)^{s/2}$. $\delta^s:H_0^{m+s}\to H_0^m$ is an isomorphism and

$$\delta^s\phi = \sum_{\alpha,|\alpha|\geq 1} \lambda_\alpha^{s/2}\phi_\alpha Y_\alpha.$$

Operator $(-\Delta)$ creates the scale Hilbert spaces $\bar{H}_0^\alpha(S^2)$, $\alpha \in R$, with scalar production

$$(f,g)_\alpha = (f,g)_{\bar{H}_0^\alpha(S)} = ((-\Delta)^{\alpha/2} f_1 (-\Delta)^{\alpha/2}\psi),$$

$$\|f\|_\alpha = \|f\|_{\bar{H}_0^\alpha(S)} = \|(-\Delta)^{\alpha/2} f\| = \Big(\sum_{n=1}^{\infty}\sum_{|m|\leq n} \lambda_n^\alpha f_{mn}^2\Big)^{1/2}.$$

Denote by $\bar{H}_0^\alpha(S)'$ the dual space of $\bar{H}_0^\alpha(S)$. It is easy to see that

$$(\bar{H}_0^\alpha(S))' = \bar{H}_0^{-\alpha}(S), \quad \alpha \geq 0.$$

Lemma 2.1. *Let $\alpha > \beta$, $f \in \bar{H}_0^\alpha(S^2)$ and λ_1 be the first eigenvalue of LB-operator, then*

1) $\|f\|_\alpha \geq \lambda_1^{\frac{\alpha-\beta}{2}} \|f\|_\beta$;
2) $\lambda_1\|f\|^2 \leq \|\nabla f\|^2 = \|(-\Delta)^{1/2} f\|^2, \quad f \in \bar{H}_0^1(S)$;
3) $\|f\|_{L^4(S)} \leq 2^{1/4}\|f\|_{L^2(S)}^{1/2} \cdot \|\nabla f\|_{L^2(S)}^{1/2}$.

The Hilbert space V_α and its dual space, which is used later, are defined by

$$V_\alpha = \bar{H}_0^\alpha(S) \times \bar{H}_0^\alpha(S),$$
$$V_\alpha' = (\bar{H}_0^\alpha(S) \times \bar{H}_0^\alpha(S))' = \bar{H}_0^{-\alpha}(S) \times \bar{H}_0^{-\alpha}(S) = V_{-\alpha},$$
$$u = (u_1, u_2) \in V_\alpha.$$

It is easily seen that

$$\|u\|_{V_\alpha} \geq \lambda_1^{\frac{\alpha-\beta}{2}} \|u\|_{V_\beta}, \quad \alpha > \beta,$$

and

$$\ldots \subset V_2 \subset V_1 \subset V_0 \subset V_1' \subset V_2' \subset \ldots,$$

where imbedding operators are compact and continuous.

Definition 2.1. The function $Z \in H^s$ is called general derivative $\delta^r\psi$ of function $\psi \in H^s$, if $\forall\phi \in C^\infty(S)$

$$(Z,\phi)_s = (\psi, \delta^r\phi)_s.$$

Lemma 2.2. *Let $s \in R$, $r \in R_+$, then*

$$\forall Z \in H_0^{s+r} \quad \|Z\|_s \leq 2^{-r/2}\|Z\|_{s+r}.$$

Corollary 2.1. *The following norms are equivalent:*

$$\|\xi\|_{H^s} \sim \|\Delta\psi\|_{H^s} \sim \|\psi\|_{H^{s+2}} \sim \|\nabla\psi\|_{H^{s+1}}.$$

3. Absorbing sets and attractors

We can reduce system (1.1)–(1.4) to the Cauchy problem for nonlinear equations of evolution in the space H

$$\frac{\partial u}{\partial t} = F(u), \quad u|_{t=0} = u_0. \tag{3.1}$$

We can define the operators $S(t) : u_0 \to u(t)$, these operators enjoy the standard resolving semigroup properties for system (3.1) and they are continuous operators

$$u(t) = S(t)u_0. \tag{3.2}$$

We assume that the set X is an invariant set, if $S(t)X = X$.

Definition 3.1. Denote by

$$\omega(A) = \bigcap_{\tau \geq 0} \overline{\bigcup_{t \geq \tau} S(t)A},$$

the ω-limit set of A.

Let $\bigcup_{t \geq t_0} S(t)A$ ($A \neq \emptyset$, $t_0 > 0$) be relatively compact in H, then $\omega(A)$ is *invariant*, *compact* set [9].

We need to define absorbing sets and attractors [3].

Definition 3.2. The set B_a is called *absorbing* for semigroup $S(t)$, if for all bounded set $B \subset H$ there exists $t_0(B)$ that $S(t)B \subset B_a$ for $t \geq t_0(B)$.

Definition 3.3. The set $A \subset H$ is called global attractor for semigroup $S(t)$, $t \geq 0$, if:

1. A is a compact invariant set;
2. A attracts every bounded set $B \subset H$, i.e., $\text{dist}_H(S(t)B, A) \to 0$, $t \to \infty$.

There exist conditions such that resolving semigroup $S(t)$ will obtain a global attractor [1, 9]. We consider in this article the Babin–Vishik theorem, used later on, which gives us such conditions.

Theorem 3.1 (Babin–Vishik [1]). *Let $S(t)$ be a semigroup and let $S(t)$ satisfy the following conditions (H – the Banach space):*

1) *there exists the compact absorbing set $B_a \in H$,*

2) *operators $S(t) : H \to H$ are continuous for $t \geq 0$,*

3) *the semigroup* $S(t)$ *is uniformly bounded in* H, *i.e.*, $\forall R > 0\ \exists C(R)$, *that* $\|S(t)u\|_H \leq C(R)$, *if* $\|u\|_H \leq R\ \forall t \geq 0$.

Then the semigroup $S(t)$ *has a global attractor in* H.

Lemma 3.1. *The trilinear form* $(J(\psi,\xi),\Delta\chi)$, *where* $(\cdot,\cdot)$ *– inner product between* $J(\psi,\xi)$ *and* $\Delta\chi$, *satisfies the following inequality:*

$$|(J(\psi,\xi),\Delta\chi)| \leq \|\nabla\psi\|_{L^4(S)} \cdot \|\nabla\xi\|_{L^4(S)} \cdot \|\Delta\chi\|.$$

Lemma 3.2 [7]. $\forall\psi \in H_0^1$ *we have that*

$$\|\psi\|_{L^p(S)} \leq K_0(\|\psi\|)^{1-\alpha}(\|\nabla\psi\|)^{\alpha},$$

where $K_0 = \max\{2, p/2\}^{\alpha}$, $\alpha = 1 - 2/p$, $p \in [2,\infty)$.

Let $\psi \in L_\infty(0,T;H_0^0) \cap L_{\alpha p}(0,T;H_0^1)$, *then*

$$\psi \in L_p(Q),$$

and

$$(\|\psi\|_{L^p(S)})^p \leq K_0^p\|\psi\|^{p(1-\alpha)}\|\nabla\psi\|^{p\alpha} \leq C\|\psi\|^{p\alpha}_{H_0^1},$$

hence

$$\|\psi\|_{L^p(Q)} \leq C\|\psi\|_{L_{p\alpha}(0,T;H_0^1)}.$$

Lemma 3.3 [7]. *Let* $f \in L_p(0,T;X)$, $\frac{\partial f}{\partial t} \in L_p(0,T;X)$, *then* f *– continuos function* $[0,T] \to X$ *(may be after correction of* f *on a zero-measure set).*

Lemma 3.4 [7]. *Let*

$$B_0 \subset B \subset B_1,$$

be the Banach spaces and there exists the compact mapping $B_0 \to B$. *Consider the space*

$$W = \{v:\ v \in L_{p_0}(0,T;B_0),\ v' \in L_{p_1}(0,T;B_1)\},$$

$T < \infty$ *and* $1 < p_i < \infty$,

$$\|v\|_W = \|v\|_{L_{p_0}(0,T;B_0)} + \|v'\|_{L_{p_1}(0,T;B_1)},$$

then the imbedding operator $W \to L_{p_0}(0,T;B)$ *is compact.*

Now we set

$$V = (\psi, \tau), \quad \gamma = 4\mu_0/\sigma,$$
$$A_0 = \begin{pmatrix} -\Delta & 0 \\ 0 & -\Delta \end{pmatrix}, \quad A_1 = \begin{pmatrix} \Delta & 0 \\ 0 & \Delta - \gamma \end{pmatrix},$$
$$A_2 = \begin{pmatrix} -k\Delta & 2k\Delta \\ k\Delta & -(2k + k_0 + \nu\gamma)\Delta + h\gamma \end{pmatrix}, \quad A_3 = \begin{pmatrix} \Delta^2 & 0 \\ 0 & \Delta^2 \end{pmatrix},$$
$$B(v, v) = (J(\psi, \Delta\psi + 2\mu) + J(\tau, \Delta\tau), J(\psi, \Delta\tau - \gamma\tau) + J(\tau, \psi + 2\mu)),$$

then we can rewrite system (1.1)–(1.4) in the form

$$A_1 \frac{\partial v}{\partial t} = -B(v, v) + A_2 v + A_3 v + F. \tag{3.3}$$

Definition 3.4 (General solution). The function

$$v = (\psi, \tau) \in L(0, T; H_0^2 \cap L_2(0, T; H_0^3)$$

is called a general solution (3.3), if $\forall \omega \in L_2(0, T; H_0^1)^2$, the following equalities hold:

$$\left(A_1 \frac{\partial v}{\partial t}, \omega\right) + (B(v, v), \omega) + (A_2 v, \omega) + \nu(A_0^{1/2} A_3^{1/2} v, A_0^{1/2}\omega) = (F, \omega), \tag{3.4}$$
$$v(0) = v_0, \tag{3.5}$$

where $v_0 \in H_0^2$.

Lemma 3.5. *Let v be a solution of (3.4)–(3.5), then*

$$b(v, v) \in L_2(0, T; H_0^{-1}), \qquad \frac{\partial A_1 v}{\partial t} \in L_2(0, T; H_0^{-1}),$$

and

$$\frac{\partial v}{\partial t} \in L_2(0, T; H_0^1), \qquad v \in C(0, T; H_0^1).$$

Theorem 3.2. *Let $F \in H_0^{-1}(S^2)$, $\nu > 0$ and $v_0 \in H_0^2(S^2)$, then there exists a unique solution v of (3.4)–(3.5) and resolving semigroup has a unique global attractor in $H_0^1(S^2)$.*

Proof. Uniqueness. Let v_1 and v_2 be different solutions of (3.4)–(3.5). Put $\delta v = v_1 - v_2$. Furthermore, by Lemmas 3.1 and 3.2 and by the Gelder ϵ-inequality, we have

$$|b(v_1, v_1) - b(v_2, v_2)| \leq \frac{\nu}{2} \|\Delta \delta v\|^2 + C_M \|\nabla \delta v\|^2.$$

Therefore, $\Delta\delta v$ satisfies

$$\frac{1}{2}\frac{\partial\|\Delta\delta v\|_{A_1}^2}{\partial t}+\nu\|\Delta\delta v\|^2\le\frac{\nu}{2}\|\Delta\delta v\|^2+C_M\|\delta v\|_{A_1}^2,$$

or

$$\frac{\partial\|\Delta\delta v\|_{A_1}^2}{\partial t}\le 2C_M\|\delta v\|_{A_1}^2,$$

and

$$\frac{\partial(e^{-2C_Mt}\|\Delta\delta v\|_{A_1}^2)}{\partial t}\le 0,$$

hence, we obtain

$$\|\Delta\delta v\|_{A_1}=0,$$

with $\|\Delta\delta v\|_{A_1}(0)=0$.

Existences. Let $W_\alpha\in H_0^1$ be basis ([3], Lemma 6.5) and we define approximate solution in the form

$$v^N=\sum_{\alpha>0}^{N}g_\alpha^N(t)W_\alpha. \tag{3.6}$$

Now, we solve (3.4)–(3.5) by the standard Galerkin method, then system (3.4)–(3.5) is reduced to the system of ordinary differential equations

$$\begin{aligned}(A_1\frac{\partial v^N}{\partial t},w_\alpha)+(b(v^N,v^N),w_\alpha)+(A_2v^N,w_\alpha)&\\ +\nu(A_0^{1/2}A_3^{1/2}v^N,A_0^{1/2}w_\alpha)=(F,w_\alpha),&\end{aligned} \tag{3.7}$$

with initial conditions

$$v^N(0)=v_0^N,\quad \|A_1v_0^N-A_1v_0\|_{L_2}\to 0. \tag{3.8}$$

From the theory of ordinary differential equation, we can conclude that there exists a unique solution v^N on interval $[0,t^N]$. Furthermore, we consider the inner product between (3.7) and g_α^N and does summation over α, because trilinear form in this case is equal to zero, then we obtain

$$\frac{1}{2}\frac{\partial|v^N|_{A_1}^2}{\partial t}+\nu a_3(v^N(t),v^N(t))+a_2(v^N(t),v^N(t))=(F,v^N(t)), \tag{3.9}$$

which implies that

$$\frac{1}{2}\frac{\partial|v^N|_{A_1}^2}{\partial t}+\nu\|A_3^{1/2}v^N\|_0+\sigma\|A_0^{1/2}v^N\|_0\le\frac{\nu}{2}\|A_3^{1/2}v^N\|_0+C_F\|F\|_{-1},$$

and

$$\|v^N\|^2_{A_1}(t) + \nu \int_0^t \|A_3^{1/2} v^N\|_{d\tau} \leq \|v_0^N\|^2_{A_1} + 2C_F \int_0^t \|F\|_{-1} \leq C_1. \quad (3.10)$$

We have that $A_3^{1/2}v^N$ is bounded in

$$L_2(0,T;H_0^1) \cap L_\infty(0,T;L_2).$$

Now, we replace w_α by $-\lambda_\alpha^{-1}w_\alpha$ in (3.7) and repeat calculations, then we obtain the following estimates:

$$\|A^{1/2o}v^N\|^2_{A_1}(t) + \nu \int_0^t \|A_3^{1/2}v^N\|_1 d\tau \leq C_2, \quad (3.11)$$

and we can take T as t^N. As in the proof of Theorem 6.1 in [7] and by the applicaton of Lemma 3.2 we obtain that $\frac{\partial v^N}{\partial t}$ is bounded in $L_2(0,T;H_0^{-1})$ and using (3.10) and (3.11), we can conclude that there exists subsequence $A_3^{1/2}v^n$ of sequence $A_3^{1/2}v^N$, that

- $A_3^{1/2}v^n \to A_3^{1/2}v$ weakly in $L_2(0,T;H_0^1)$,
- $A_3^{1/2}v^n \to A_3^{1/2}v$ $*$-weakly in $L_\infty(0,T;L_2)$,
- $A_3^{1/2}v^n \to A_3^{1/2}v$ in $L_2(0,T;L_2)$,
- $\frac{\partial A_3^{1/2}v^n}{\partial t} \to \frac{\partial A_3^{1/2}v}{\partial t}$ weakly in $L_2(0,T;H_0^{-1})$.

We infer from the first and the fourth points that $A_3^{1/2}v^n(0) \to A_3^{1/2}v(0)$ weakly in H_0^{-1} and $A_3^{1/2}v(0) = A_3^{1/2}v_0$. By Lemmas 3.1 and 3.2, we obtain that the bilinear form $b(v,v) \in L_2(0,T;H_0^{-1}$, hence, using the first point, we obtain that

$$\int_0^T (b(v_N, w_\alpha), v^N)dt \to \int_0^T (b(v, w_\alpha), v^N)dt. \quad (3.12)$$

Now, we have that (3.7) converges to (3.4) as $n \to \infty$ $\forall w_\alpha$ and hence, $\forall \omega \in L_2(0,T;H_0^1)$.

A priori estimates. We consider the inner product between (3.3) and v, and we obtain

$$\frac{1}{2}\frac{\partial \|v\|^2_{A_1}}{\partial t} - \int_S (B(v,v),v)dS + \int_S (A_2 v, v)dS + \nu\|\Delta v\|^2_{A_1} = \int_S (F,v)dS. \quad (3.13)$$

As the trilinear form has the antisymmetrical property, hence the term $\int_S (B(v,v),v)dS$ is equal to zero. Hence, we obtain

$$\frac{1}{2}\frac{\partial \|v\|^2_{A_1}}{\partial t} + \int_S (A_2 v, v)dS + \nu\|\Delta v\|^2_{A_1} = \int_S (F,v)dS. \quad (3.14)$$

By the Gelder ϵ-inequality and Lemma 2.1, from (3.14), we have the following inequalities:

$$\frac{\partial\|v\|^2_{A_1}}{\partial t} + \nu_1\|v\|^2_{A_1} \leq \|\theta^*\|^2_{-1}/(2\lambda_1\nu), \tag{3.15}$$

$$\|v\|^2_{A_1} \leq \|v\|^2_{A_1}(0)e^{-\nu_1 t} + \|\theta^*\|^2_{-1}/(2\lambda_1\nu)\cdot(1-e^{-\nu_1 t}), \tag{3.16}$$

where $\nu_1 > 0$. Then we can easily see that $\exists\rho_1^2$,

$$\|v\|^2_{A_1} \leq \rho_1^2, \tag{3.17}$$

$\forall t \geq t_1$. Now, we consider the inner product between (3.3) and Δv and repeart calculations, then we obtain following estimate:

$$\|v\|^2_{A_2} \leq \|v\|^2_{A_2}(0)e^{-\nu_2 t} + \|\theta^*\|^2_{-1}/(2\lambda_1\nu)\cdot(1-e^{-\nu_2 t}), \tag{3.18}$$

where $\nu_2 > 0$. Therefore, $\exists\rho_1^2$, such that

$$\|v\|^2_{A_2} \leq \rho_2^2, \tag{3.19}$$

$\forall t \geq t_2$.

Then there exists $\exists t_3$, that (3.18) and (3.19) are simultaneously satisfied. By the application the Babin–Vishik theorem, we obtain the existence of a global attractor for resolving semigroup $S(t)$ of system (1.1)–(1.4) in $H_0^1(S^2)$. □

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