Bull. Nov. Comp. Center, Math. Model. in Geoph., 6 (2000), 31–40

Laguerre spectral method as applied to numerical modeling of viscoelastic seismic problems

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The wave propagation in real media can be described within the framework of the theory of linear viscoelasticity. The presence of convolutional integral in Boltzmann's superposition principle poses the main difficulties in implementing the direct numerical methods in time domain. The paper presents an efficient algorithm, based on the application of the spectral Laguerre method for approximation of temporal derivatives as applied to the problem of seismic wave propagation in the heterogeneous viscoelastic medium.

Introduction

The paper presents the new efficient algorithm based on the application of the spectral Laguerre method for approximation of temporal derivatives as applied to the problem of seismic wave propagation in the heterogeneous viscoelastic medium. This approach is an analogue to the frequency domain forward modeling, where instead of the frequency w we have the number m – the degree of the Laguerre polynomials. In contrast to conventional approaches [1–3], the Laguerre method has some advantages in viscoelastic modeling. This does not require introduction of the memory variables [1] which obey additional time evolution equations and increase the dimension of the system under study. This technique permits us to consider the most general convolutional relations between stress and strain tensors, expressed by an arbitrary relaxation function in Boltzmann's superposition principle. Application of the Laguerre transform together with finite differences along the spatial coordinates reduces the solution of the original problem to a system of linear algebraic equations with a sparse matrix, independent of the number m. Only the right-hand side of the system has the recurrent dependence on the parameter m. Therefore, we can use fast methods for solving the obtained system with a great number of the right-hand sides. As it takes place, the matrix is only once transformed as compared to the frequency-domain forward modeling. It essentially decreases computer costs of our algorithm.

1. Some constitutive relations of the viscoelastic media

The most general stress-strain relation for a viscoelastic material [4] is expressed by

$$\sigma_{ij}(\boldsymbol{x},t) = \boldsymbol{C}_{ijkl}(\boldsymbol{x},t) * \dot{\varepsilon}_{kl}(\boldsymbol{x},t), \tag{1}$$

where σ_{ij} and ε_{ij} are elements of the stress and the strain tensor, respectively, and $*$ represents the time convolution. The dot above the variable represents a time derivative. In a pure elastic medium, $\boldsymbol{C}_{ijkl}(\boldsymbol{x})$ are the elements of Hook's tensor. In viscoelastic medium, $\boldsymbol{C}_{ijkl}(\boldsymbol{x},t)$ are the elements of the tensor depending on the entire response history. Equation (1) is the formulation of Boltzmann's superposition principle.

A more appropriate model of the relaxation mechanism is the "standard linear solid" model represented by the stress-strain relation as a differential equation [5]

$$\sigma_{ij} + \tau_\sigma \dot{\sigma}_{ij} = \boldsymbol{M}^R_{ijkl}(\boldsymbol{x})(\varepsilon_{kl} + \tau_\varepsilon \dot{\varepsilon}_{kl}), \tag{2}$$

where τ_σ and τ_ε denote the stress and the strain relaxation times, respectively, for one relaxation mechanism, $\boldsymbol{M}^R_{ijkl}(\boldsymbol{x})$ is a relaxed modulus at each point $(\boldsymbol{x})$ of the medium. After superimposing all the relaxation mechanisms, the relation between $\boldsymbol{C}_{ijkl}$ and $\boldsymbol{M}^R_{ijkl}$ is given by

$$\boldsymbol{C}_{ijkl}(\boldsymbol{x},t) = \boldsymbol{M}^R_{ijkl}(\boldsymbol{x})\left[1 - \sum_{s=1}^{L}\left(1 - \frac{\tau^s_\varepsilon}{\tau^s_\sigma}\right)\exp\left(-\frac{t}{\tau^s_\sigma}\right)\right]. \tag{3}$$

Here τ^s_σ and τ^s_ε denote the stress and the strain relaxation times for the s-th relaxation mechanism, L is the number of relaxation mechanisms. Performing the time derivatives in equation (1), and using (3), yields

$$\dot{\sigma}_{ij}(\boldsymbol{x},t) = \boldsymbol{M}^U_{ijkl}(\boldsymbol{x})\dot{\varepsilon}_{kl}(\boldsymbol{x},t) + \boldsymbol{M}^R_{ijkl}(\boldsymbol{x})\sum_{s=1}^{L}\Phi^s(t) * \dot{\varepsilon}_{kl}(\boldsymbol{x},t), \tag{4}$$

where the response function of the medium is expressed by

$$\Phi^s(t) = \frac{1}{\tau^s_\sigma}\left(1 - \frac{\tau^s_\varepsilon}{\tau^s_\sigma}\right)\exp\left(-\frac{t}{\tau^s_\sigma}\right)H(t).$$

In (4) the relaxed modulus $\boldsymbol{M}^R_{ijkl}$ and the unrelaxed modulus $\boldsymbol{M}^U_{ijkl}$ satisfy

$$\boldsymbol{M}^U_{ijkl}(\boldsymbol{x}) = \boldsymbol{M}^R_{ijkl}(\boldsymbol{x})\left[1 - \sum_{s=1}^{L}\left(1 - \frac{\tau^s_\varepsilon}{\tau^s_\sigma}\right)\right]. \tag{5}$$

In terms of velocities, a relaxed modulus gives the zero-frequency phase velocity, whereas an unrelaxed modulus gives the high-frequency limit of the phase velocity. Equations (4) and (5) are suitable for the numerical modeling in question.

2. Statement of the problem and description of the method

The algorithm is discussed on an example of solution of the first order elastic equations in the cylindrical coordinates (r, θ, z) for the 3D axially-symmetric, vertical heterogeneous half-space $z \geq 0$. The selected physical model can be described by the following system:

$$\begin{aligned}
\rho\frac{\partial u_r}{\partial t} &= \frac{\partial \sigma_r}{\partial r} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r}, \\
\rho\frac{\partial u_z}{\partial t} &= \frac{\partial \sigma_{rz}}{\partial r} + \frac{\partial \sigma_z}{\partial z} + \frac{\sigma_{rz}}{r}, \\
\frac{\partial \sigma_r}{\partial t} &= (\Lambda + 2M)\frac{\partial u_r}{\partial r} + \Lambda\Big(\frac{\partial u_z}{\partial z} + \frac{u_r}{r}\Big), \\
\frac{\partial \sigma_z}{\partial t} &= (\Lambda + 2M)\frac{\partial u_z}{\partial z} + \Lambda\Big(\frac{\partial u_r}{\partial r} + \frac{u_r}{r}\Big), \\
\frac{\partial \sigma_\theta}{\partial t} &= (\Lambda + 2M)\frac{u_r}{r} + \Lambda\Big(\frac{\partial u_z}{\partial z} + \frac{\partial u_r}{\partial r}\Big), \\
\frac{\partial \sigma_{rz}}{\partial t} &= M\Big(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\Big).
\end{aligned} \tag{6}$$

Here σ_{ij} denotes the component of symmetric stress tensor, u_i denotes a velocity component. Λ and M are integral operators of the form:

$$\Lambda x(t) = \lambda x(t) - \hat{\lambda}\int_{-\infty}^{t} x(\tau)\chi(t-\tau)\,d\tau, \quad Mx(t) = \mu x(t) - \hat{\mu}\int_{-\infty}^{t} x(\tau)g(t-\tau)\,d\tau.$$

The elastic constants $\lambda(z)$, $\mu(z)$ and the non-elastic constants $\hat{\lambda}(z)$, $\hat{\mu}(z)$ are arbitrary functions of the variable z; $\chi(z,t)$, $g(z,t)$ are relaxation functions.

The problem is solved with zero initial data

$$u_r|_{t=0} = u_z|_{t=0} = \sigma_r|_{t=0} = \sigma_z|_{t=0} = \sigma_\theta|_{t=0} = \sigma_{rz}|_{t=0} = 0 \tag{7}$$

and the following boundary conditions:

$$\sigma_{rz}|_{z=0} = 0, \quad \sigma_z|_{z=0} = F(r)f(t), \tag{8}$$

where $f(t)$ represents the time variation of the source, $F(r)$ is a function of distribution of a source on the plane $z = 0$. We can choose $F(r)$ in the form $F(r) = n_0^2/2\pi(1 + n_0^2r^2)^{3/2}$, to be suitable to simulate a point source when $n_0 \to \infty$.

At the first step, let us make use the representation of the solution of (6)–(8) as a combination of the Fourier–Bessel series [6]:

$$\left\{ \begin{array}{c} u_r \\ \sigma_{rz} \end{array} \right\} = \frac{2}{a^2} \sum_{n=1}^{\infty} \left\{ \begin{array}{c} W_5(k_n, z, t) \\ W_2(k_n, z, t) \end{array} \right\} \frac{J_1(k_n r)}{[J_0(k_n a)]^2}, \tag{9}$$

$$\left\{ \begin{array}{c} u_z \\ \sigma_z \end{array} \right\} = \frac{2}{a^2} \sum_{n=1}^{\infty} \left\{ \begin{array}{c} W_6(k_n, z, t) \\ W_1(k_n, z, t) \end{array} \right\} \frac{J_0(k_n r)}{[J_0(k_n a)]^2}, \tag{10}$$

$$\sigma_\theta = \frac{2}{a^2} \sum_{n=1}^{\infty} \frac{1}{r} W_3(k_n, z, t) \frac{J_1(k_n r)}{[J_0(k_n a)]^2} + \frac{2}{a^2} \sum_{n=1}^{\infty} W_4(k_n, z, t) \frac{J_0(k_n r)}{[J_0(k_n a)]^2}, \tag{11}$$

$$\begin{aligned} \sigma_r = {} & \frac{2}{a^2} \sum_{n=1}^{\infty} [k_n W_3(k_n, z, t) + W_4(k_n, z, t)] \frac{J_0(k_n r)}{[J_0(k_n a)]^2} - \\ & \frac{2}{a^2} \sum_{n=1}^{\infty} \frac{1}{r} W_3(k_n, z, t) \frac{J_1(k_n r)}{[J_0(k_n a)]^2}, \end{aligned} \tag{12}$$

where J_0, J_1 are the Bessel functions of the first kind, and k_n are the roots of the transcendental equation $J_1(k_n a) = 0$. We choose a parameter a to be rather large to consider the wave field up to the time $t < T$, where T is the minimal time propagation of the leading wave front from the reflecting surface $r = a$.

After applying (9)–(12) to problem (6)–(8) we arrive to the following equations:

$$\begin{aligned} \frac{\partial W_1}{\partial t} &= (\Lambda + 2M) \frac{\partial W_6}{\partial z} + \Lambda k_n W_5, \\ \frac{\partial W_2}{\partial t} &= M \left(\frac{\partial W_5}{\partial z} - k_n W_6 \right), \\ \frac{\partial W_3}{\partial t} &= 2M W_5, \\ \frac{\partial W_4}{\partial t} &= \Lambda \left(\frac{\partial W_6}{\partial z} + k_n W_5 \right), \\ \rho \frac{\partial W_5}{\partial t} &= \frac{\partial W_2}{\partial z} - k_n^2 W_3 - k_n W_4, \\ \rho \frac{\partial W_6}{\partial t} &= \frac{\partial W_1}{\partial z} + k_n W_2 \end{aligned} \tag{13}$$

with the initial data

$$W_s|_{t=0} = 0, \quad s = 1, \ldots, 6, \tag{14}$$

and the boundary conditions

$$W_2|_{z=0} = 0, \qquad W_1 \Big|_{z=0} = \frac{1}{2\pi} \exp(k_n/n_0) f(t). \tag{15}$$

At the second step, we apply to problem (13)–(15) the integral Laguerre transform with respect to the time coordinate [7, 8]:

$$\bar{W}_s^m(k_n, z) = \int\limits_0^\infty W_s(k_n, z, t)(ht)^{-\alpha/2} l_m^\alpha(ht)\, d(ht), \quad s = 1, \ldots, 6, \tag{16}$$

with the inverse formulas

$$W_s(k_n, z, t) = (ht)^{\alpha/2} \sum_{m=0}^\infty \frac{m!}{(m+\alpha)!} \bar{W}_s^m(k_n, z) l_m^\alpha(ht), \quad s = 1, \ldots, 6, \tag{17}$$

where $l_m^\alpha(ht)$ are the orthonormal Laguerre functions expressed by the classical Laguerre polynomials $L_m^\alpha(ht)$ [9]. Here we select an integer parameter $\alpha \geq 1$ to satisfy the initial data (7). Let us consider the Laguerre series for the relaxation functions $g(t)$ and $\chi(t)$:

$$\left\{ \begin{matrix} g(z,t) \\ \chi(z,t) \end{matrix} \right\} = \sum_{m=0}^\infty \left\{ \begin{matrix} g_m(z) \\ \chi_m(z) \end{matrix} \right\} l_m^0(ht) \tag{18}$$

with the inverse formulas

$$\left\{ \begin{matrix} g_m(z) \\ \chi_m(z) \end{matrix} \right\} = \int\limits_0^\infty \left\{ \begin{matrix} g(z,t) \\ \chi(z,t) \end{matrix} \right\} l_m^0(ht)\, d(ht). \tag{19}$$

A certain form of the functions $g(t)$ and $\chi(t)$ is defined by the mechanism of attenuation. For example, we can define them as $\chi(t) = \phi_1(t)$ and $g(t) = \phi_2(t)$ for the general standard linear solid rheology [5]:

$$\phi_\nu(t) = \sum_{l=1}^{L_\nu} \frac{1}{\tau_{\sigma,l}^\nu} \left(\frac{\tau_{\varepsilon,l}^\nu}{\tau_{\sigma,l}^\nu} - 1 \right) e^{-t/\tau_{\sigma,l}^\nu}, \quad \nu = 1, 2, \tag{20}$$

where L is the number of relaxation mechanisms, $\tau_{\varepsilon,l}$ and $\tau_{\sigma,l}$ are the strain and the stress relaxation times of the l-th mechanism. In this case, the relation between the unrelaxed modulus M_U and the relaxed modulus M_R is of the form:

$$M_U^\nu = M_R^\nu \left[1 - \sum_{l=1}^{L_\nu} \left(1 - \frac{\tau_{\varepsilon,l}^\nu}{\tau_{\sigma,l}^\nu} \right) \right], \tag{21}$$

where $M_U^1 = \lambda$, $M_U^2 = \mu$ and $M_R^1 = \hat{\lambda}$, $M_R^2 = \hat{\mu}$. After applying the Laguerre integral transform (16) to problem (13)–(15) we arrive to the following equations:

$$
\begin{aligned}
&\frac{h}{2}\bar{W}_1^m - (\Lambda^0 + 2M^0)\frac{\partial \bar{W}_6^m}{\partial z} - k_n\Lambda^0\bar{W}_5^m = f_1^{m-1},\\
&\frac{h}{2}\bar{W}_2^m - M^0\frac{\partial \bar{W}_5^m}{\partial z} + k_n M^0\bar{W}_6^m = f_2^{m-1},\\
&\frac{h}{2}\bar{W}_3^m - 2M^0\bar{W}_5^m = f_3^{m-1},\\
&\frac{h}{2}\bar{W}_4^m - \Lambda^0\frac{\partial \bar{W}_6^m}{\partial z} - k_n\Lambda^0\bar{W}_5^m = f_4^{m-1},\\
&\frac{h}{2}\bar{W}_5^m - \frac{1}{\rho}\frac{\partial \bar{W}_2^m}{\partial z} + \frac{k_n}{\rho}\left(k_n\bar{W}_3^m + \bar{W}_4^m\right) = f_5^{m-1},\\
&\frac{h}{2}\bar{W}_6^m - \frac{1}{\rho}\left(\frac{\partial \bar{W}_1^m}{\partial z} + k_n\bar{W}_2^m\right) = f_6^{m-1},
\end{aligned}
\tag{22}
$$

where

$$
\begin{aligned}
\Lambda^0 &= \lambda - \frac{\chi_0}{h}\hat{\lambda} = \hat{\lambda}\left[1 + h\sum_{l=1}^{L_1}\frac{\tau_{\varepsilon,l}^1 - \tau_{\sigma,l}^1}{2 + h\tau_{\sigma,l}^1}\right],\\
M^0 &= \mu - \frac{g_0}{h}\hat{\mu} = \hat{\mu}\left[1 + h\sum_{l=1}^{L_2}\frac{\tau_{\varepsilon,l}^2 - \tau_{\sigma,l}^2}{2 + h\tau_{\sigma,l}^2}\right].
\end{aligned}
$$

The coefficients g_m and χ_m in the above formulas are analytically defined by substituting (20) into (19). The right-hand side of system (22) is iteratively defined by the formulas:

$$
\begin{aligned}
f_1 &= -h\sum_{j=0}^{m-1}\bar{W}_1^j + \hat{\lambda}\sum_{j=0}^{m-1}a_j\left(\frac{\partial \bar{W}_6^{m-j-1}}{\partial z} + k_n\bar{W}_5^{m-j-1}\right) +\\
&\quad 2\hat{\mu}\sum_{j=0}^{m-1}b_j\frac{\partial \bar{W}_6^{m-j-1}}{\partial z},\\
f_2 &= -h\sum_{j=0}^{m-1}\bar{W}_2^j + \hat{\mu}\sum_{j=0}^{m-1}b_j\left(\frac{\partial \bar{W}_5^{m-j-1}}{\partial z} - k_n\bar{W}_6^{m-j-1}\right),\\
f_3 &= -h\sum_{j=0}^{m-1}\bar{W}_3^j + 2\hat{\mu}\sum_{j=0}^{m-1}b_j\bar{W}_5^{m-j-1},\\
f_4 &= -h\sum_{j=0}^{m-1}\bar{W}_4^j + \hat{\lambda}\sum_{j=0}^{m-1}a_j\left(\frac{\partial \bar{W}_6^{m-j-1}}{\partial z} + k_n\bar{W}_5^{m-j-1}\right),\\
f_5 &= -h\sum_{j=0}^{m-1}\bar{W}_5^j, \qquad f_6 = -h\sum_{j=0}^{m-1}\bar{W}_6^j,
\end{aligned}
\tag{23}
$$

where

$$a_j = 4h\sum_{l=1}^{L_1} \frac{\tau^1_{\varepsilon,l} - \tau^1_{\sigma,l}}{(2+h\tau^1_{\sigma,l})^2}\left(\frac{2-h\tau^1_{\sigma,l}}{2+h\tau^1_{\sigma,l}}\right)^j, \quad b_j = 4h\sum_{l=1}^{L_2} \frac{\tau^2_{\varepsilon,l} - \tau^2_{\sigma,l}}{(2+h\tau^2_{\sigma,l})^2}\left(\frac{2-h\tau^2_{\sigma,l}}{2+h\tau^2_{\sigma,l}}\right)^j.$$

Now the boundary conditions take the form:

$$\bar{W}_2^m|_{z=0} = 0, \quad \bar{W}_1^m|_{z=0} = \frac{1}{2\pi}\exp(k_n/n_0)\int_0^\infty f(t)(ht)^{-\alpha/2} l_m^\alpha(ht)\, d(ht). \tag{24}$$

Let us note that in the numerical solution, the system of equations (22) reduces to the system of the four equations. Also, zero boundary conditions are introduced at $z = b$ the form:

$$\bar{W}_5^m|_{z=b} = 0, \qquad \bar{W}_6^m|_{z=b} = 0. \tag{25}$$

When reducing equations (13) we have taken advantage of the theorem, which we have proved for our needs:

Theorem. *Let two arbitrary functions be represented as a Laguerre function series:*

$$f(t) = (ht)^{\alpha/2}\sum_{n=0}^{\infty}\frac{n!}{(n+\alpha)!} f_n l_n^\alpha(ht), \quad \phi(t) = (ht)^{\frac{\beta}{2}}\sum_{k=0}^{\infty}\frac{k!}{(k+\beta)!}\phi_k l_k^\beta(ht).$$

Then, the function $\varphi(t) = \int_0^t f(\tau)\phi(t-\tau)\,d\tau$ *can be also represented as a Laguerre function series*

$$\varphi(t) = (ht)^{(\alpha+\beta)/2}\sum_{m=0}^{\infty}\frac{m!}{(m+\alpha+\beta)!}\varphi_m l_m^{\alpha+\beta}(ht),$$

where

$$\varphi_m = \int_0^\infty (ht)^{-(\alpha+\beta)/2} l_m^{\alpha+\beta}(ht)\varphi(t)d(ht) = \frac{1}{h}\phi_0 f_m + \frac{1}{h}\sum_{j=0}^{m-1}(\phi_{m-j} - \phi_{m-j-1})f_j.$$

Problem (22)–(25) is reduced to a system of the linear algebraic equations with the help of finite difference approximation with respect to the coordinate z. The sparse matrix of the obtained system is independent of number m, only the right-hand side has the recurrent dependence on the parameter m. We use fast methods such as the Cholesky method for the solution of this system with a great number of the right-hand sides. As a result, after finding $\bar{W}_s^m(k_n, z)$, $s = 1, \ldots, 6$, it is sufficient to substitute them in the inverse formulas (9)–(12) and (17) to obtain the solution of the original problem (6)–(8).

3. An example of numerical calculation

The numerical calculations were carried out for a model of the medium with a thin absorbing layer above the isotropic elastic half-space. The physical parameters of this medium model are the following:

- the upper layer: $\rho_1 = 1$ g/cm^3, $c_{p1}(\nu = 0) = 1.5$ km/s, $c_{s1}(\nu = 0) = 1$ km/s;
- the lower half-space: $\rho_2 = 2$ g/cm^3, $c_{p2} = 3$ km/s, $c_{s2} = 2$ km/s.

The upper layer thickness is 0.5 km. The after-effect functions for this layer are described by two relaxation mechanisms ($L_p = L_s = 2$). The values of relaxation times are set as follows:

- for P-wave $\tau_{\varepsilon,1} = 0.4107$, $\tau_{\varepsilon,2} = 0.06661$, $\tau_{\sigma,1} = 0.401$, $\tau_{\sigma,2} = 0.06504$;
- for S-wave $\tau_{\varepsilon,1} = 0.416$, $\tau_{\varepsilon,2} = 0.06529$, $\tau_{\sigma,1} = 0.4012$, $\tau_{\sigma,2} = 0.063$.

The relaxation times were selected so that the values of the quality factor $Q_P = 60$ for P-wave and $Q_S = 40$ for S-wave be constant in the frequency band of the signal simulated in the source.

The amplitude spectrum of the source and the quality factors are displayed in Figure 1. Expressions for the quality factors and the phase velocities in viscoelastic media can be found in [10].

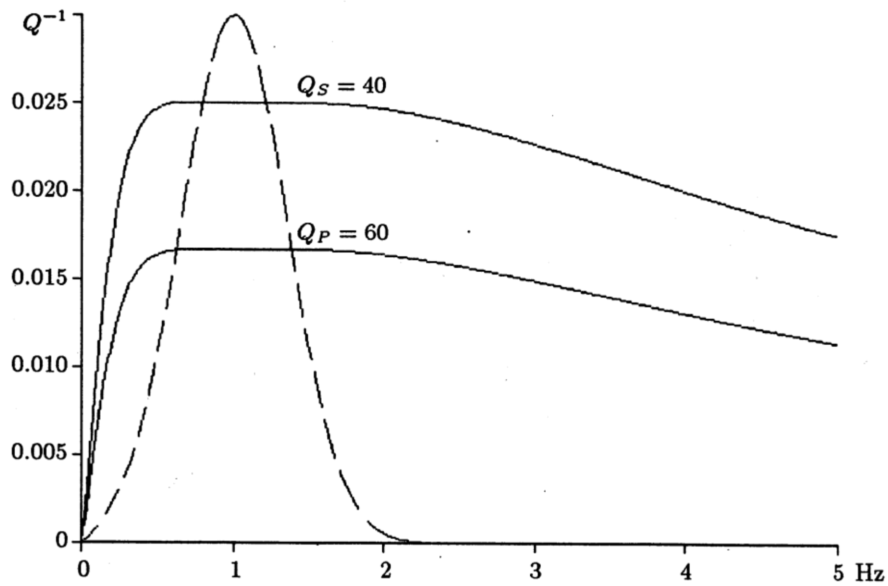

Figure 1. Graphs of the quality factors $Q_p(\nu)$, $Q_s(\nu)$. The dashed line represents the amplitude spectrum of the source

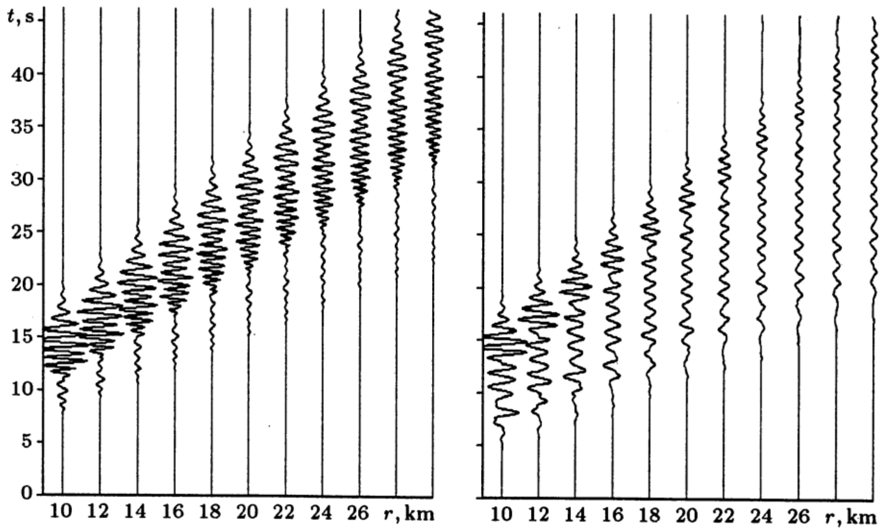

Figure 2. Calculation of seismotraces: the elastic model to the left and the viscoelastic model to the right

Figure 2 illustrates the results of the numerical calculation of the vertical components of the displacement velocities U_z on the free surface $z = 0$ for different epicentral distances. The left figure represents seismotraces for the elastic medium model; the right figure – seismotraces for the model with the given after-effect functions. The wave field was simulated by the vertical type source with the coordinates $r_0 = 0$, $z_0 = 0$ km. The function $f(t)$ represents the time variation of the source taken in the form:

$$f(t) = \exp\left[-\frac{(2\pi\nu(t-t_0))^2}{\gamma^2}\right]\sin(2\pi\nu(t-t_0)), \tag{26}$$

where $\gamma = 4$, $\nu = 1$ Hz, $t_0 = 1.5$ s.

In this case, the wave field has a complicated interference character. The interference waves, propagating on the layer surface, both in elastic and viscoelastic media, form a number of steady-state groups (modes), each propagating with its own velocity. In the viscoelastic case, the amplitude of such oscillation considerably attenuates.

Conclusion

We have presented the spectral Laguerre method for the viscoelastic modeling. This approach is an analogue to the frequency-domain modeling but the resulting linear system with the right-hand side has a sparse matrix independent of the number m – the degree of the Laguerre polynomials. This essentially decreases computer costs of viscoelastic modeling as compared with the frequency-domain modeling.

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