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On an algorithm of domain decomposition based on finite integral Fourier–Bessel transforms

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The paper presents an algorithm of domain decomposition when solving the forward dynamic seismic problem in the polar coordinate system. The new algorithm is based on combination of the method of direct and of finite integral Fourier transforms described in [1, 2]. As seen from the studies conducted, if on some sites of the medium the wave propagation velocity is a constant function, the efficiency of the algorithm can be essentially increased using the domain decomposition method. Particularly, for the domains with constant velocity, the solution can be written down in the explicit form. We consider the proposed decomposition algorithm on the example of the following problem.

1. Statement of the problem

Let us consider the solution to the wave equation in the polar coordinate system

$$\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \varphi^2} = \frac{1}{v_p^2(r,\varphi)}\frac{\partial^2 U}{\partial t^2} + \frac{4\pi}{r}\delta(r-r_0)\delta(\varphi-\varphi_0)f(t) \quad (1)$$

on the half-space $0 \le \varphi \le \pi$, $0 \le r < \infty$. The problem is considered with zero initial data

$$U\Big|_{t=0} = \frac{\partial U}{\partial t}\Big|_{t=0} = 0 \quad (2)$$

and with the boundary conditions

$$\frac{\partial U}{\partial \varphi}\Big|_{\varphi=0} = \frac{\partial U}{\partial \varphi}\Big|_{\varphi=\pi} = 0. \quad (3)$$

We assume the velocity in the medium to be a piecewise-continuous function $v_p(r,\varphi)$ of the two coordinates r, φ on the interval $r_1 < r < r_2$ and $v_p = \text{const}$ on $r \le r_1$ and $r \ge r_2$, i.e., the function has

$$v_p(r,\varphi) = \begin{cases} v_1, & r \le r_1 \\ v_p(r,\varphi), & r_1 < r < r_2 \\ v_2, & r \ge r_2. \end{cases}$$

2. Theoretical aspects of the method

At the first stage of solving problem (1)–(3) let us make use of the finite integral cosine-Fourier transform

$$R_n(r,t) = \int_0^\pi U(r,\varphi,t)\cos(n\varphi)\,d\varphi \tag{4}$$

with the inversion formula

$$U(r,\varphi,t) = \frac{1}{\pi}R_0(r,t) + \frac{2}{\pi}\sum_{n=1}^{\infty} R_n(r,t)\cos(n\varphi). \tag{5}$$

After applying the integral transform (4), (5) the new problem for $R_n(r,t)$ is of the form

$$\begin{gathered}\sum_{m=0}^{N}\left[\frac{\partial^2 R_m}{\partial r^2} + \frac{1}{r}\frac{\partial R_m}{\partial r} - \frac{m^2}{r^2}R_m\right]c(n,m,r) = \frac{\partial^2 R_n}{\partial t^2} + f(r,n,t),\\ R_n\Big|_{t=0} = \frac{\partial R_n}{\partial t}\Big|_{t=0} = 0, \quad n = 0,1,\dots,N,\end{gathered} \tag{6}$$

where

$$f(r,n,t) = \frac{4\pi}{r}v_p^2(r,\varphi_0)\delta(r-r_0)\cos(n\varphi_0)f(t),$$

$$c(n,m,r) = \begin{cases} \dfrac{1}{\pi}\displaystyle\int_0^\pi v_p^2(r,\varphi)\cos(n\varphi)\,d\varphi, & m = 0,\\ \dfrac{2}{\pi}\displaystyle\int_0^\pi v_p^2(r,\varphi)\cos(n\varphi)\cos(m\varphi)\,d\varphi, & m = 1,2,\dots,N. \end{cases}$$

At the next step we consider the solution to this problem on the intervals $[0,r_1]$, $[r_1-\Delta r, r_2+\Delta r]$ and $[r_2, A]$ separately.

For this purpose we introduce the following definitions $Q_n^1(t) = R_n(r_1,t)$, $Q_n^2(t) = R_n(r_1-\Delta r,t)$, $Q_n^3(t) = R_n(r_2+\Delta r,t)$, $Q_n^4(t) = R_n(r_2,t)$. Thus, we assume that

$$R_n(r,t) = \begin{cases} S_n(r,t), & r \le r_1,\\ P_n(r,t), & r_1-\Delta r \le r \le r_2+\Delta r,\\ W_n(r,t), & r \ge r_2, \end{cases}$$

and for each interval we obtain the appropriate problem (6).

On the interval $[0, r_1]$:

$$\begin{cases} v_1^2 \left[\dfrac{\partial^2 S_n}{\partial r^2} + \dfrac{1}{r} \dfrac{\partial S_n}{\partial r} - \dfrac{n^2}{r^2} S_n \right] = \dfrac{\partial^2 S_n}{\partial t^2} + f(r,n,t), \\ S_n \Big|_{t=0} = \dfrac{\partial S_n}{\partial t} \Big|_{t=0} = 0, \quad S_n \Big|_{r=r_1} = Q_n^1; \end{cases} \tag{7}$$

On the interval $[r_1 - \Delta r, r_2 + \Delta r]$:

$$\begin{cases} \sum\limits_{m=0}^{N} \left[\dfrac{\partial^2 P_m}{\partial r^2} + \dfrac{1}{r} \dfrac{\partial P_m}{\partial r} - \dfrac{m^2}{r^2} P_m \right] c(r,n,m) = \dfrac{\partial^2 P_n}{\partial t^2} + f(r,n,t), \\ P_n \Big|_{t=0} = \dfrac{\partial P_n}{\partial t} \Big|_{t=0} = 0, \quad P_n \Big|_{r=r_1-\Delta r} = Q_n^2, \quad P_n \Big|_{r=r_2+\Delta r} = Q_n^3; \end{cases} \tag{8}$$

On the interval $[r_2, A]$:

$$\begin{cases} v_2^2 \left[\dfrac{\partial^2 W_n}{\partial r^2} + \dfrac{1}{r} \dfrac{\partial W_n}{\partial r} - \dfrac{n^2}{r^2} W_n \right] = \dfrac{\partial^2 W_n}{\partial t^2} + f(r,n,t), \\ W_n \Big|_{t=0} = \dfrac{\partial W_n}{\partial t} \Big|_{t=0} = 0, \quad W_n \Big|_{r=r_2} = Q_n^4. \end{cases} \tag{9}$$

In problems (7)–(9),

$$f(r,n,t) = \frac{4\pi}{r} v_p^2(r,\varphi_0) \delta(r - r_0) \cos(n\varphi_0) f(t)$$

if r_0 belongs to the respective interval, and $f(r,n,t) \equiv 0$ otherwise.

Further let us make use of the analytical solution to the homogeneous medium and of the numerical-analytical solution to the 2D-inhomogeneous medium. From the condition of continuity of the solution $R_n(r,t)$ of problem (6), we arrive at the system

$$\begin{aligned} P_n(r_1, t; Q_n^2, Q_n^3) &= Q_n^1(t), \\ S_n(r_1 - \Delta r, t; Q_n^1) &= Q_n^2(t), \\ W_n(r_2 + \Delta r, t; Q_n^4) &= Q_n^3(t), \\ P_n(r_2, t; Q_n^2, Q_n^3) &= Q_n^4(t). \end{aligned} \tag{10}$$

The functions $Q_n^1(t)$, $Q_n^2(t)$, $Q_n^3(t)$, $Q_n^4(t)$ forms the solution to this system. After that we can define the functions $S_n(r,t;Q_n^1)$, $W_n(r,t;Q_n^4)$, and $P_n(r,t;Q_n^2,Q_n^3)$.

At the final stage, by substituting the given values in the appropriate solution to problems (7)–(9) and using the inversion formula (5), we obtain the solution $U(r,\varphi,t)$ to the original problem (1)–(3).

Now we consider the main stages of the method in more detail.

2.1. Analytical solution to homogeneous medium

Let us consider problems (7) and (9). It is possible to obtain their solutions by using the finite integral Hankel transforms [3]. Let us apply to problem (7) the integral transform

$$\bar{S}_n(\kappa_{in}, t) = \int_0^{r_1} r S_n(r, t) J_n(\kappa_{in} r) dr \tag{11}$$

with the inversion formula

$$S_n(r, t) = \frac{2}{r_1^2} \sum_{i=1}^{\infty} \bar{S}_n(\kappa_{in}, t) \frac{J_n(\kappa_{in} r)}{[J_n'(\kappa_{in} r_1)]^2}, \tag{12}$$

where J_n is the Bessel function of the first kind, J_n' is a derivative of J_n and κ_{in} are the positive roots of an equation $J_n(\kappa_{in} r_1) = 0$.

After applying transform (11), (12), problem (7) reduces to the form

$$\begin{cases} \dfrac{\partial^2 \bar{S}_n}{\partial t^2} + \kappa_{in}^2 v_1^2 \bar{S}_n = \phi_n(\kappa_{in}, t) \\ \bar{S}_n\Big|_{t=0} = \dfrac{\partial \bar{S}_n}{\partial t}\Big|_{t=0} = 0. \end{cases} \tag{13}$$

Here

$$\phi_n(\kappa_{in}, t) = 4\pi v_1^2 \cos(n\varphi_0) J_n(\kappa_{in} r_0) f(t) - \kappa_{in} v_1^2 r_1 J'_n(\kappa_{in} r_1) Q_n^1(t).$$

The solution to this problem is written down in the form

$$\bar{S}_n(\kappa_{in}, t) = \frac{1}{\kappa_{in} v_1} \int_0^t \phi_n(\kappa_{in}, \tau) \sin(\kappa_{in} v_1 (t - \tau))\, d\tau. \tag{14}$$

Substituting this value in the inversion formula (12), we come to the solution to problem (7).

For problem (9) we introduce an additional boundary condition

$$W_n(r, t)\Big|_{r=A} = 0$$

and take advantage of the appropriate integral Hankel transform

$$\bar{W}_n(\mu_{in}, t) = \int_{r_2}^{A} r W_n(r, t) \left[J_n(\mu_{in} r) Y_n(\mu_{in} A) - J_n(\mu_{in} A) Y_n(\mu_{in} r)\right] dr \tag{15}$$

with the inversion formula

$$W_n(r,t) = \frac{\pi^2}{2}\sum_{i=1}^{\infty}\frac{\mu_{in}^2 J_n^2(\mu_{in}r_2)}{J_n^2(\mu_{in}A) - J_n^2(\mu_{in}r_2)}\Big[J_n(\mu_{in}r)Y_n(\mu_{in}A) - J_n(\mu_{in}A)Y_n(\mu_{in}r)\Big]\bar{W}_n(\mu_{in},t), \tag{16}$$

where Y_n is the Bessel function of the second kind and μ_{in} are the positive roots of the transcendental equation

$$J_n(\mu_{in}r_2)Y_n(\mu_{in}A) - J_n(\mu_{in}A)Y_n(\mu_{in}r_2) = 0.$$

After applying transform (15), (16), problem (9) reduces to the form

$$\begin{cases} \dfrac{\partial^2 \bar{W}_n}{\partial t^2} + \mu_{in}^2 v_2^2 \bar{W}_n = \psi_n(\mu_{in},t), \\ \bar{W}_n\Big|_{t=0} = \dfrac{\partial \bar{W}_n}{\partial t}\Big|_{t=0} = 0. \end{cases} \tag{17}$$

Here

$$\psi_n(\mu_{in},t) = 4\pi v_2^2\cos(n\varphi_0)\left[J_n(\mu_{in}r_0)Y_n(\mu_{in}A) - J_n(\mu_{in}A)Y_n(\mu_{in}r_0)\right]f(t) + \frac{2}{\pi}v_2^2 Q_n^4(t)\frac{J_n(\mu_{in}A)}{J_n(\mu_{in}r_2)}$$

The solution to this problem is written in the form

$$\bar{W}_n(\mu_{in},t) = \frac{1}{\mu_{in}v_2}\int_0^t \psi_n(\mu_{in},\tau)\sin\left(\mu_{in}v_2(t-\tau)\right)\,d\tau. \tag{18}$$

Substituting this value in the inversion formula (16), we obtain the solution to problem (9).

2.2. Numerical-analytical solution to 2D-inhomogeneous medium

Let us consider the solution to problem (8) following [1]. Introduce the uniform grid in the variable r:

$$\omega = \{r_i = r_1 - \Delta r + ih,\ i = 1,\dots,K;\ r_2 + \Delta r = r_1 - \Delta r + (K+1)h\}.$$

We present the vector of the solution in the form

$$\vec{P}(r,t) = \left(P_0(r,t), P_1(r,t), \dots, P_N(r,t)\right)^T.$$

Using an approximation of the first and the second derivatives with respect to r with the second order, problem (8) will be written down in the vector form

$$\begin{cases} \dfrac{\partial^2 \vec{P}}{\partial t^2} + C\,A\,\vec{P} = \vec{F}, \\ \vec{P}\Big|_{t=0} = \dfrac{\partial \vec{P}}{\partial t}\Big|_{t=0} = 0. \end{cases} \tag{19}$$

Here $\vec{F} = \left(\vec{F}_1, \vec{F}_2, \ldots, \vec{F}_K\right)$, where

$$\vec{F}_i(t) = \begin{cases} \dfrac{v_1^2}{h}(\dfrac{1}{h} - \dfrac{1}{2(r1-\Delta r+h)}) \left(Q_0^2(t), Q_1^2(t), \ldots, Q_N^2(t)\right)^T, & i=1, \\ -\dfrac{4\pi v_p^2(r_0,\varphi_0)}{r_0 h} f(t)\,(\cos(0), \cos(\varphi_0), \ldots, \cos(N\varphi_0))^T, & \\ & i = \dfrac{r_0 - (r_1 - \Delta r)}{h}, \\ \dfrac{v_2^2}{h}\left(\dfrac{1}{h} + \dfrac{1}{2(r2+\Delta r - h)}\right) \left(Q_0^3(t), Q_1^3(t), \ldots, Q_N^3(t)\right)^T, & i = K, \\ 0, & i \neq 1, \quad i \neq K, \quad i \neq \dfrac{r_0 - (r_1 - \Delta r)}{h}. \end{cases}$$

The matrix C is block diagonal, and the matrix A is block three-diagonal:

$$C = \begin{pmatrix} C_1 & & & & 0 \\ & C_2 & & & \\ & & \ddots & & \\ & & & C_{K-1} & \\ 0 & & & & C_K \end{pmatrix},$$

$$A = \begin{pmatrix} A_1 & D_1 & & & 0 \\ B_1 & A_2 & D_2 & & \\ & \ddots & \ddots & \ddots & \\ & & B_{K-2} & A_{K-1} & D_{K-1} \\ 0 & & & B_{K-1} & A_K \end{pmatrix},$$

where i-th blocks of $N \times N$ dimension are defined as follows:

$$C_i = \begin{pmatrix} c(0,0,r_i) & c(1,0,r_i) & \ldots & c(N,0,r_i) \\ c(0,1,r_i) & c(1,1,r_i) & \ldots & c(N,1,r_i) \\ \vdots & \vdots & \ddots & \vdots \\ c(0,N,r_i) & c(1,N,r_i) & \cdots & c(N,N,r_i) \end{pmatrix},$$

$$A_i = \operatorname{diag}\left\{\frac{2}{h^2}, \frac{2}{h^2} + \frac{1}{(r_1 - \Delta r + ih)^2}, \ldots, \frac{2}{h^2} + \frac{N^2}{(r_1 - \Delta r + ih)^2}\right\},$$
$$i = 1, \ldots, K;$$

$$B_i = -\frac{1}{h}\left(\frac{1}{h} - \frac{1}{2(r_1 + \Delta r + (i+1)h)}\right)E, \qquad i = 1, 2, \ldots, K-1,$$

$$D_i = -\frac{1}{h}\left(\frac{1}{h} + \frac{1}{2(r_1 + \Delta r + ih)}\right)E, \qquad i = 1, 2, \ldots, K-1.$$

The algorithm for solving a similar problem is described in detail in [1]. Therefore we consider here only its general scheme. Let us present a matrix $\tilde{A} = CA$ as $\tilde{A} = T\Lambda T^{-1}$, where T is a matrix of eigenvectors, and $\Lambda = \operatorname{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_L,\}$ consists of eigenvalues ($L = K(N+1)$). Then, using the replacement $\vec{Y} = T^{-1}\vec{P}$, system (19) will be transformed to the form

$$\begin{cases} \dfrac{\partial^2 \vec{Y}}{\partial t^2} + \Lambda \vec{Y} = \vec{\Phi} \\ \vec{Y}\Big|_{t=0} = \dfrac{\partial \vec{Y}}{\partial t}\Big|_{t=0} = 0, \end{cases} \tag{20}$$

where $\vec{\Phi} = T^{-1}\vec{F}$.

The solution to the components $\vec{Y}$ is written by the expression

$$Y_l(t) = \frac{1}{\sqrt{\lambda_l}} \int_0^t \Phi_l(\tau) \sin\left(\sqrt{\lambda_l}(t-\tau)\right) d\tau, \quad l = 1, 2, \ldots, L.$$

Hence, taking into account the replacement $\vec{P} = T\vec{Y}$, the solution to problem (19) will finally have the form

$$P_l(t) = \sum_{i=1}^{L} t_{l,i} \sum_{j=1}^{L} \frac{t_{j,i}}{\sqrt{\lambda_i}} \int_0^t F_j(\tau) \sin\left(\sqrt{\lambda_i}(t-\tau)\right) d\tau, \tag{21}$$

where $t_{j,i}$ are elements of the matrix T.

2.3. Determination of the values $Q_n^1(t), Q_n^2(t), Q_n^3(t), Q_n^4(t)$ and scheme of solving the original problem

Using the terms from Subsections 2.1 and 2.2, let us introduce additional functions. For the solution to problem (19) at $r = r_1$

$$\begin{aligned} p_{n,m}^1(t) &= a \sum_{l=1}^{L} t_{K_1+n+1,l}\, t_{m+1,l}\, g_l(t), \\ p_{n,m}^2(t) &= a \sum_{l=1}^{L} t_{K_1+n+1,l}\, t_{L-N+m,l}\, g_l(t), \end{aligned} \tag{22}$$

where

$$K_1 = \frac{\Delta r - h}{h} N, \quad a = \frac{v_1^2}{h}\left(\frac{1}{h} - \frac{1}{2(r_1 - \Delta r + h)}\right),$$

and

$$p^3_{n,m}(t) = \begin{cases} c_m \sum_{l=1}^{L} t_{K_1+n+1,l} t_{K_0+m+1,l} g_l(t), & r_1 - \Delta r < r_0 < r_2 - \Delta r, \\ 0, & r_0 \le r_1 - \Delta r, \; r_0 \ge r_2 + \Delta r, \end{cases}$$

where

$$K_0 = \frac{r_0 - (r_1 - \Delta r) - h}{h} N, \quad c_m = -\frac{4\pi}{r_0 h} v_p^2(r_0, \varphi_0) \cos(m\varphi_0).$$

Here $g_l(t) = \frac{1}{\sqrt{\lambda_l}} \sin\left(\sqrt{\lambda_l} t\right)$.

Similarly, for the solution at $r = r_2$

$$\begin{aligned} p^4_{n,m}(t) &= b \sum_{l=1}^{L} t_{K_2+n+1,l} t_{m+1,l} g_l(t), \\ p^5_{n,m}(t) &= b \sum_{l=1}^{L} t_{K_2+n+1,l} t_{L-N+m,l} g_l(t), \end{aligned} \tag{23}$$

where

$$K_2 = \frac{r_2 - (r_1 - \Delta r) - h}{h} N, \quad b = \frac{v_2^2}{h} \left(\frac{1}{h} + \frac{1}{2(r_2 + \Delta r - h)} \right),$$

$$p^6_{n,m}(t) = \begin{cases} c_m \sum_{l=1}^{L} t_{K_2+n+1,l} t_{K_0+m+1,l} g_l(t), & r_1 - \Delta r < r_0 < r_2 - \Delta r, \\ 0, & r_0 \le r_1 - \Delta r, \quad r_0 \ge r_2 + \Delta r. \end{cases}$$

Using the solution to problem (7) at $r = r_1 - \Delta r$, we also define

$$s^1_n(t) = -\frac{2v_1}{r_1} \sum_{i=1}^{\infty} \frac{J_n\left(\kappa_{in}(r_1 - \Delta r)\right)}{J'_n(\kappa_{in} r_1)} \sin(\kappa_{in} v_1 t) \tag{24}$$

and

$$s^2_n(t) = \begin{cases} \frac{8\pi v_1}{r_1^2} \cos(n\varphi_0) \sum_{i=1}^{\infty} \frac{J_n\left(\kappa_{in}(r_1 - \Delta r)\right) J_n\left(\kappa_{in} r_0\right)}{\kappa_{in} \left[J'_n(\kappa_{in} r_1)\right]^2} \sin(\kappa_{in} v_1 t), & r_0 \le r_1, \\ 0, & r_0 > r_1. \end{cases}$$

Similarly, for the solution to problem (9) at $r = r_2 + \Delta r$ we define

$$w_n^1(t) = \pi v_2 \sum_{i=1}^{\infty} \frac{\mu_{in}^2 J_n(\mu_{in} r_2) J_n(\mu_{in} A)}{J_n^2(\mu_{in} A) - J_n^2(\mu_{in} r_2)} X_n^i(r_2 + \Delta r) \sin(\kappa_{in} v_2 t) \tag{25}$$

and

$$w_n^2(t) = \begin{cases} d_n \sum_{i=1}^{\infty} \frac{\mu_{in}^2 J_n^2(\mu_{in} r_2)}{J_n^2(\mu_{in} A) - J_n^2(\mu_{in} r_2)} X_n^i(r_2 + \Delta r) X_n^i(r_0) \sin(\mu_{in} v_2 t), & r_0 \geq r_2, \\ 0, & r_0 < r_2. \end{cases}$$

Here $d_n = 2\pi^3 v_2 \cos(n\varphi_0)$, $X_n^i(r) = J_n(\mu_{in} r) Y_n(\mu_{in} A) - J_n(\mu_{in} A) Y_n(\mu_{in} r)$.

From the condition of continuity of solution (10), we obtain the system

$$\begin{aligned} Q_n^1(t) &= \sum_{m=0}^{N} \left[\int_0^t Q_m^2(\tau) p_{n,m}^1(t-\tau) d\tau + \int_0^t Q_m^3(\tau) p_{n,m}^2(t-\tau) d\tau + f_m^1(t) \right], \\ Q_n^4(t) &= \sum_{m=0}^{N} \left[\int_0^t Q_m^2(\tau) p_{n,m}^4(t-\tau) d\tau + \int_0^t Q_m^3(\tau) p_{n,m}^5(t-\tau) d\tau + f_m^2(t) \right], \\ Q_n^2(t) &= \int_0^t Q_n^1(\tau) s_n^1(t-\tau)\, d\tau + \int_0^t f(\tau) s_n^2(t-\tau)\, d\tau, \\ Q_n^3(t) &= \int_0^t Q_n^4(\tau) w_n^1(t-\tau)\, d\tau + \int_0^t f(\tau) w_n^2(t-\tau)\, d\tau, \end{aligned} \tag{26}$$

where $n = 0, 1, 2, \ldots, N$.

Here

$$f_m^1(t) = \int_0^t f(\tau) p_{n,m}^3(t-\tau) d\tau, \quad f_m^2(t) = \int_0^t f(\tau) p_{n,m}^6(t-\tau) d\tau,$$

$f(t)$ is a bland-limited source function.

Solving this system we find $Q_n^1(t)$, $Q_n^2(t)$, $Q_n^3(t)$, and $Q_n^4(t)$. Then, substituting these values in the appropriate formulas of the solutions to problems (7)–(9) we determine $R_n(r, t)$. And further, by the inversion formula (5) we can calculate the solution to the original problem (1)–(3).

3. Some aspects of numerical calculations

We consider some features of numerical realization of our algorithm. We should find values of the roots of transcendental equations, used for calculation in the solutions to problems (7) and (9). We can use their analytical representation in the form of a row according to the known formulas [4]. The values of the roots are defined precisely enough by a small number of terms of the row. So i-th root of the equation $J_n(x) = 0$, in order of absolute values is defined as

$$x_n^{(i)} = \beta - \frac{l-1}{8\beta} - \frac{4(l-1)(7l-31)}{3(8\beta)^3} - \frac{32(l-1)(83l^2-982l+3779)}{15(8\beta)^5} - \dots,$$

where $\beta = \frac{\pi}{4}(2n+4i-1)$, $l = 4n^2$.

Similarly, for the equation $J_n(x)Y_n(\rho x) - J_n(\rho x)Y_n(x) = 0$ $(\rho > 0)$, the i-th root is equal to

$$x_n^{(i)} = \sigma + \frac{a}{\sigma} + \frac{b-a^2}{\sigma^3} + \frac{c-4ab+2a^3}{\sigma^5} + \dots,$$

where

$$\sigma = \frac{i\pi}{\rho-1}, \quad a = \frac{m-1}{8\rho}, \quad b = \frac{4(m-1)(m-25)(\rho^3-1)}{38\rho^3(\rho-1)},$$
$$c = \frac{32(m-1)(m^2-114m+1073)(\rho^5-1)}{58\rho^5(\rho-1)}, \quad m = 4n^2.$$

The number of the required roots of these equations for a certain $n = 0, 1, \dots, N$, used in calculation of the functions $s_n^1(t)$, $s_n^2(t)$, $w_n^1(t)$, $w_n^2(t)$ by formulas (25), (26) and in construction of the solutions $S_n(r,t)$, $W_n(r,t)$ to problems (7), (9), is defined from the Fourier spectrum band of the source depending on $f(t)$. On this basis, it is possible to reduce the number of the summed up terms when calculating the functions $p_n^1(t)$, $p_n^2(t)$, $p_n^3(t)$, $p_n^4(t)$, $p_n^5(t)$, $p_n^6(t)$ and constructing the solution $P_n(r,t)$, having chosen from the determined eigenvalues of problem (19) only those which correspond to the wave field spectrum.

The solution to system (26) which contains integral equations as convolution type is numerically done in t with the help of the known methods of integral representation by the quadrature sums [5]. Thus, one should note the stability of the solution.

The main factors, effecting the error of definition of the functions $Q_n^1(t)$, $Q_n^2(t)$, $Q_n^3(t)$, $Q_n^4(t)$, and consequently the final result are the accuracy of the finite difference approximation along the coordinate r when solving problem (19) and the chosen width of the interval Δr. As the analysis of numerical calculations has shown, in the case of difference approximation with second order of accuracy, the optimal Δr is a quarter of the spatial wave length in the medium, corresponding to $\min\{v_1, v_2\}$.

Conclusion

The offered algorithm of domain decomposition may be applied to media with a more complicated structure of the velocity profile, if there are sites, where the velocity is constant. This algorithm permits us to reduce essentially the computer time and to decrease the required volume of the main memory. The main volume of calculations for the given algorithm falls on construction of the numerical-analytical solution for a 2D-inhomogeneous medium when calculating the eigenproblem of the matrix of system (19). As is known, in numerical calculations the N times increase of the matrix brings about the increase of the computer time by N^3 times. The use of the analytical solution on the intervals $[0, r_1]$, $[r_2, R]$ allows, if necessary, the increase of the boundaries of these domains without essential increase of the volume and of time of calculations.

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