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The Bernoulli equation for an incompressible two-fluid medium at pressure phases equilibrium with constant saturation of phases

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Abstract. In this paper, the Bernoulli equation for an incompressible two-fluid medium with equilibrium of pressure phases in the reversible hydrodynamic approximation is constructed.

Keywords: incompressible fluid, Bernoulli equation, saturation.

The development of the advanced computational modeling for compressible multi-phase flows is of interest in a number of scientific and engineering disciplines and in many industrial applications. Although in recent years the intensive efforts in the multiphase flow modeling have been made still many basic physical, mathematical, and computational issues are largely unresolved. The classical approach to the development of multiphase models is based on the assumption that a multiphase flow can be considered as a set of interacting continua and described as an averaged continuous medium in which the behavior of each phase is governed by the conservation laws of mass, momentum and energy, while the interfacial interaction is taken into account through differential and algebraic source terms in the phase conservation laws [1, 2].

The study of flows of viscous compressible / incompressible liquids based on solving a complete system of equations of the two-velocity hydrodynamics is actual. A very limited number of cases admitting an analytical integration of the Navier–Stokes equations is considered in published works [3–5]. For an analytical solution of hydrodynamic problems, there is, in fact, only the perturbation method, whose development and application have recently been the major achievements. Numerical methods stand apart.

Both the previous and current research efforts in relation to the multiphase flow modeling mostly concentrate on the two-phase computational models. These include, in particular, a single-pressure model for the two-phase compressible flows, which is still used as a basic model in some industrial computer codes. The governing equations used in the basic single-pressure model are of a mixed hyperbolic/elliptic type thus making the initial boundary value problem mathematically ill-posed. Consequently, computations performed with this model on coarse meshes or using dissipative numerical schemes yield reasonable solutions, but when a mesh is sufficiently refined or more accurate numerical methods are used, the solution does not converge [6].

The objective of the present paper is to construct the Bernoulli equations for a stationary system of equations of the two-velocity hydrodynamics with equilibrium of pressure phases. On the one hand, these solutions can be useful for testing numerical methods of solving equations of the two-velocity hydrodynamics, on the other hand, they are used to find the fluid outflow velocity through the opening in the wall or in the bottom of a vessel.

1. Equations of the two-velocity hydrodynamics with pressure phases equilibrium

In papers [7] and [8] there is constructed a nonlinear two-velocity model for the motion of a fluid saturated through a deformable porous medium based on conservation laws, invariance of equations with respect to the Galilean transformations and the thermodynamic consistency condition. The two-velocity two-fluid hydrodynamic theory with the pressure equilibrium condition in the subsystems was constructed [6]. In the isothermal case the equations of motion of a two-velocity medium in the non-dissipative case with equilibrium of pressure phases in the system have the form [9–11]:

$$\frac{\partial \bar\rho}{\partial t} + \operatorname{div}(\tilde\rho\tilde{\boldsymbol v} + \rho\boldsymbol v) = 0, \qquad \frac{\partial \tilde\rho}{\partial t} + \operatorname{div}(\tilde\rho\tilde{\boldsymbol v}) = 0, \tag{1}$$

$$\bar\rho\left(\frac{\partial \boldsymbol v}{\partial t} + (\boldsymbol v, \nabla)\boldsymbol v\right) = -\nabla p + \frac{\tilde\rho}{2}\nabla(\tilde{\boldsymbol v} - \boldsymbol v)^2 + \bar\rho\boldsymbol f, \tag{2}$$

$$\bar\rho\left(\frac{\partial \tilde{\boldsymbol v}}{\partial t} + (\tilde{\boldsymbol v}, \nabla)\tilde{\boldsymbol v}\right) = -\nabla p - \frac{\rho}{2}\nabla(\tilde{\boldsymbol v} - \boldsymbol v)^2 + \bar\rho\boldsymbol f, \tag{3}$$

where $\tilde{\boldsymbol v}$ and $\boldsymbol v$ are the velocity vectors of the subsystems of the two-velocity continuum with the corresponding partial densities $\tilde\rho$ and ρ, $\bar\rho = \tilde\rho + \rho$ is the total density of the two-velocity continuum, $p = p(\bar\rho, (\tilde{\boldsymbol v} - \boldsymbol v)^2)$ is the equation of state of the two-velocity continuum, and $\boldsymbol f$ is the mass force vector per unit mass.

In the absence of the mass forces $\boldsymbol f = 0$, the system of equations (1)–(3) has the solution $\boldsymbol v = 0$, $\tilde{\boldsymbol v} = 0$, $\rho = \rho^0$, $\tilde\rho = \tilde\rho^0$, $p = p^0$ for a mixture of fluids at rest with the uniform pressure $p = p^0$, the partial densities ρ^0, $\tilde\rho^0$, and the temperature T.

We rewrite equations (2) and (3) in the equivalent form

$$\bar\rho\left(\frac{\partial \boldsymbol v}{\partial t} + \frac{1}{2}\nabla(v^2) - \boldsymbol v \times \operatorname{rot}\boldsymbol v\right) = -\nabla p + \frac{\tilde\rho}{2}\nabla(\tilde{\boldsymbol v} - \boldsymbol v)^2 + \bar\rho\boldsymbol f, \tag{4}$$

$$\bar\rho\left(\frac{\partial \tilde{\boldsymbol v}}{\partial t} + \frac{1}{2}\nabla(\tilde v^2) - \tilde{\boldsymbol v} \times \operatorname{rot}\tilde{\boldsymbol v}\right) = -\nabla p - \frac{\rho}{2}\nabla(\tilde{\boldsymbol v} - \boldsymbol v)^2 + \bar\rho\boldsymbol f, \tag{5}$$

From these equations, it is possible to derive other equations that determine the variation of the vortices in the course of time. For this, we apply the operator rot to both parts of equations (4), (5). As a result, we obtain

$$\frac{\partial \mathbf{\Omega}}{\partial t} - \operatorname{rot}(\boldsymbol{v} \times \mathbf{\Omega}) = -\operatorname{rot}\Big(\frac{\nabla p}{\bar\rho}\Big) + \operatorname{rot}\Big(\frac{\tilde\rho}{2\bar\rho}\nabla(\tilde{\boldsymbol{v}} - \boldsymbol{v})^2\Big) + \operatorname{rot}\boldsymbol{f},$$
$$\frac{\partial \tilde{\mathbf{\Omega}}}{\partial t} - \operatorname{rot}(\tilde{\boldsymbol{v}} \times \tilde{\mathbf{\Omega}}) = -\operatorname{rot}\Big(\frac{\nabla p}{\bar\rho}\Big) - \operatorname{rot}\Big(\frac{\rho}{2\bar\rho}\nabla(\tilde{\boldsymbol{v}} - \boldsymbol{v})^2\Big) + \operatorname{rot}\boldsymbol{f}.$$

Here $\mathbf{\Omega} = \boldsymbol{v}$, $\tilde{\mathbf{\Omega}} = \tilde{\boldsymbol{v}}$.

2. The Bernoulli equations for an incompressible two-fluid medium at pressure phases equilibrium with constant saturation of phases

In the case of homogeneous incompressible media, that is, under the conditions $\rho^f = \text{const}$ and $\tilde\rho^f = \text{const}$, where ρ^f, $\tilde\rho^f$ are the physical densities of the phases and constant bulk saturation of the substances composing the two-phase continuum $\Rightarrow \rho = \text{const}$, $\tilde\rho = \text{const} \Rightarrow$

$$\operatorname{div}\boldsymbol{v} = 0, \qquad \operatorname{div}\tilde{\boldsymbol{v}} = 0.$$

In other words, the vectors $\boldsymbol{v}$ and $\tilde{\boldsymbol{v}}$ are solenoidal [12].

In the stationary case, the system of equations (4), (5) takes the form

$$\boldsymbol{v} \times \operatorname{rot}\boldsymbol{v} = \nabla\Big(\frac{1}{2}\,v^2 + \frac{1}{\bar\rho}\,p - \frac{\tilde\rho}{2\bar\rho}\,(\tilde{\boldsymbol{v}} - \boldsymbol{v})^2\Big) - \boldsymbol{f}, \tag{6}$$
$$\tilde{\boldsymbol{v}} \times \operatorname{rot}\tilde{\boldsymbol{v}} = \nabla\Big(\frac{1}{2}\,\tilde v^2 + \frac{1}{\bar\rho}\,p + \frac{\rho}{2\bar\rho}\,(\tilde{\boldsymbol{v}} - \boldsymbol{v})^2\Big) - \boldsymbol{f}. \tag{7}$$

From these relations for the potential mass force with the potential F, multiplying, respectively, (6) by $\boldsymbol{v}$ and (7) by $\tilde{\boldsymbol{v}}$, we obtain

$$\Big(\boldsymbol{v}, \nabla\Big(\frac{1}{2}\,v^2 + \frac{1}{\bar\rho}\,p - \frac{\tilde\rho}{2\bar\rho}\,(\tilde{\boldsymbol{v}} - \boldsymbol{v})^2 + F\Big)\Big) = 0,$$
$$\Big(\tilde{\boldsymbol{v}}, \nabla\Big(\frac{1}{2}\,\tilde v^2 + \frac{1}{\bar\rho}\,p + \frac{\rho}{2\bar\rho}\,(\tilde{\boldsymbol{v}} - \boldsymbol{v})^2 + F\Big)\Big) = 0.$$

Hence, we obtain the Bernoulli equations

$$\frac{1}{2}\,v^2 + \frac{p}{\bar\rho} - \frac{\tilde\rho}{2\bar\rho}\,(\tilde{\boldsymbol{v}} - \boldsymbol{v})^2 + F = C, \tag{8}$$
$$\frac{1}{2}\,\tilde v^2 + \frac{p}{\bar\rho} + \frac{\rho}{2\bar\rho}\,(\tilde{\boldsymbol{v}} - \boldsymbol{v})^2 + F = \tilde C, \tag{9}$$

where C, $\tilde{C}$ are arbitrary constants.

Equations (8) and (9) admit a passage to the limit to the Bernoulli equation in the single-velocity hydrodynamics of incompressible media when the velocity of the fluid and the physical densities coincide.

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