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Theorem of the mean for inhomogeneous poroelastic static system

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Abstract. Relations of the mean for a vector of displacement of an elastic porous body and pore pressure for an inhomogeneous poroelastic static system, when mass forces are present and energy dissipation is absent, are obtained.

1. Introduction

Direct and inverse theorems about the mean for the mathematical physics equations represent not only theoretical [1], but a practical interest as well. Also, relations of the mean are most useful in computational mathematics as they give an effective method of constructing difference schemes. In Monte Carlo methods, the mean value theorems play a special role as they are basic for constructing algorithms of a random walk by spheres [2, 3]. For many basic equations and system equations such theorems have been proved (see [1, 2]). In [4, 5], a relation of the mean for the inhomogeneous system of the Lame equations is obtained.

The simulation of two-phase flows in heterogenous porous media is widely used in oil production. For example, the simulation of a reservoir is intended for reconstructing a geological history of a sedimentary basin and, in particular, of dislocation of a hydrocarbon component on the geological time scale. The simulation of a reservoir deals with understanding and prediction of the fluid flows occurring in the oil production processes. On the other hand, the simulation of two-phase flows in porous media plays an important role for the prediction of earthquakes preparation as it is an energy intensive process.

In the given paper, using the method proposed in [4, 5], the relations of the mean for inhomogeneous equations of a poroelastic static system are obtained. Namely, the relations of the mean for a vector of displacement of an elastic porous body and pore pressure are obtained. The knowledge of these values is sufficient, on the one hand, for the evaluation of reservoirs in oil production and, on the other hand, for the definition of a dilatancy area in the earthquakes prediction problems.

2. Area of dilatancy

The interaction between regional and local tectonic forces in seismic-prone zones can lead to the appearance of areas of a high concentration of tectonic stresses. After a certain time, destruction of the medium resulting in an earthquake takes place in some of these areas. Although the earthquake preparation process lasts sufficiently long (several years), it is an energy intensive process. A considerable rheological change in the medium takes place, and anomalous zones of geophysical fields of various nature are formed. Cracks opening in the zones with increased values of shearing and tensile stresses is the most universal mechanism of developing changes in a medium. Such zones are formed in the vicinity of the sources of future earthquakes if in this case the spatial distribution of forces is non-uniform. The majority of seismologists consider that the initial stage of opening cracks and the subsequent state of the medium, when the destruction process is developing, are associated with the dilatancy of the medium [6, 7].

Dilatancy is a nonlinear loosening of a medium due to formation of cracks due to a shear. This takes place when tangential stresses exceed a certain threshold. A dilatancy area is considered to incorporate a set of elastic porous medium points, for which at a given stress field $\{\sigma_{ij}\}$ the following condition is fulfilled:

$$D_\tau \equiv \tau - \alpha(P + \rho g z) - Y \geq 0, \tag{1}$$

where ρ is the density of rocks, g is the acceleration of gravity, z is the depth of a point, P is the hydrodynamic pressure, α is the internal friction coefficient, Y is the cohesion of rocks, τ is the intensity of tangential stresses

$$\tau = \frac{\sqrt{3}}{2}\Big[(\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{11} - \sigma_{33})^2 + 6\,(\sigma_{12}^2 + \sigma_{13}^2 + \sigma_{23}^2)\Big]^{1/2}.$$

Condition (1) coincides with the Schleicher–Nadai criterion of material destruction under the action of shearing loads. It satisfactorily describes the beginning of the rock destruction process. It can also be used at the "pre-destruction" stage (when loading constitutes up to 60–90 % of the critical value) for the qualitative description of the shape of areas with intensification of cracks opening.

3. Statement of the problem

Let us assume that the bounded domain $\widetilde{\Omega} \subset R^3$ is filled with a homogeneous isotropic elastic porous medium. The elastic porous static state of the medium $\widetilde{\Omega}$ in the presence of mass sources and in the absence of dissipation of energy is described by the system of the differential equations [8–10]:

$$\frac{\rho_{0,s}}{\rho_0}\frac{\partial P}{\partial x_i}+\sum_{k=1}^{3}\frac{\partial \bar{h}_{ik}}{\partial x_k}=\rho_{0,s}f_i,\quad \frac{\rho_{0,l}}{\rho_0}\frac{\partial P}{\partial x_i}=\rho_{0,l}f_i,\quad i=1,2,3. \tag{2}$$

Here $\bar{h}_{ik}$ is a stress tensor, P is the pore pressure, $\rho_0=\rho_{0,l}+\rho_{0,s}$, $\rho_{0,l}$ and $\rho_{0,s}$ are partial densities of fluid and an elastic porous body, respectively, $\boldsymbol{f}=(f_1,f_2,f_3)$ is the mass force. The total stress tensor of the elastic porous body looks like

$$\sigma_{ik}=-\bar{h}_{ik}-P\delta_{ik}, \tag{3}$$

$$\bar{h}_{ik}=-\mu\Big(\frac{\partial U_i}{\partial x_k}+\frac{\partial U_k}{\partial x_i}\Big)-\Big(\lambda-\frac{\rho_{0,s}}{\rho_0}K\Big)\delta_{ik}\operatorname{div}\boldsymbol{U}+\frac{\rho_{0,l}}{\rho_0}K\delta_{ik}\operatorname{div}\boldsymbol{V}, \tag{4}$$

$$P=\big(K-(\rho_0^2\hat{\alpha}+K/\rho_0)\rho_{0,s}\big)\operatorname{div}\boldsymbol{U}-(\rho_0^2\hat{\alpha}+K/\rho_0)\rho_{0,l}\operatorname{div}\boldsymbol{V}, \tag{5}$$

where δ_{ik} is the Kronecker delta, $\boldsymbol{U}=(U_1,U_2,U_3)$ is the vector of displacement of the elastic porous body, $\boldsymbol{V}=(V_1,V_2,V_3)$ is the vector of displacement of fluid, $\operatorname{div}\boldsymbol{U}=U_{1,1}+U_{2,2}+U_{3,3}$, $U_{i,j}=\dfrac{\partial U_i}{\partial x_j}$, $K=\lambda+\dfrac{2}{3}\mu$, $\hat{\alpha}$, λ, μ are the constants from the equation of state [9].

4. A system of differential equations about the vector of displacement of an elastic porous body and pore pressure

Let us exclude from system (2) the thermodynamic degrees of freedom, using their expressions (4) and (5). Then, using the second equation of system (2), we exclude a displacement of fluid from the first equation of system (2). Further, we act with a divergence operator on both sides of the second equation of system (2). As a result, we will obtain an inhomogeneous system of second order differential equations of the vector of displacement of the elastic porous body $\boldsymbol{U}$ and the pore pressure P:

$$L\boldsymbol{U}=-\frac{K}{\hat{\alpha}+K}\rho_0\boldsymbol{f},\qquad \Delta P=F. \tag{6}$$

In system (6), $F=\rho_0\operatorname{div}\boldsymbol{f}$, $L\boldsymbol{U}=\mu\Delta\boldsymbol{U}+\tilde{\lambda}\nabla\operatorname{div}\boldsymbol{U}$, $\tilde{\lambda}=\lambda-\frac{K}{\hat{\alpha}+K}K$.

Thus, system (2) was reduced to two independent equations concerning the vector of displacement of the elastic porous body $\boldsymbol{U}$ and the pore pressure P. Theorems of the mean for these equations are proved in [1, 4, 5, 11]. From the results of these papers follow:

Theorem 1. *Let $\Omega\subset R^3$ be a bounded domain, restricted by a smooth enough surface $\partial\Omega$, $\boldsymbol{U}\in C^2(\Omega)\cap C^1(\overline{\Omega})$, $L\boldsymbol{U}$ being absolutely integrable in Ω. Then the following representation is valid*:

$$U_k(\boldsymbol{r}) = \int_{\partial\Omega} \big\{\Gamma_k(\boldsymbol{r}-\boldsymbol{q})\cdot T_n[\boldsymbol{U}(\boldsymbol{q})] - \boldsymbol{U}(\boldsymbol{q})\cdot T_n[\Gamma_k(\boldsymbol{r}-\boldsymbol{q})]\big\}\, dS_q - \int_{\Omega} \Gamma_k(\boldsymbol{r}-\boldsymbol{q})\cdot L\boldsymbol{U}(\boldsymbol{q})\, dV_q, \quad k=1,2,3, \quad \forall \boldsymbol{r}\in\Omega, \tag{7}$$

where

$$\Gamma_k(\boldsymbol{r}) = \sum_{i=1}^{3} U_{ik}\mathrm{e}_i = \frac{1}{16\pi\mu(1-\nu)}\Big[(3-4\nu)\frac{e_k}{r} + \frac{\boldsymbol{r}}{r^3}x_k\Big],$$

$$T_n(\boldsymbol{U}) = \sum_{i,j=1}^{3} \tilde{\sigma}_{ij} n_i \mathrm{e}_j = 2\mu\frac{\partial \boldsymbol{U}}{\partial n} + \tilde{\lambda}\boldsymbol{n}\,\mathrm{div}\,\boldsymbol{U} + \mu\boldsymbol{n}\times(\nabla\times\boldsymbol{U}).$$

Here $\tilde{\sigma}_{ij} = \tilde{\lambda}\,\mathrm{div}\,\boldsymbol{U}\delta_{ij} + \mu(U_{i,j}+U_{j,i})$ *is a stress tensor of the elastic medium and* $\nu = \dfrac{\tilde{\lambda}}{2(\tilde{\lambda}+\mu)}$.

Lemma 1. *For an arbitrary function* $\boldsymbol{U}\in C^2(U_R)\cap C^1(\overline{U}_R)$, *the equality*

$$\int_{U_R} \boldsymbol{U}(\boldsymbol{q}) dV_q = \frac{1-2\nu}{4(2-3\nu)}\left\{\frac{1}{\mu}\int_{U_R}(q^2-R^2)L\boldsymbol{U}(\boldsymbol{q})\,dV_q + 2R\int_{\partial U_R}[\boldsymbol{U}(\boldsymbol{q}) + \frac{\boldsymbol{q}(\boldsymbol{q}\cdot\boldsymbol{U})}{R^2}]\,dS_q\right\}$$

is valid.

Theorem 2 (about mean). *Let* Ω *be an arbitrary domain,* $\boldsymbol{U}\in C^3(\Omega)$, $P\in C^2(\Omega)$ *are a solution of system* (6). *Then for any ball* $U_R(\boldsymbol{r})\subseteq\Omega$ *the following equalities are valid*:

$$\begin{aligned}\boldsymbol{U}(\boldsymbol{r}) = {} & \frac{3}{16\pi R^2(2-3\nu)}\int_{\partial U_R(\boldsymbol{r})}\left\{(1-4\nu)\boldsymbol{U}(\boldsymbol{q}) + 5\frac{\boldsymbol{p}(\boldsymbol{p}\cdot\boldsymbol{U}(\boldsymbol{q}))}{R^2}\right\}dS_q - \\ & \frac{\rho_0 K}{\rho_0^3\hat{\alpha}+K}\frac{1}{16\pi\mu(1-\nu)}\int_{U_R(\boldsymbol{r})}\left\{(3-4\nu)\Big(\frac{1}{R}-\frac{1}{p}\Big)\boldsymbol{f}(\boldsymbol{q}) + \Big(\frac{1}{R^3}-\frac{1}{p^3}\Big)\boldsymbol{p}(\boldsymbol{p}\cdot\boldsymbol{f}(\boldsymbol{q}))\right\}dV_q - \\ & \frac{\rho_0 K}{\rho_0^3\hat{\alpha}+K}\frac{3-2\nu}{32\pi R^3\mu(1-\nu)(2-3\nu)}\int_{U_R(\boldsymbol{r})}(R^2-p^2)\boldsymbol{f}(\boldsymbol{q})\,dV_q,\end{aligned} \tag{8}$$

$$P(\boldsymbol{r}) = \frac{1}{4\pi R^2}\int_{\partial U_R(\boldsymbol{r})} P(\boldsymbol{q})\,dS_q - \frac{\rho_0}{4\pi}\int_{U_R(\boldsymbol{r})}\Big(\frac{1}{R}-\frac{1}{p}\Big)\,\mathrm{div}\,\boldsymbol{f}(\boldsymbol{q})\,dV_q, \tag{9}$$

where $\boldsymbol{p} = \boldsymbol{q}-\boldsymbol{r}$.

Theorem 3 (inverse theorem about mean). *Let $\Omega \subset R^3$ be an arbitrary domain $\boldsymbol{U} \in C^3(\Omega)$, $P \in C^2(\Omega)$, $\boldsymbol{f} \in C^1(\Omega)$ and for any ball $U_R(\boldsymbol{r}) \subseteq \Omega$ these functions satisfy relations of the mean* (8), (9). *Then the functions $\boldsymbol{U}$ and P are solutions of system* (6).

Proof. Following [13], it is sufficient to prove the statement of the theorem for $\boldsymbol{U}$ in the case $\boldsymbol{r} = 0$. The proof for the pore pressure P will be carried out in a similar manner. Clearly, in the ball $\overline{U}_R \subseteq \Omega$ for $\boldsymbol{U} \in C^3(\overline{U}_R)$ representation (7) is valid:

$$U_k(0) = \int_{\partial U_R} \big\{ \Gamma_k(\boldsymbol{q}) \cdot T_n[\boldsymbol{U}(\boldsymbol{q})] - \boldsymbol{U}(\boldsymbol{q}) \cdot T_n[\Gamma_k(\boldsymbol{q})] \big\}\, dS_q - \int_{U_R} \Gamma_k(\boldsymbol{q}) \cdot L\boldsymbol{U}(\boldsymbol{q})\, dV_q, \quad k = 1, 2, 3. \tag{10}$$

According to (10) with the use of the above lemma, similarly to [7, 13] we obtain

$$U_k(0) = \frac{3}{16\pi R^2(2-3\nu)} \int_{\partial U_R} \{(1-4\nu)U_k + 5\frac{x_k(\boldsymbol{q} \cdot \boldsymbol{U}(\boldsymbol{q}))}{R^2}\}\, dS_q + \int_{U_R} \Big\{ \big(\hat{\Gamma}_k(\boldsymbol{q}) - \Gamma_k(\boldsymbol{q})\big) L\boldsymbol{U}(\boldsymbol{q}) + \frac{(3-2\nu)(R^2-p^2)}{32\pi R^3 \mu(1-\nu)(2-3\nu)} LU_k \Big\}\, dV_q, \tag{11}$$

where

$$\hat{\Gamma}_k(\boldsymbol{r}) = \frac{1}{16\pi\mu(1-\nu)} \Big[(3-4\nu)\frac{e_k}{R} + \frac{\boldsymbol{r}}{R^3} x_k \Big].$$

Substituting into (11) relation about the mean (8), we obtain

$$\int_{U_R} \Big\{ \big(\hat{\Gamma}_k(\boldsymbol{q}) - \Gamma_k(\boldsymbol{q})\big)(L\boldsymbol{U}(\boldsymbol{q}) - \boldsymbol{g}(\boldsymbol{q})) + \frac{(3-2\nu)(R^2-p^2)}{32\pi R^3 \mu(1-\nu)(2-3\nu)} (LU_k - g_k) \Big\} dV_q = 0, \tag{12}$$

where

$$g_k = -\frac{K}{\rho_0^3 \hat{\alpha} + K} \rho_0 f_k, \quad k = 1, 2, 3.$$

Using the definition of the functions Γ_k and $\hat{\Gamma}_k$ we obtain from (12)

$$\int_{U_R}\Big\{\Big(\frac{x_k}{R^3}-\frac{x_k}{r^3}\Big)\big(\boldsymbol{r}\cdot(L\boldsymbol{U}(\boldsymbol{q})-\boldsymbol{g}(\boldsymbol{q}))\big)+$$
$$\Big((3-4\nu)\Big(\frac{1}{R}-\frac{1}{r}\Big)+\frac{3-2\nu}{2(2-3\nu)}\Big(\frac{1}{R}-\frac{r^2}{R^3}\Big)\Big)(LU_k-g_k)\Big\}\,dV_q=0. \quad (13)$$

Equality (13) is valid for all sufficiently small R, then having divided it into R^2 and passing to a limit at $R\to 0$, with allowance for the equalities [13]

$$\frac{1}{R^2}\int_{U_R}\Big(\frac{1}{r}-\frac{1}{R}\Big)\boldsymbol{g}(\boldsymbol{r})dV_r\to\frac{2\pi}{3}\boldsymbol{g}(0),$$
$$\frac{1}{R^2}\int_{U_R}\Big(\frac{1}{R}-\frac{r^2}{R^3}\Big)\boldsymbol{g}(\boldsymbol{r})dV_r\to\frac{8\pi}{15}\boldsymbol{g}(0),$$
$$\frac{1}{R^2}\int_{U_R}\Big(\frac{1}{r^3}-\frac{1}{R^3}\Big)x_k(\boldsymbol{r}\cdot\boldsymbol{g}(\boldsymbol{r}))dV_r\to\frac{2\pi}{5}g_k(0),$$

we find that

$$C[L\boldsymbol{U}_k(0)-\boldsymbol{g}_k(0)]=0,$$

where $C=\frac{2(1-\nu)(2\nu-1)}{2-3\nu}$ for $\nu<1/2$, hence

$$L\boldsymbol{U}(0)-\boldsymbol{g}(0)=0.$$

The theorem is proved. □

Given the fields of the stresses and the pore pressure, it is possible to define a dilatancy area in a porous medium according to formula (1).

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