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Investigation of one dynamic system arising in a two-fluid medium

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Abstract. The Cauchy problem for a one-dimensional homogeneous system of the Hopf-type equations arising in a two-fluid medium is considered. It is believed that energy dissipation occurs only due to the friction coefficient (analogous to Darcy) and the Cauchy data are given in the form of a finite trigonometric Fourier series. Recursive systems of ordinary differential equations for amplitudes are obtained.

Keywords: two-fluid medium, viscosity, Darcy coefficient, direct problem, Fourier series, relative momentum.

Introduction

The hydrodynamic model of a multiphase fluid is well known and described in detail in textbooks and monographs on continuum mechanics [1–7]. The Hopf equation is one of the simplest hyperbolic models in which the Cauchy problem for the Hopf equation is equivalent to the implicit function theorem. This equation only takes into account advective processes in the model. If we take into account the effects of interfacial friction, then, as shown in a two-fluid medium [8–23], a system of hyperbolic Hopf type arises. The system of equations of two-velocity hydrodynamics and the system of the Hopf-type equations have much in common. This system takes into account the following effects: the presence of a quadratic nonlinearity in the velocities of the subsystems, associated with advective processes and corresponding to the dependence of the speed of sound propagation on the amplitude of sound waves, and the presence of a diffusion term linear in the relative momentum on the right sides, associated with interphase processes friction and responsible for the attenuation of sound waves.

As for the properties of the solutions, they are completely different. In a system of the Hopf-type equations with a vanishing coefficient, an analogue of the Darcy coefficient, both strong (shock waves) and weak discontinuities are formed, while the solution of a two-speed hydrodynamics system does not have such features. However, the scope of applicability of this system is by no means limited to the examples given; such systems arise in many problems, which determines its significance [24]. Therefore, nonlinear models are of particular value, although they are greatly simplified in comparison with the original system of two-speed hydrodynamics, but retain its important features.

1. Formulation of the problem

Let us consider the process of propagation of nonlinear waves in a two-fluid medium, described by a one-dimensional homogeneous system of equations [8, 9, 14, 15, 16, 22–24]:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -b(u - \tilde{u}), \tag{1}$$

$$\frac{\partial \tilde{u}}{\partial t} + \tilde{u}\frac{\partial \tilde{u}}{\partial x} = \varepsilon b(u - \tilde{u}), \tag{2}$$

where u and $\tilde{u}$ are the velocities of subsystems with the corresponding partial densities ρ and $\tilde{\rho}$, b is a positive constant responsible for friction in the system (analogue of the Darcy coefficient), $\varepsilon = \rho/\tilde{\rho}$ is a dimensionless positive constant.

Let us consider the Cauchy problem for a system of the Hopf-type equations with periodic initial data in the form of a trigonometric polynomial [25]:

$$u|_{t=0} = \sum_{j=-N}^{N} \alpha_j e^{ijx}, \tag{3}$$

$$\tilde{u}|_{t=0} = \sum_{j=-N}^{N} \tilde{\alpha}_j e^{ijx} \tag{4}$$

Note that for any constant C, the solution $u(t,x)$ and $\tilde{u}(t,x)$ of Cauchy problem for system (1), (2) can be replaced by another solution $U(t,x)$ and $\tilde{U}(t,x)$ with the initial data shifted by C:

$$U(t,x) = C + u(t, x + Ct), \qquad \tilde{U}(t,x) = C + \tilde{u}(t, x + Ct).$$

These functions satisfy system (1), (2). This one-parameter group of symmetries allows us to select problems (3), (4) with the initial data $\alpha_0 = 0$ and $\tilde{\alpha}_0 = 0$ and move from a system of Hopf-type equations to an integrated system of equations

$$\frac{\partial v}{\partial t} + \frac{1}{2}\frac{\partial v^2}{\partial x} = -b(v - \tilde{v}), \qquad u = \frac{\partial v}{\partial x}, \tag{5}$$

$$\frac{\partial \tilde{v}}{\partial t} + \frac{1}{2}\frac{\partial \tilde{v}^2}{\partial x} = \varepsilon b(v - \tilde{v}), \qquad \tilde{u} = \frac{\partial \tilde{v}}{\partial x}. \tag{6}$$

An approximate solution of the latter system is constructed by the Galerkin method [25]:

$$\frac{\partial v}{\partial \tau} + \frac{1}{2}\left\{\frac{\partial v^2}{\partial x}\right\} = -b(v - \tilde{v}),$$
$$v(\tau, x) = \sum_{n=1}^{N} \frac{1}{n} a_n(\tau) e^{inx} + \sum_{n=1}^{N} \frac{1}{n} b_n(\tau) e^{-inx}, \tag{7}$$
$$\frac{\partial \tilde{v}}{\partial \tau} + \frac{1}{2}\left\{\frac{\partial \tilde{v}^2}{\partial x}\right\} = \varepsilon b(v - \tilde{v}),$$
$$\tilde{v}(\tau, x) = \sum_{n=1}^{N} \frac{1}{n} \tilde{a}_n(\tau) e^{inx} + \sum_{n=1}^{N} \frac{1}{n} \tilde{b}_n(\tau) e^{-inx}. \tag{8}$$

Curly brackets here indicate the rejection of higher harmonics with numbers greater than N [25]. Equating the coefficients for e^{imx}, $|m| \le N$, in the equations for $v(\tau, x)$ and $\tilde{v}(\tau, x)$ leads to a closed system of differential equations for $4N$ unknown functions $a_n(\tau)$, $\tilde{a}_n(\tau)$ and $b_n(\tau)$, $\tilde{b}_n(\tau)$, $n = 1, \ldots, N$:

$$a'_n(\tau) = f_n(\boldsymbol{a}, \boldsymbol{b}) - b(a_n - \tilde{a}_n), \qquad b'_n(\tau) = g_n(\boldsymbol{a}, \boldsymbol{b}) - b(b_n - \tilde{b}_n), \tag{9}$$
$$\tilde{a}'_n(\tau) = \tilde{f}_n(\tilde{\boldsymbol{a}}, \tilde{\boldsymbol{b}}) + \varepsilon b(a_n - \tilde{a}_n), \qquad \tilde{b}'_n(\tau) = \tilde{g}_n(\tilde{\boldsymbol{a}}, \tilde{\boldsymbol{b}}) + \varepsilon b(b_n - \tilde{b}_n), \tag{10}$$

where the prime is differentiation with respect to time denoted by τ.

The initial data for the dynamic system (9), (10) due to the relation $u = \dfrac{\partial v}{\partial x}$ and $\tilde{u} = \dfrac{\partial \tilde{v}}{\partial x}$ are given in the form

$$a_n(0) = -i\alpha_n, \qquad b_n(0) = i\alpha_{-n},$$
$$\tilde{a}_n(0) = -i\tilde{\alpha}_n, \qquad \tilde{b}_n(0) = i\tilde{\alpha}_{-n}.$$

For real solutions, the conditions $\bar{a}_n = b_n$ and $\bar{\tilde{a}}_n = \tilde{b}_n$ must be satisfied.

In the dynamic system (9), (10), the functions $f_n(\boldsymbol{a}, \boldsymbol{b})$, $\tilde{f}_n(\tilde{\boldsymbol{a}}, \tilde{\boldsymbol{b}})$ and $g_n(\boldsymbol{a}, \boldsymbol{b})$, $\tilde{g}_n(\tilde{\boldsymbol{a}}, \tilde{\boldsymbol{b}})$ are homogeneous quadratic polynomials of their arguments. As an illustration, we give the form of the function $\tilde{f}_N(\tilde{\boldsymbol{a}}, \tilde{\boldsymbol{b}})$ depending on the parity of the number N:

$$\tilde{f}_N(\tilde{\boldsymbol{a}}, \tilde{\boldsymbol{b}}) = \begin{cases} -N(\tilde{a}_1\tilde{a}_{N-1} + \tilde{a}_2\tilde{a}_{N-2} + \ldots + \tilde{a}_m\tilde{a}_{m+1}), & N = 2m+1, \\ -N(\tilde{a}_1\tilde{a}_{N-1} + \tilde{a}_2\tilde{a}_{N-2} + \ldots + \tilde{a}_{m-1}\tilde{a}_{m+1}) - m\tilde{a}_m^2, & N = 2m. \end{cases}$$

In particular, when $N = 5$ the dynamic system (9), (10) has the following form:

$$a'_1 = b_1a_2 + b_2a_3 + b_3a_4 + b_4a_5 - b(a_1 - \tilde{a}_1),$$
$$b'_1 = a_1b_2 + a_2b_3 + a_3b_4 + a_4b_5 - b(b_1 - \tilde{b}_1),$$
$$a'_2 = 2(b_1a_3 + b_2a_4 + b_3a_5) - a_1^2 - b(a_2 - \tilde{a}_2),$$
$$b'_2 = 2(a_1b_3 + a_2b_4 + a_3b_5) - b_1^2 - b(b_2 - \tilde{b}_2),$$

$$
\begin{aligned}
a_3' &= 3(b_1a_4 + b_2a_5) - 3a_1a_2 - b(a_3 - \tilde a_3),\\
b_3' &= 3(a_1b_4 + a_2b_5) - 3b_1b_2 - b(b_3 - \tilde b_3),\\
a_4' &= 4b_1a_5 - 4a_1a_3 - 2a_2^2 - b(a_4 - \tilde a_4),\\
b_4' &= 4a_1b_5 - 4b_1b_3 - 2b_2^2 - b(b_4 - \tilde b_4),\\
a_5' &= -5(a_1a_4 + a_2a_3) - b(a_5 - \tilde a_5),\\
b_5' &= -5(b_1b_4 + b_2b_3) - b(b_5 - \tilde b_5),\\
\tilde a_1' &= \tilde b_1\tilde a_2 + \tilde b_2\tilde a_3 + \tilde b_3\tilde a_4 + \tilde b_4\tilde a_5 + \varepsilon b(a_1 - \tilde a_1),\\
\tilde b_1' &= \tilde a_1\tilde b_2 + \tilde a_2\tilde b_3 + \tilde a_3\tilde b_4 + \tilde a_4\tilde b_5 + \varepsilon b(b_1 - \tilde b_1),\\
\tilde a_2' &= 2(\tilde b_1\tilde a_3 + \tilde b_2\tilde a_4 + \tilde b_3\tilde a_5) - \tilde a_1^2 + \varepsilon b(a_2 - \tilde a_2),\\
\tilde b_2 &= 2(\tilde a_1\tilde b_3 + \tilde a_2\tilde b_4 + \tilde a_3\tilde b_5) - \tilde b_1^2 + \varepsilon b(b_2 - \tilde b_2),\\
\tilde a_3 &= 3(\tilde b_1\tilde a_4 + \tilde b_2\tilde a_5) - 3\tilde a_1\tilde a_2 + \varepsilon b(a_3 - \tilde a_3),\\
\tilde b_3' &= 3(\tilde a_1\tilde b_4 + \tilde a_2\tilde b_5) - 3\tilde b_1\tilde b_2 + \varepsilon b(b_3 - \tilde b_3),\\
\tilde a_4 &= 4\tilde b_1\tilde a_5 - 4\tilde a_1\tilde a_3 - 2\tilde a_2^2 + \varepsilon b(a_4 - \tilde a_4),\\
\tilde b_4' &= 4\tilde a_1\tilde b_5 - 4\tilde b_1\tilde b_3 - 2\tilde b_2^2 + \varepsilon b(b_4 - \tilde b_4),\\
\tilde a_5' &= -5(\tilde a_1\tilde a_4 + \tilde a_2\tilde a_3) + \varepsilon b(a_5 - \tilde a_5),\\
\tilde b_5' &= -5(\tilde b_1\tilde b_4 + \tilde b_2\tilde b_3) + \varepsilon b(b_5 - \tilde b_5).
\end{aligned}
$$

It is easy to see that the dynamic system (9), (10) admits a reduction $b_n = a_n$, $\tilde b_n = \tilde a_n$ which, following [25], we call the *main* reduction. This reduction corresponds to odd-in-variable solutions of the system of equations (1), (2) and initial data of the form

$$
\begin{aligned}
a_n(0) &= -i\alpha_n, \qquad & b_n(0) &= i\alpha_{-n} = -i\alpha_n,\\
\tilde a_n(0) &= -i\tilde\alpha_n, \qquad & \tilde b_n(0) &= i\tilde\alpha_{-n} = -i\tilde\alpha_n,
\end{aligned}
$$

for $n = 1, \dots, N$.

As noted in [25], the principal reduction allows approximate solutions of a system of the Hopf-type equations to satisfy additional zero Dirichlet boundary conditions:

$$
\begin{aligned}
\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} &= -b(u - \tilde u), \qquad & 0 \le x \le 2\pi,\\
u(t,0) = u(t,2\pi) &= 0, \qquad & t \ge 0,\\
\frac{\partial \tilde u}{\partial t} + \tilde u\frac{\partial \tilde u}{\partial x} &= \varepsilon b(u - \tilde u), \qquad & 0 \le x \le 2\pi,\\
\tilde u(t,0) = \tilde u(t,2\pi) &= 0, \qquad & t \ge 0.
\end{aligned}
$$

Let us now consider the question of integrating a system of equations of the Hopf type. An analogue of a particular solution to a system of the

Hopf-type equations is the following solution to equation (7), (8) for $N = 2$:

$$
\begin{aligned}
a_1' &= b_1 a_2 - b(a_1 - \tilde{a}_1), & b_1' &= a_1 b_2 - b(b_1 - \tilde{b}_1),\\
a_2' &= -a_1^2 - b(a_2 - \tilde{a}_2), & b_2' &= -b_1^2 - b(b_2 - \tilde{b}_2),\\
\tilde{a}_1' &= \tilde{b}_1 \tilde{a}_2 + \varepsilon b(a_1 - \tilde{a}_1), & \tilde{b}_1' &= \tilde{a}_1 \tilde{b}_2 + \varepsilon b(b_1 - \tilde{b}_1),\\
\tilde{a}_2' &= -\tilde{a}_1^2 + \varepsilon b(a_2 - \tilde{a}_2), & \tilde{b}_2 &= -\tilde{b}_1^2 + \varepsilon b(b_2 - \tilde{b}_2).
\end{aligned}
$$

A particular solution of this dynamic system is the following:

$$
a_1 = \tilde{a}_1 = b_1 = \tilde{b}_1 = 0,
$$

$$
\begin{aligned}
a_2 &= \frac{c_2 + c_1 e^{-(1+\varepsilon)b\tau}}{1+\varepsilon}, & \tilde{a}_2 &= \frac{c_2 - \varepsilon c_1 e^{-(1+\varepsilon)b\tau}}{1+\varepsilon},\\
b_2 &= \frac{c_4 + c_3 e^{-(1+\varepsilon)b\tau}}{1+\varepsilon}, & \tilde{b}_2 &= \frac{c_4 - \varepsilon c_3 e^{-(1+\varepsilon)b\tau}}{1+\varepsilon}.
\end{aligned}
$$

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