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Numerical–analytical method for solving the forward dynamical seismic problems

G.V. Konyukh, Yu.V. Krivtsov, B.G. Mikhailenko

When solving the multi-dimensional problems of theory of the seismic wave propagation, the numerical-analytical method, based on the combination of finite integral transforms with the finite-difference method is gaining in importance in the last few years (see [1–3]). In this paper, instead of the finite difference method with respect to one spatial coordinate and time, we propose to use the finite difference approximation with respect to only the spatial coordinate with a subsequent analytical solution of the obtained system of ordinary differential equations.

1. Statement of the problem

Let us illustrate the main stages of the method on the wave equation in the Cartesian coordinate system

$$\frac{\partial^2 U}{\partial z^2} + \frac{\partial^2 U}{\partial x^2} = \frac{1}{v_p^2(z)} \frac{\partial^2 U}{\partial t^2} + \delta(x - x_0)\delta(z - z_0)f(t). \tag{1}$$

Here $v_p(z)$ is a piecewise-continuous function of the coordinate z; x_0, z_0 are the coordinates of the source simulated by the right-hand side of equation (1); $f(t)$ is a time signal in the source.

The problem is solved with zero initial data

$$U\big|_{t=0} = \frac{\partial U}{\partial t}\Big|_{t=0} = 0, \tag{2}$$

and the boundary conditions on the free surface in the form

$$\frac{\partial U}{\partial z}\Big|_{z=0} = 0. \tag{3}$$

Assume that the function $U(z, x, t)$ possesses sufficient smoothness for using the subsequent transformations.

2. Numerical-analytical method for solving problem

For solving problem (1)–(3) let us make use of the finite integral cosine-Fourier transform

$$R(z,n,t) = \int_0^a U(z,x,t)\cos\Big(\frac{n\pi x}{a}\Big)\,dx, \tag{4}$$

with the inversion formula

$$U(z,x,t) = \frac{1}{a}R(z,0,t) + \frac{2}{a}\sum_{n=1}^{\infty} R(z,n,t)\cos\Big(\frac{n\pi x}{a}\Big). \tag{5}$$

The equation obtained after the transformation contains the terms $\frac{\partial U}{\partial x}\big|_{x=0}$, $\frac{\partial U}{\partial x}\big|_{x=a}$. Let us introduce new additional boundary conditions

$$\frac{\partial U}{\partial x}\Big|_{x=0} = \frac{\partial U}{\partial x}\Big|_{x=a} = U\big|_{z=b} = 0, \tag{6}$$

and consider the wave field up to the time $t < T$, where T is the minimal time of propagation of the leading wave front from the reflecting surfaces $x = a$, $z = b$. We are able to do it due to hyberbolicity of the problem. The new boundary problem for $R(z,n,t)$ is of the form

$$\frac{\partial^2 R}{\partial z^2} - k_n^2 R = \frac{1}{v_p^2(z)}\frac{\partial^2 R}{\partial t^2} + \cos\Big(\frac{n\pi x_0}{a}\Big)\delta(z - z_0)f(t), \tag{7}$$

$$\frac{\partial R}{\partial z}\Big|_{z=0} = R\big|_{z=b} = 0, \tag{8}$$

$$R\big|_{t=0} = \frac{\partial R}{\partial t}\Big|_{t=0} = 0, \tag{9}$$

where

$$k_n^2 = \Big(\frac{n\pi}{a}\Big)^2, \qquad n = 1, 2, \ldots.$$

For solving the system of problems (7)–(9) let us make use of the finite difference approximation with respect to the coordinate Z. As a result, the original equation is transformed to the system of ordinary differential equations. To this end let us introduce in the variable z the uniform difference grid

$$\omega = \{z_i = (i-1)h; \quad i = 1, \ldots, N+1; \quad b = Nh\}.$$

If $v_p(z)$ is continuous at the node $z_i \in \omega$, we assume $v_i = v_p(z_i)$, otherwise $v_i = (v_{i+1} + v_{i-1})/2$. The coordinate z_0, which determines the location of a source, is calculated by the formula

$$z_0 = (l-1)h.$$

Determination of the functions $R_i(n,t)$ on the lines $z = z_i$ reduces to solving the Cauchy problem for the system of N linear differential second order equations. Write down the system in the vector form

$$\frac{d^2\bar{Z}}{dt^2} + A_n\bar{Z} = f(t)\bar{F}, \tag{10}$$

$$\bar{Z}\big|_{t=0} = \frac{d\bar{Z}}{dt}\Big|_{t=0} = 0. \tag{11}$$

Note, that the original system has been reduced to the form of (10)–(11) by means of the preliminary replacement of the variables

$$\bar{Z}(n,t) = D\bar{R}(n,t), \quad D = \text{diag}\Big\{\frac{1}{v_1}, \frac{\sqrt{2}}{v_2}, \ldots, \frac{\sqrt{2}}{v_N}\Big\},$$
$$\bar{R}(n,t) = (R_1(n,t), \ldots, R_N(n,t))^T,$$

providing the symmetry of the matrix A_n. In equation (10) the vector $\bar{F}$ is determined by the components

$$F_i = 0, \quad i = 1, \ldots, N; \quad i \neq l;$$
$$F_l = -\cos\Big(\frac{n\pi x_0}{a}\Big)\frac{\sqrt{2}}{h}v_l, \quad i = l, \quad l \neq 1,$$
$$F_l = -\cos\Big(\frac{n\pi x_0}{a}\Big)\frac{v_l}{h}, \quad i = l, \quad l = 1.$$

Let us distinguish the dependence on the parameter n in the square matrix A_n by representing it as a sum of two constants for these medium matrices

$$A_n = A + k_n^2 B, \tag{12}$$

where B is a diagonal matrix

$$B = \text{diag}\{v_1^2, v_2^2, \ldots, v_N^2\},$$

and A is a three-diagonal symmetric positive matrix

$$A = \frac{1}{h^2}\begin{pmatrix} 2v_1^2 & -\sqrt{2}v_1v_2 & \cdots & \cdots & 0 \\ -\sqrt{2}v_1v_2 & 2v_2^2 & -v_2v_3 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & -v_{N-1}v_N & 2v_{N-1}^2 & -v_{N-1}v_N \\ 0 & \cdots & \cdots & -v_{N-1}v_N & 2v_N^2 \end{pmatrix}.$$

Using the orthonormal decomposition [4]

$$A_n = Q\,\text{diag}\{\lambda_1, \ldots, \lambda_N\}Q^{-1}, \tag{13}$$

and replacing the variables

$$\bar{Y}(n,t) = Q^{-1}\bar{Z}(n,t),$$

problem (10)–(11) falls into N independent Cauchy problems for each component Y_i of the vector $\bar{Y}(n,t)$:

$$\frac{d^2Y_i}{dt^2} + \lambda_i Y_i = \varphi_i = F_l Q_{l,i} f(t), \tag{14}$$

$$Y_i\big|_{t=0} = \frac{dY_i}{dt}\Big|_{t=0} = 0. \tag{15}$$

The solution to (14)–(15) is written down in the form

$$Y_i(t) = \int_0^t \varphi_i(\tau)\frac{1}{\sqrt{\lambda_i}}\sin(\sqrt{\lambda_i}(t-\tau))d\tau. \tag{16}$$

Depending on the type of function of the signal $f(t)$ the values $Y_i(t)$ can be obtained either analytically or numerically by calculating the integral in (16). Let, for example,

$$f(t) = \exp\Big(-\frac{(2\pi f_0(t-t_0))^2}{\gamma^2}\Big)\sin(2\pi f_0(t-t_0)),$$

where t_0, f_0, γ are certain constants. Then for all $2t_0 \le t < T$ formula (16) takes the form

$$Y_i(t) \approx \frac{F_l Q_{l,i}\gamma}{4\sqrt{\pi\lambda_i}f_0}\cos(\sqrt{\lambda_i}(t-t_0)) \times$$
$$\Big\{\exp\Big(-\Big[\frac{(2\pi f_0+\sqrt{\lambda_i})\gamma}{4\pi f_0}\Big]^2\Big) - \exp\Big(-\Big[\frac{(2\pi f_0-\sqrt{\lambda_i})\gamma}{4\pi f_0}\Big]^2\Big)\Big\}.$$

After we have found $\bar{Y}(n,t)$, it is sufficient to return to the original variable

$$\bar{R}(n,t) = D^{-1}Q\bar{Y}(n,t),$$

and then to find the solution $U(x,z_i,t)$ of the original problem by formula (5).

Let us turn our attention to the following. In our opinion, the basic stage of the problem solution by the above-discussed method, is the development of an efficient algorithm for constructing expansion decomposition (13) for the succession of the matrices A_n, $n = 1,2,\ldots$, of the form of (12). For determination of the matrices Q, which diagonalyze A_n, we have proposed the following algorithm.

3. Determination of eigenvalues and eigenvectors of the matrices $A_n = A + k_n^2 B$

3.1. Statement of the problem of calculating the matrices Q for all k_n.

Let

$$W = \frac{1}{k_n^2} A_n = \frac{1}{k_n^2} A + B.$$

For determining the matrix Q which diagonalyzes the matrix W, let us present the problem of calculation of eigenvalues and eigenvectors of the matrix W as

$$(W - \lambda E)\bar{y} = \Big(\frac{1}{k_n^2} A + B - \frac{\lambda}{k_n^2} E\Big)\bar{y} = 0. \tag{17}$$

The succession $\frac{1}{k_n^2} \to 0$ at $n \to \infty$ is monotonous. Therefore the difference $(\frac{1}{k_n^2} - \frac{1}{k_j^2})$ is insufficient for rather large n, j. This allows us to make use of the results of the perturbation theory [5] for solving problem (17). Knowing the solution of the problem for a certain term $\frac{1}{k_c^2}$ for the close to it numbers $\frac{1}{k_n^2}$ eigenvectors and eigenvalues can be found in the following way. Transform equation (17)

$$\Big(\Big(\frac{1}{k_n^2} - \frac{1}{k_c^2}\Big)A + \Big(\frac{1}{k_c^2} A + B\Big) - \frac{\lambda}{k_n^2} E\Big)\bar{y} = 0. \tag{18}$$

Rewrite (18) in the form

$$(\mu A + B' - \lambda' E)\bar{y} = 0, \tag{19}$$

and present eigenvalues and eigenvectors of problem (19) as expansion in μ series

$$\lambda_k'(\mu) = \sum_{s=0}^{\infty} \mu^s \nu_{ks}, \qquad \bar{y}_k(\mu) = \sum_{s=0}^{\infty} \mu^s T_{k,s}, \qquad k = 0, \ldots, N, \tag{20}$$

$$\nu_{k0} = \lambda_k'(0), \qquad \bar{y}_k(0) = T_{k,0}, \tag{21}$$

$\lambda_k'(0)$, $\bar{y}_k(0)$ are eigenvalues and eigenvectors of the matrix B'.

It is known [6] that the series in (20) are convergent, if the value of perturbation $\| \mu A \|$ is lesser than half the distance d from the eigenvalue $\lambda_k'(0)$ to the set of eigenvalues of the operator B', i.e.,

$$\Big|\Big(\frac{1}{k_n^2} - \frac{1}{k_c^2}\Big)\Big| = \mu \le \frac{d}{2 \| A \|}. \tag{22}$$

The perturbation theory [5] yields the recurrent formulas for determination of ν_{ks}, $T_{k,s}$:

$$\nu_{ks} = \bar{y}_k(0)^T A T_{k,s-1}, \qquad T_{k,1} = -S_k A T_{k,0}, \tag{23}$$

$$T_{k,s} = S_k\Big(\sum_{n=1}^{s-1} \nu_{kn} T_{k,s-n} - A T_{k,s-1}\Big), \tag{24}$$

$$S_k = \sum_{s=1, s\neq k}^{N} \frac{P_s}{\lambda_s - \lambda_k}, \qquad P_s = \bar{y}_s(0)\bar{y}_s^T(0). \tag{25}$$

The QL algorithm requires for computation of all eigenvalues and eigenvectors approximately [7] $15.3N^2$ multiplications, $3.4N^2$ divisions, $1.7(N-1)$ square roots and $15.3N^2$ additions. Computational volume according to formulas (20), (21) of the perturbation theory, if restricted by K first terms, is much less, but requires to determine the coefficients ν_{ks}, $T_{k,s}$, $s = 1, \ldots, K$ first.

3.2. Algorithm for calculation of matrix Q for all k_n

Algorithm consists of three steps.

1. At the first step, for $k_c = k_M$, the problem (17) of determination of eigenvalues and eigenvectors for symmetrical, three-diagonal, real W is solved. QL-algorithm is applied for this.
2. At the second step, eigenvalues and eigenvectors, found for matrices $(\frac{1}{k_c^2}A + B)$, are used for calculation of coefficients of series ν_{ks}, $T_{k,s}$ according to formulas (23)–(25).
3. At the third step for all k_n, $n = c-1, c-2, \ldots$, such that $\mu = (\frac{1}{k_n^2} - \frac{1}{k_c^2})$ satisfy condition (22), eigenvalues and eigenvectors are calculated by formulas of the perturbation theory (20), (21). If for some $n = c - L$ condition of convergence (22) is not satisfied, the value L is checked. If $L = 1$, then the first step of the algorithm is executed with $k_c = k_{c-L}$. If $L > 1$, then the second step of the algorithm is executed with $k_c = k_{c-L+1}$.

Algorithm has to be executed, until matrices Q for all k_n are found.

4. Conclusion

To conclude, let us dwell on the advantages of the given algorithm. This algorithm allows the calculation of a wave field at any moment of time without recurrent recalculation procedure from one time level to another as it takes place when we use the finite difference approximation with respect to time. In addition, this approach makes it possible to obtain solutions for many sources without essential computer costs, because we need to diagonalize the matrix A_n only one time for all the sources.

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