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Application of the Laguerre integral transforms for solving dynamic seismic problems

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When solving the forward seismic problems in inhomogeneous media it appeared effective to use algorithms based on a combination of finite integral Fourier, Fourier–Bessel or Legendre transforms along one or two spatial coordinates with the finite difference technique along the remaining coordinate. The development of such an approach for vertically and radially inhomogeneous media is given in [1–2], for 2D and 3D inhomogeneous media in [3–7]. When modeling seismic fields in the media with attenuation the Fourier transform along the temporal coordinate was used, and the obtained boundary problem was solved by the sweep method [9].

In the given paper, the efficiency of application of the Laguerre integral transform along the temporal coordinate for the equations of the first and second order with respect to time is considered. An aspect of exact satisfaction of the initial data for these equations is investigated also. The analytical solution for wave fields propagation in the homogeneous media is obtained. The solution is represented as a series of the Laguerre functions. Advantages of the Laguerre integral transform as compared to the Fourier transform are discussed when solving the forward seismic problems in 2D inhomogeneous media.

1. Laguerre integral transform

Let us introduce the integral transform

$$F_m = \int_0^\infty F(t)(ht)^{-\frac{\alpha}{2}} l_m^\alpha(ht)\,dt, \tag{1}$$

$$F(t) = \sum_{m=0}^\infty F_m (ht)^{\frac{\alpha}{2}} l_m^\alpha(ht), \tag{2}$$

where $l_m^\alpha(ht)$ are the orthonormal Laguerre functions

$$\int_0^\infty l_m^\alpha(ht) l_n^\alpha(ht)\,dt = \delta_{mn}. \tag{3}$$

The Laguerre functions $l_m^\alpha(ht)$ are expressed by the classical Laguerre polynomials $L_m^\alpha(ht)$ (see [8]). We select the parameter α to be integer and positive, then

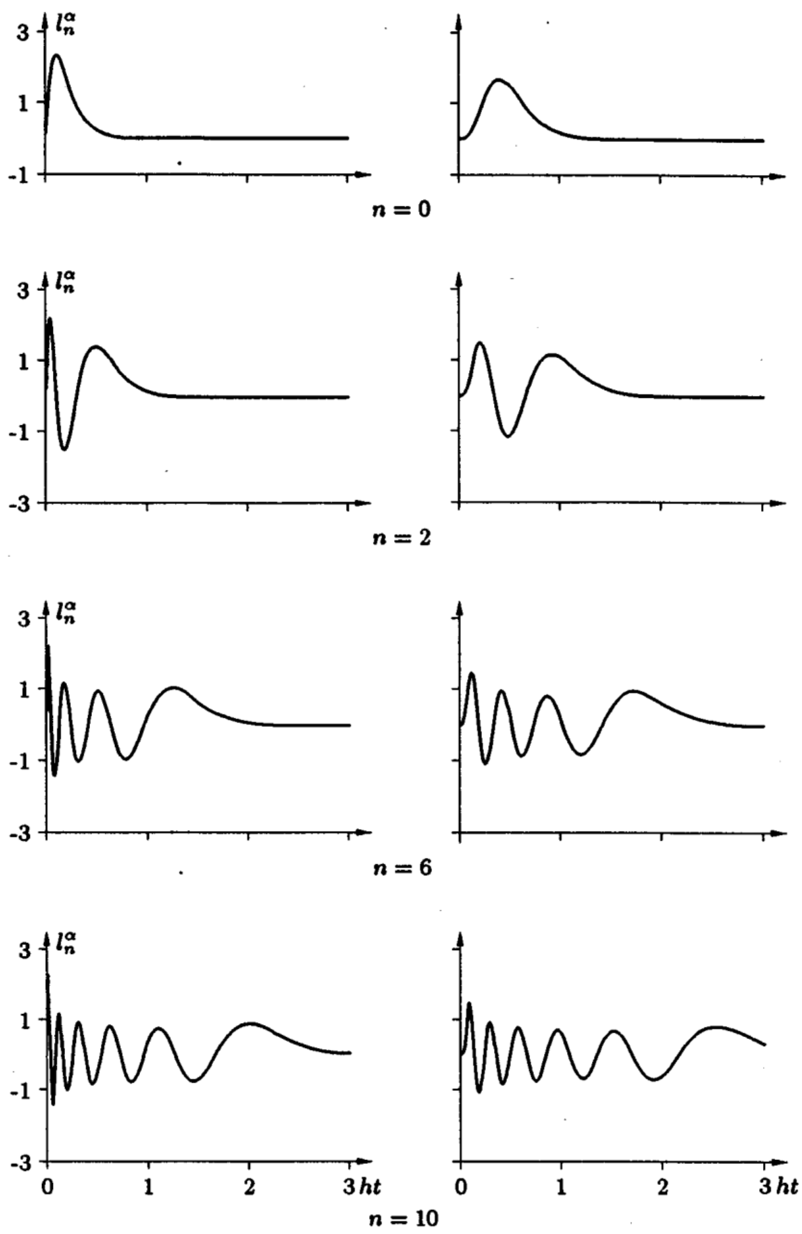

Figure 1. Graphs of Laguerre functions depending on n for the parameter $\alpha = 5$ (on the left) and 10 (on the right)

$$l_m^\alpha(ht) = \sqrt{\frac{hm!}{(m+\alpha)!}}(ht)^{\frac{\alpha}{2}}e^{-\frac{ht}{2}}L_m^\alpha(ht). \tag{4}$$

In formulas (1)–(4), $m = 0, 1, 2, \dots$. In addition, the new shift parameter $h > 0$ is introduced whose sense and efficiency of application is discussed below. Further, for the substantiation of validity of application of decomposition (1), (2) to seismic problems, we shall assume all the functions to be, at least, piecewise–continuous and to have some limitations on their behaviour in 0 and ∞ (see [8]).

In Figure 1 the graphs of the Laguerre functions for various parameters m and α are presented. As is seen from the figure, the Laguerre functions are essentially distinct from zero only on the limited segment of the axis t. With the increase of number m, which specifies the quantity of half-cycles of the Laguerre functions, its duration is increased.

2. Application of the Laguerre integral transform for the 1D acoustic problems

For simplicity, let us illustrate application of the Laguerre integral transform (1), (2) on an example of the solution to a system of 1D equations of the first order.

The distribution of an acoustic wave in the homogeneous medium is described by a system of equations:

$$\frac{\partial P}{\partial t} - \rho v_p^2 \frac{\partial u_z}{\partial z} = 0, \tag{5}$$

$$\frac{\partial u_z}{\partial t} - \frac{1}{\rho}\frac{\partial P}{\partial z} = 0. \tag{6}$$

The problem is solved with zero initial data

$$P|_{t=0} = 0, \qquad u_z|_{t=0} = 0 \tag{7}$$

and the boundary conditions

$$P|_{z=0} = f(t), \quad u_z|_{z=0} = -\frac{1}{\rho v_p} f(t). \tag{8}$$

We assume that the functions $P(z,t)$ and $u_z(z,t)$ tend to zero at $t \to \infty$. We apply to problem (5)–(8) the Laguerre integral transform:

$$Q_n(z) = \int_0^\infty P(z,t)(ht)^{-\frac{\alpha}{2}} l_n^\alpha(ht)\,dt, \tag{9}$$

$$R_n(z) = \int_0^\infty u_z(z,t)(ht)^{-\frac{\alpha}{2}} l_n^\alpha(ht)dt, \tag{10}$$

$$f_n = \int_0^\infty f(t)(ht)^{-\frac{\alpha}{2}} l_n^\alpha(ht)dt \tag{11}$$

with the inversion formulas

$$P(z,t) = \sum_{n=0}^\infty Q_n(z)(ht)^{\frac{\alpha}{2}} l_n^\alpha(ht), \tag{12}$$

$$u_z(z,t) = \sum_{n=0}^\infty R_n(z)(ht)^{\frac{\alpha}{2}} l_n^\alpha(ht), \tag{13}$$

$$f(t) = \sum_{n=0}^\infty f_n(ht)^{\frac{\alpha}{2}} l_n^\alpha(ht). \tag{14}$$

Let us obtain a system of the differential equations to satisfy the functions $Q_n(z)$, $R_n(z)$. Multiply equation (5) by $(ht)^{-\alpha/2} l_n^\alpha(ht)$ and integrate it over the variable t from 0 to ∞:

$$\int_0^\infty \left(\frac{\partial P}{\partial t} - \rho v_p^2 \frac{\partial u_z}{\partial z}\right)(ht)^{-\frac{\alpha}{2}} l_n^\alpha(ht)dt =$$

$$-\rho v_p^2 \frac{\partial R_n}{\partial z} + (ht)^{-\frac{\alpha}{2}} l_n^\alpha(ht)P(z,t)\Big|_0^\infty - \int_0^\infty P(z,t)\frac{d}{dt}\left[(ht)^{-\frac{\alpha}{2}} l_n^\alpha(ht)\right]dt. \tag{15}$$

In order to satisfy the initial conditions (7), we show that the term preceding the function $P(z,t)$ in the right-hand side of formula (15) at $t = 0$ and $t \to \infty$ is limited and does not tend to ∞. Denote

$$I_0 = (ht)^{-\frac{\alpha}{2}} l_n^\alpha(ht) = \sqrt{\frac{hn!}{(n+\alpha)!}} e^{-\frac{ht}{2}} L_n^\alpha(ht). \tag{16}$$

We have (see [8])

$$I_0 = \sqrt{\frac{hn!}{n+\alpha)!\pi}}(ht)^{-\beta} n^{\frac{\alpha}{2}-\frac{1}{4}} \left[\cos\left(2\sqrt{nht} - \beta\pi\right) + O\left(\frac{1}{\sqrt{n}}\right)\right] \tag{17}$$

at $n \to \infty$, where $\beta = (2\alpha + 1)/4$.

As it is seen from formulas (16), (17), the expression denoted by I_0 has the limited value at $t = 0$ and at $t \to \infty$. In this case we can satisfy the initial data (7) by setting $P|_{t=0} = 0$ in equation (15) and assuming $P|_{t\to\infty} = 0$. The integral term in the right-hand side of equation (15) is transformed to the form

$$
\begin{aligned}
I_1 &= \int_0^\infty P(z,t)\frac{d}{dt}\left[(ht)^{-\frac{\alpha}{2}} l_n^\alpha(ht)\right] dt \\
&= \int_0^\infty P(z,t)\frac{d}{dt}\left[\sqrt{\frac{hn!}{(n+\alpha)!}} e^{-\frac{ht}{2}} L_n^\alpha(ht)\right] dt \\
&= \int_0^\infty P(z,t)\sqrt{\frac{hn!}{(n+\alpha)!}}\left[-\frac{h}{2}e^{-\frac{ht}{2}} L_n^\alpha(ht) - he^{-\frac{ht}{2}}\sum_{k=0}^{n-1} L_k^\alpha(ht)\right] dt \\
&= -\frac{h}{2}Q_n(z) - h\sqrt{\frac{n!}{(n+\alpha)!}}\sum_{k=0}^{n-1}\sqrt{\frac{(k+\alpha)!}{k!}}Q_k(z).
\end{aligned} \tag{18}
$$

Here the following recurrence relation was used:

$$
\frac{d}{dt}L_n^\alpha(ht) = -h\sum_{k=0}^{n-1} L_k^\alpha(ht). \tag{19}
$$

Having conducted similar transformations for equation (6), we arrive at a new system of equations of the form

$$
\frac{h}{2}Q_n + h\sqrt{\frac{n!}{(n+\alpha)!}}\sum_{k=0}^{n-1}\sqrt{\frac{(k+\alpha)!}{k!}}Q_k - \rho v_p^2\frac{\partial R_n}{\partial z} = 0, \tag{20}
$$

$$
\frac{h}{2}R_n + h\sqrt{\frac{n!}{(n+\alpha)!}}\sum_{k=0}^{n-1}\sqrt{\frac{(k+\alpha)!}{k!}}R_k - \frac{1}{\rho}\frac{\partial Q_n}{\partial z} = 0 \tag{21}
$$

with the boundary conditions

$$
Q_n(z)|_{z=0} = f_n, \qquad R_n(z)|_{z=0} = -\frac{1}{\rho v_p}f_n. \tag{22}
$$

The exact solution to (20)–(22) is written down in the form

$$
\begin{aligned}
Q_n(z) &= \sqrt{\frac{n!}{(n+\alpha)!}}\left[\sum_{k=0}^{n}\sqrt{\frac{(k+\alpha)!}{k!}}f_k L_{n-k}^0\left(h\frac{z}{v_p}\right) - \right. \\
&\qquad \left.\sum_{k=0}^{n-1}\sqrt{\frac{(k+\alpha)!}{k!}}f_k L_{n-k-1}^0\left(h\frac{z}{v_p}\right)\right],
\end{aligned} \tag{23}
$$

$$
R_n(z) = -\frac{1}{\rho v_p}Q_n(z) \tag{24}
$$

Finally, the solution to problem (5)–(8) looks like

$$P(z,t) = (ht)^{\frac{\alpha}{2}} \sum_{n=0}^{\infty} Q_n(z) l_n^{\alpha}(ht), \tag{25}$$

$$u_z(z,t) = -\frac{1}{\rho v_p} P(z,t), \tag{26}$$

where f_k is determined by formula (11).

Let us simulate the source as the function

$$f(t) = \exp\left[-\frac{(2\pi f_0(t-t_0))^2}{\gamma^2}\right] \sin(2\pi f_0(t-t_0)), \tag{27}$$

where $\gamma = 4$, $f_0 = 1$, $t_0 = 1.5$ s. Consider the behaviour of the function $Q_n(z)$ defined by formula (23) depending on the number n of the Laguerre function at the fixed parameter h and the distance z. The graphs of the normal function $Q_n(z)$ are shown in Figure 2. They are presented at distances of $z = 5\lambda$ and $z = 10\lambda$, where λ is the dominating wavelength in the medium (Figure 2a). Impulses of the plane wave reconstructed by formula (25) at the same distances are indicated in Figure 2b. The parameters $h = 20$, $\alpha = 2$ are selected for the calculation. In Figure 3 the graphs of the function $Q_n(z)$ are given at $z = 10\lambda$ for $h = 40$ and $h = 60$. As follows from the figure, with the increase of the parameter h the spectral function $Q_n(z)$ is shifted to the right, and the width of the spectrum decreases. For the signal selected the optimal value of the parameter h is within 20 to 30. With the increase of frequency of the signal $f(t)$ it is necessary to linearly increase the value of the parameter h for attaining the optimal convergence of a series.

Let us discuss a question of satisfaction of the initial conditions of the problem for 1D second order wave equation when using the Laguerre integral transform along the temporal coordinate. The distribution of an acoustic wave in the homogeneous medium is described by the equation

$$\frac{\partial^2 P}{\partial z^2} = \frac{1}{v_p^2}\frac{\partial^2 P}{\partial t^2}, \tag{28}$$

with zero initial data

$$P\Big|_{t=0} = \frac{\partial P}{\partial t}\Big|_{t=0} = 0, \tag{29}$$

and the boundary conditions

$$P\Big|_{z=0} = f(t). \tag{30}$$

We apply to problem (28)–(30) the Laguerre integral transform (9), (12). Multiplying both parts of equation (28) by $(ht)^{\frac{\alpha}{2}} l_n^{\alpha}(ht)$ and integrating by parts over the variable t from 0 to ∞ we have

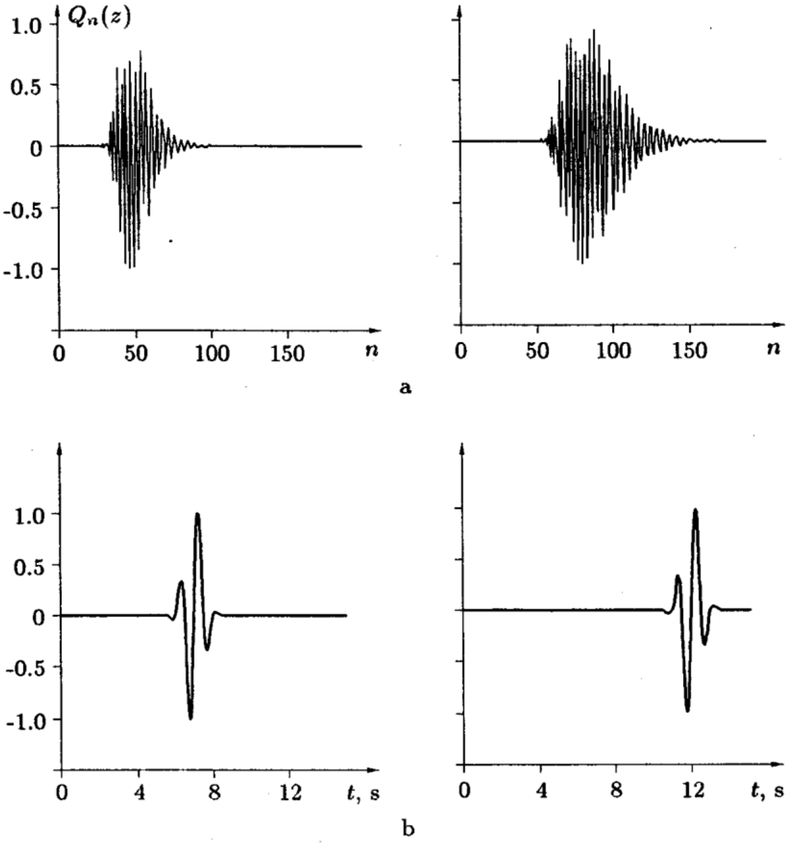

Figure 2. Graphs of the functions $Q_n(z)$ for $z = 5\lambda$ and $z = 10\lambda$ (a) and reconstructed impulses of the plane wave at the same distances (b)

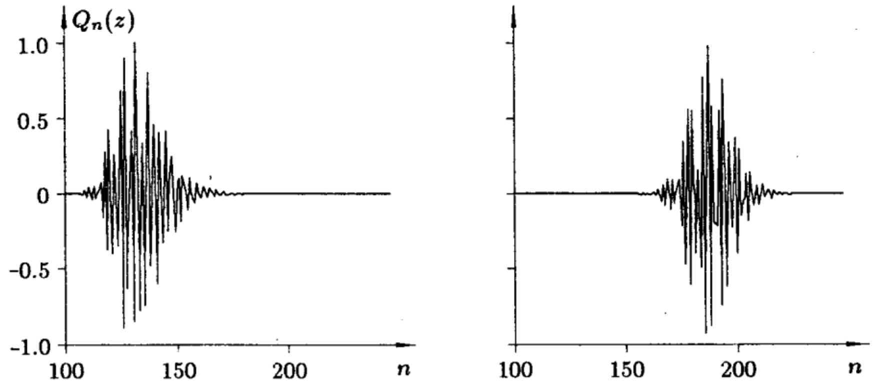

Figure 3. Graphs of the function $Q_n(z)$ for $z = 10\lambda$ for the parameter $h = 40$ (on the left) and 60 (on the right)

$$\int_0^\infty \left(\frac{\partial^2 P}{\partial z^2} - \frac{1}{v_p^2}\frac{\partial^2 P}{\partial t^2}\right)(ht)^{-\frac{\alpha}{2}} l_n^\alpha(ht)dt = \frac{\partial^2 Q_n}{\partial z^2} - \frac{1}{v_p^2}\left((ht)^{-\frac{\alpha}{2}} l_n^\alpha(ht)\frac{dP}{dt}\bigg|_0^\infty \right.$$

$$\left. -\frac{d}{dt}\left[(ht)^{-\frac{\alpha}{2}} l_n^\alpha(ht)\right] P\bigg|_0^\infty + \int_0^\infty P\frac{d^2}{dt^2}\left[(ht)^{-\frac{\alpha}{2}} l_n^\alpha(ht)\right] dt\right). \quad (31)$$

We can satisfy zero initial data (29) if in equation (31) we set $P|_{t=0} = \frac{\partial P}{\partial t}|_{t=0} = 0$. In this case it is necessary for us to make sure that the expressions preceding the function P and its derivative do not tend to ∞ at $t = 0$ and $t \to \infty$. As for the expression preceding the derivative $\frac{\partial P}{\partial t}$, we have shown it earlier (see formulas (16), (17)). Similarly, it is possible to show that the expression preceding the function P will have a finite value if we take advantage of the formula

$$\frac{d}{dt}\left[(ht)^{-\frac{\alpha}{2}} l_n^\alpha(ht)\right] = -\sqrt{\frac{hn!}{(n+\alpha)!}}\left(\frac{h}{2}e^{-\frac{ht}{2}} L_n^\alpha(ht) + he^{-\frac{ht}{2}}\sum_{k=0}^{n-1} L_k^\alpha(ht)\right). \quad (32)$$

Equation (31) with taken into account (29) takes the form

$$v_p^2\frac{\partial^2 Q_n}{\partial z^2} - \int_0^\infty P\left[\sqrt{\frac{hn!}{(n+\alpha)!}}\left(\frac{h^2}{4}e^{-\frac{ht}{2}} L_n^\alpha(ht) + h^2 e^{-\frac{ht}{2}}\sum_{k=0}^{k-1}(n-k)L_k^\alpha(ht)\right)\right] = 0. \quad (33)$$

Here we made use of the recurrence relation

$$\frac{d^2}{dt^2}\left[L_n^\alpha(ht)\right] = \frac{d}{dt}\left[-h\sum_{k=0}^{n-1} L_k^\alpha(ht)\right] = h^2\sum_{k=0}^{n-1}\sum_{j=0}^{k-1} L_k^\alpha(ht)$$

$$= h^2\sum_{k=0}^{n-2}(n-k-1)L_k^\alpha(ht). \quad (34)$$

Finally, an equation for the functions $Q_n(z)$ looks like

$$\frac{\partial^2 Q_n}{\partial z^2} - \frac{h^2}{4v_p^2}Q_n - \frac{h^2}{v_p^2}\sqrt{\frac{n!}{(n+\alpha)!}}\sum_{k=0}^{n-1}(n-k)\sqrt{\frac{(k+\alpha)!}{k!}}Q_k = 0. \quad (35)$$

The boundary condition (30) will be transformed as follows:

$$Q_n|_{z=0} = f_n. \quad (36)$$

Note, also, that for the decomposition to the Laguerre functions (12) satisfy the initial data (29), it is necessary to select the parameter $\alpha \geq 2$. It is easy to make sure in it if we substitute a series (12) in the initial data (29) and differentiate it.

Let us consider a solution to equation (28) in the case, when the velocity $v_p(z)$ is an arbitrary function of the coordinate z. For the sake of convenience we simulate a seismic plane wave in the right-hand side of equation (28), and replace the Dirichlet boundary condition on the free surface for the Neumann condition. After application of the Laguerre integral transform (9), (12), the problem is written down as

$$\frac{\partial Q_n}{\partial z^2} - \frac{h^2}{4v_p^2(z)} Q_n = \varphi_n(z), \tag{37}$$

with the boundary conditions

$$\left.\frac{\partial Q_n}{\partial z}\right|_{z=0} = 0, \quad Q_n\Big|_{z=b} = 0, \tag{38}$$

where

$$\varphi_n(z) = \delta(z - z_0) f_n + \frac{h^2}{v_p^2(z)} \sqrt{\frac{n!}{(n+\alpha)!}} \sum_{k=0}^{n-1} (n-k) \sqrt{\frac{(k+\alpha)!}{k!}} Q_k.$$

We solve problem (37), (38) numerically. For this purpose we introduce in the variable z the uniform difference grid:

$$\omega = \{z_i = (i-1)\Delta z; \quad i = 1, \ldots, N+1, \ b = N\Delta z\}.$$

Approximating the spatial derivative with the second order of accuracy, we come to a system of the linear algebraic equations, which in the vector form is written as

$$A\vec{U}_n = \vec{F}_n. \tag{39}$$

Here $\vec{U}_n = (Q_{n_1}, Q_{n_2} \ldots, Q_{n_N})^T$ and the vector of the right-hand side is defined by the components

$$(\vec{F}_n)_1 = -\frac{1}{2}\varphi_n(z_1), \quad (\vec{F}_n)_i = -\varphi_n(z_i), \quad i = 2, \ldots, N.$$

The matrix A is three-diagonal, symmetric and positive definite:

$$A = \frac{1}{\Delta z^2} \begin{bmatrix} a_1 & -1 & & & 0 \\ -1 & a_2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & a_{N-1} & -1 \\ 0 & & & -1 & a_N \end{bmatrix}, \tag{40}$$

with the diagonal entries

$$a_1 = 1 + \frac{1}{2}\left(\frac{h\Delta z}{2v_p(z_1)}\right)^2, \quad a_i = 2 + \frac{1}{2}\left(\frac{h\Delta z}{2v_p(z_i)}\right)^2, \quad i = 2, \ldots, N.$$

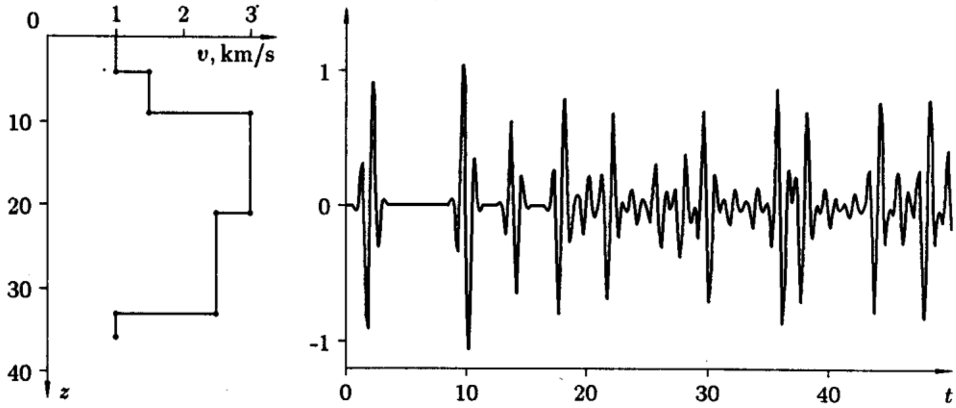

Figure 4. Model of a medium and a seismogram of reflected plane waves. Distances are given in wavelengths, time – in periods

In system (39) the matrix A does not depend on the parameter n. The dependence on the parameter n is recurrent and defined only by the right-hand side of equation (39). It enables us to solve very fast the problem according to the Cholesky scheme, when the system of the algebraic equations is considered for many right-hand sides, and the matrix A is decomposed only once. Note, that if one takes advantage of the Fourier transform along the temporal coordinate, we shall obtain a matrix A, dependent on the temporal frequency. It would considerably increase our computer costs.

A model of the medium and calculation of the synthetic seismogram at falling the seismic plane wave is presented in Figure 4.

3. Application of the Laguerre transform for 2D acoustic problems

Let us consider the inhomogeneous wave equation

$$\frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 u}{\partial x^2} - \frac{1}{v_p^2(x,z)}\frac{\partial^2 u}{\partial t^2} = \delta(x - x_0)\delta(z - z_0)f(t). \tag{41}$$

We search for its solution satisfying zero initial data

$$u\Big|_{t=0} = 0, \qquad \frac{\partial u}{\partial t}\Big|_{t=0} = 0 \tag{42}$$

and the boundary conditions on the free surface

$$\frac{\partial u}{\partial z}\Big|_{z=0} = 0. \tag{43}$$

We assume $v_p(x,z)$ to be a piecewise-continuous function of two coordinates. The source, with the coordinates x_0, z_0 is simulated by the right-hand side of equation (41), a temporal signal in the source being given by the function $f(t)$.

For solving (41), we take advantage of the finite integral cosine–Fourier transform

$$R(z,n,t) = \int_0^a u(z,x,t)\cos\left(\frac{n\pi x}{a}\right)dx \tag{44}$$

with the inversion formula

$$u(z,x,t) = \frac{1}{a}R(z,0,t) + \frac{2}{a}\sum_{n=1}^{\infty} R(z,n,t)\cos\left(\frac{n\pi x}{a}\right). \tag{45}$$

If we introduce the new additional boundary conditions

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = \left.\frac{\partial u}{\partial x}\right|_{x=a} = \left.u\right|_{z=b} = 0 \tag{46}$$

and consider the wave field up to the time $t \le T$, where T is the minimal time of propagation of the wave front from the reflecting surfaces $x = a$, $z = b$, the new boundary value problem for $R(z,n,t)$ will be written in the form (see [3])

$$\frac{\partial^2 R(z,n,t)}{\partial z^2} - k_n^2 R(z,n,t) - \sum_{l=0}^{\infty} c(l,n,z)\frac{\partial^2 R(z,l,t)}{\partial t^2} = \cos(k_n x_0)\delta(z-z_0)f(t) \tag{47}$$

$$\left.\frac{\partial R(z,n,t)}{\partial z}\right|_{z=0} = \left.R(z,n,t)\right|_{z=b} = 0, \tag{48}$$

$$\left.\frac{\partial R(z,n,t)}{\partial t}\right|_{t=0} = \left.R(z,n,t)\right|_{t=0} = 0, \tag{49}$$

where

$$c(l,n,z) = \int_0^a \frac{1}{v_p^2(x,z)}\cos(k_l x)\cos(k_n x)dx \tag{50}$$

$$k_n = \left(\frac{n\pi}{a}\right), \qquad n = 0,1,2,\dots . \tag{51}$$

Let us apply to problem (47)–(49) the Laguerre integral transform along the variable t:

$$Q(z,n,m) = \int_0^{\infty} R(z,n,t)(ht)^{-\frac{\alpha}{2}} l_m^{\alpha}(ht)dt \tag{52}$$

$$R(z,n,t) = \sum_{m=0}^{\infty} Q(z,n,m)(ht)^{\frac{\alpha}{2}} l_m^{\alpha}(ht). \tag{53}$$

Repeating the mathematics from Section 3, we obtain the problem for the decomposition coefficients $Q(z,l,m)$ (53):

$$\frac{\partial^2 Q(z,n,m)}{\partial z^2} - k_n^2 Q(z,n,m) -$$
$$\sum_{l=0}^{\infty} c(l,n,z)\left[\frac{h^2}{4}Q(z,l,m) + h^2\sqrt{\frac{m!}{(m+\alpha)!}}\sum_{j=0}^{m-1}(m-j)\sqrt{\frac{(j+\alpha)!}{j!}}Q(z,l,j)\right]$$
$$= \cos(k_n x_0)\delta(z-z_0)f_m, \tag{54}$$

with the boundary conditions

$$\left.\frac{\partial Q(z,n,m)}{\partial z}\right|_{z=0} = Q(z,n,m)\Big|_{z=b} = 0, \tag{55}$$

where $f_m = \int_0^{\infty} f(t)(ht)^{-\frac{\alpha}{2}} l_m^{\alpha}(ht)dt$.

Problem (54), (55) is reduced to the system of algebraic equations with the help of the difference approximation of derivatives with respect to the coordinate z. The obtained system with many right-hand sides, as earlier, is solved with the help of the Cholesky algorithm.

4. Conclusion

The method of solution to the seismic forward problems with the help of the Laguerre integral transform along the temporal coordinate is proposed. As compared to the classical Fourier and Fourier–Bessel or the Legendre transforms, the application of the Laguerre integral transform results in a system of equations, in which the partition parameter is included in the recurrent form only in the right-hand side. After reducing the problem to a system of algebraic equations with many right-hand sides, the fast algorithms of solution are used on the basis of matrix decomposition according to the Cholesky scheme.

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