Bull. Nov. Comp. Center, Math. Model. in Geoph., 26 (2024), 1–7

On an inverse problem arising in the theory of propagation of nonlinear waves

B.Kh. Imomnazarov, Kh.Kh. Imomnazarov, A.Kh. Mukimov

Abstract. In this paper, we consider a one-dimensional inverse source problem for the Hopf equation. We show that it is uniquely solvable in the class of finite smoothness.

Introduction

The study of wave propagation for media in which the proper pressure difference at a given time can be neglected (briefly, pressureless media) is of both mathematical and applied interest. A direct model of such media is the equations of gas dynamics in which the pressure P is formally set equal to zero. From the point of view of applications, pressureless media arise in the description of various physical phenomena, such as the evolution of multiphase flows, the movement of dispersed media, in particular, dust particles or droplets, the phenomenon of cumulation, the movement of granular media, etc. Examples of various gas-dynamic problems using pressureless media can be found, for example, in the classical monographs [1–3].

In this paper, we consider a one-dimensional inverse source problem for the Hopf equation. We show that it is uniquely solvable in the class of finite smoothness.

1. Inverse source problem for the Hopf equation

Let us consider a one-dimensional dynamic inverse problem of determining the function $u(t,x)$, $g(t)$, if the inhomogeneous Hopf equation and the following conditions are satisfied:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = f(x)g(t), \quad t>0, \;\; x \in R, \tag{1}$$

$$u|_{t=0} = u_0(x), \quad x \in R, \tag{2}$$

$$u|_{x=0} = \varphi(t), \quad t>0. \tag{3}$$

The function $f(x)$ is given. Without loss of generality, we can assume that $f(0) = 1$. Let us assume that the agreement condition is satisfied

$$\varphi(0) = u_0(0). \tag{4}$$

Using (3), we eliminate the unknown function $g(t)$ from (1) and obtain the Cauchy problem for the loaded equation. For this purpose, we formally differentiate with respect to x both parts of (1) and introduce the notation $v = u_x$. Thus, we arrive at the following problem for the loaded equation [4]

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v^2 = F(t,x) + G(t,x)v(t,0), \tag{5}$$

$$v(0,x) = u_0'(x), \tag{6}$$

$$\frac{\partial u}{\partial x} = v, \tag{7}$$

$$u(t,0) = \varphi(t), \tag{8}$$

where $F(t,x) = f'(x)\varphi'(t)$ and $G(t,x) = f'(x)\varphi(t)$.

2. Existence theorem

Theorem 1. *Let the functions $u_0(x), f(x) \in C^1[-1,1]$, $\varphi(t) \in C^1[0,1]$ and $f(0) = 1$. Then a solution to problem* (1)–(4) *exists and*

$$u(t,x), u_t(t,x), u_x(t,x) \in L_\infty(\Omega), \quad g(t) \in L_\infty(0,t_0).$$

Proof. First, we prove the solvability of problem (5)–(8). To do this, we use the fixed point method. First, we assume that the input data $f(x)$, $u_0(x)$ and $\varphi(t)$ are sufficiently smooth functions.

Fixed point method. Let $z(t,x)$, $h(t)$ be some functions. Instead of system (5), (6) we consider the system

$$\frac{\partial v}{\partial t} + z\frac{\partial v}{\partial x} + v^2 = F(t,x) + G(t,x)h(t), \tag{9}$$

$$v(0,x) = u_0'(x), \tag{10}$$

And then another problem

$$\frac{\partial u}{\partial x} = v, \tag{11}$$

$$u(t,0) = \varphi(t). \tag{12}$$

The sets $Z = \{(z,h)\}$ and $\Omega \subset R^2$ will depend on some positive parameters M, V, δ, τ, which will be specified later.

Let us put it this way

$$M = 1 + 2\Big(\sup_{|x|<1} |u_0(x)| + \sup_{0<t<1} |\varphi(t)|\Big), \quad V = 1 + 2\sup_{|x|<1} |u_0'(x)|.$$

Next, for some quantities $0 < \delta < 1$ and $0 < \tau < \delta/M$ we assume

$$\Omega = \{(t,x) : \ 0 < t < \tau, \ |x| < \delta - tM\}.$$

The domain Ω is a trapezoid with base $(-\delta, \delta)$ and height τ. Further, $Z \subset C(\overline{\Omega}) \times C[0,\tau]$ is the set of such pairs $(z(t,x), h(t))$ that

$$\|z\|_{C(\overline{\Omega})} \le M, \quad \|z_x\|_{C(\overline{\Omega})} \le V,$$

$$z(t,0) = \varphi(t), \quad \|h\|_{C[0,\tau]} \le V.$$

Let us consider system (9)–(12) with coefficients $(z(t,x), h(t)) \in Z$. Let us put

$$F_0 = \sup_{|x|<1,\, 0<t<1} |F(t,x)|, \quad G_0 = \sup_{|x|<1,\, 0<t<1} |G(t,x)|.$$

The problem for $v(t,x)$ is equivalent to a nonlinear system

$$\frac{\partial v}{\partial t}(t,y) + v^2(t,y) = F(t,y) + G(t,y)h(t), \quad \frac{\partial y}{\partial t}(t,x) = z(t,y), \tag{13}$$

$$v(0,x) = u_0'(x), \quad y(0,x) = x. \tag{14}$$

Locally in time, this system is solvable and, for small t, the estimate holds

$$\Big|\frac{\partial v}{\partial t}\Big| \le v^2 + F_0 + G_0 V.$$

Since $|u_0'(x)| < V/2$, then on some interval $(0, t_0)$ the estimate $|v(t,x)| < V$ holds, which means

$$|v(t,x)| \le V/2 + \tau(V^2 + F_0 + G_0 V).$$

We choose τ so small that $\tau(V^2 + F_0 + G_0 V) < V/2$. Then $\tau \le t_0$. This means that system (13)–(14) is solvable in the domain $(0,\tau) \times (-\delta, \delta)$ and $|v(t,x)| \le V$. Further, the following estimate holds:

$$|y(t,x) - x| \le Mt.$$

This means that $y(t,-\delta) \le -\delta + Mt$, $y(t,\delta) \ge \delta - Mt$. By the theorem on the continuous dependence of the solution on the parameter, we conclude that the values $y(t,x)$ cover the interval $(-\delta + Mt, \delta - Mt)$ when x runs through the interval $(-\delta, \delta)$. In addition, differentiability with respect to the parameter takes place, which means that this covering is univalent. Consequently, system (9)–(12) is solvable in the domain Ω. In addition, the estimate holds

$$|u(t,x)| \le |\varphi(t)| + \delta V \le M/2 + \delta V.$$

We define the operator L as follows: for given pairs $(z(t,x), h(t))$ from Z, we solve problem (9)–(12). Using the functions u and v found, we determine the functions $z(t,x) = u(t,x)$ and $h(t) = v(t,0)$.

We choose the value δ so that $\delta V \le M/2$. Then inside Ω

$$|u(t,x)| \le M.$$

Thus, for the given choice of parameters δ, τ, system (9)–(12) is solvable in the domain Ω, the functions $v(t,x)$, $u(t,x)$, $u_x(t,x)$ are continuous and the inequalities are satisfied

$$\|u\|_{C(\overline{\Omega})} \le M, \quad \|u_x\|_{C(\overline{\Omega})} \le V,$$

$$u(t,0) = \varphi(t), \quad \|v(t,0)\|_{C[0,\tau]} \le V.$$

Therefore, the operator L is well defined and $L(Z) \subset Z$.

Now we will show that the operator L is completely continuous. To do this, we will show $|v_x(t,x)| \le N$ that for some constant N. It is at this point that additional smoothness of the input data is required. We will prove this statement later. For now, we assume that such an estimate has already been obtained. Then

$$|v_t(t,x)| \le MN + V^2 + F_0 + VG_0 = N_1.$$

Hence, $v(t,0)$ is Lipschitz with constant N_1. Further, $|u_{xx}(t,x)| = |v_x(t,x)| \le N$, and $|u_t(t,x)| \le |\varphi'(t)| + \delta N_1$. These estimates show that pairs $(u(t,x), v(t,0))$ have some excess smoothness, and therefore the set of such pairs is compactly embedded in Z. Consequently, by the Schauder theorem [5] the operator has a fixed point.

Thus, problem (5)–(8) is solvable, provided that the input functions are smooth. We will show that in this case the function $u(t,x)$ is a solution to problem (1)–(4). Indeed, from the construction of the solution it follows that the trace $v(t,0)$ is defined. Then

$$\frac{\partial}{\partial x}\Big(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - f(x)g(t)\Big) = 0, \quad g(t) = \varphi'(t) + \varphi(t)v(t,0).$$

Hence,

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - f(x)g(t) = H(t)$$

with some function $H(t)$. Putting in the last equality $x = 0$, we obtain $H(t) \equiv 0$. So, the function $u(t,x)$ is a solution to problem (1)–(4).

Now we can get rid of the assumption about the increased smoothness of the input data. We only needed it to evaluate $v_x(t,x)$ and prove the **compatness** of the operator L.

To do this, we note that the following estimates hold:

$$\|u\|_{C(\overline{\Omega})} \le M, \quad \|u_x\|_{C(\overline{\Omega})} \le V, \quad \|u_x(t,0)\|_{C[0,\tau]} \le V.$$

In addition, from equation (1) we easily obtain

$$g(t) = \varphi'(t) + \varphi(t)v(t,0),$$

$$|u_t| \le MV + |f(x)|(|\varphi'(t)| + |\varphi(t)|V).$$

Moreover, the quantities M, V and the domain Ω depend only on the norms of the functions $u_0(x)$, $u_0'(x)$, $\varphi(t)$, $\varphi'(t)$, $f(x)$.

Consider a sequence of smooth functions $u_0^\tau(x)$, $\varphi^\tau(t)$, $f^\tau(x)$. For them we solve problem (5)–(8) and obtain a solution to problem (1)–(4) $\{u^\tau\}$, $\{g^\tau\}$. In this case $u^\tau(t,x), u_t^\tau(t,x), u_x^\tau(t,x) \in L_\infty(\Omega)$, $g^\tau(t) \in L_\infty(0,t_0)$.

Since $(L_1(\Omega))' = L_\infty(\Omega)$, by the Banach-Alaoglu theorem [6] the functions u_t^τ, u_x^τ, g^τ *-weakly converge to the corresponding functions u_t, u_x, g. And the functions u^τ converge strongly in $C(\Omega)$ to u. After this, it only remains to pass to the limit in system (1)–(4).

Estimate for $v_x(t,x)$. So, let the input data be smooth. Let us differentiate equation (9) with respect to x and denote by $w(t,x) = v_x(t,x)$:

$$\frac{\partial w}{\partial t} + z\frac{\partial w}{\partial x} + z_x w + 2vw = F_x(t,x) + G_x(t,x)h(t). \tag{15}$$

There is an inequality on the characteristic

$$\left|\frac{\partial w}{\partial t}\right| \le 3V|w| + F_1 + G_1 V.$$

Hence (by Gronwall's lemma) when $t < \tau$

$$|w(t,x)| \le ((F_1 + G_1 V)\tau + |w(0,x)|)e^{3V\tau} \le \tilde{N}.$$

The value $\tilde{N}$ depends on some additional derivatives of the input data, but this is not important. The main thing is that such a value exists. □

Corollary. Let the conditions of Theorem 1 be satisfied. Then the solution to problem (1)–(4) $u(t,x)$ is Lipschetz.

3. Uniqueness theorem

We will prove the uniqueness of the solution to problem (1)–(4) under the following conditions:

$$f(x) \in C^1[-1,1], \quad g(t) \in L_1(0,t_0). \tag{16}$$

Theorem 2. *Let the function* $u(t,x)$ *be continuously differentiable and* $f(x)$, $g(t)$ *satisfy condition* (16) *and* $f(0) = 1$. *Then the solution to problem* (1)–(4) *is unique.*

Proof. Let us have two solutions $u_1(t,x)$, $u_2(t,x)$. It is clear that the parameters t_0 and V are different for them, but we will take the smaller of t_0 and the larger of V.

Let us denote

$$w = u_1 - u_2, \quad s = (u_1 + u_2)/2, \quad h = g_1 - g_2.$$

In this case $|s|, |s_x| \le V$. As before, from equation (1) we easily obtain an equation on the characteristics

$$\frac{\partial w}{\partial t} + s_x(t, x(t))w = f(x(t))h(t), \tag{17}$$

$$\frac{\partial x}{\partial t} = s(t, x(t)), \tag{18}$$

$$w(0) = 0, \quad x(0) = x_0. \tag{19}$$

We obtain the Volterra equation of the first kind for the function $h(t)$. Let $z > 0$. Consider the characteristic passing through the point $(t,x) = (z,0)$. At this point $u_1(z,0) = u_2(z,0) = \varphi(z)$, and therefore we have the problem (17)–(18) with initial conditions

$$w(z) = 0, \quad x(z) = 0.$$

Besides this, obviously, $w(0) = 0$ (initial data).

In fact, it would be worth writing $w(t,z)$, $x(t,z)$, but we do not do this so as not to clutter the presentation. We have a linear equation with respect to w. Let us introduce the function

$$A(t,z) = \int_0^t s_x(\tau, x(\tau, z))\, d\tau.$$

Let us multiply both sides of (17) by $e^{A(t,z)}$, integrate over t from 0 to z and get

$$0 = \int_0^z e^{A(t,z)} f(x(t,z))h(t)\, dt.$$

This is the desired Volterra equation of the first kind, with kernel $K(t,z) = e^{A(t,z)} f(x(t,z))$. On the diagonal we have

$$e^{-Vz} < K(z,z) < e^{Vz}.$$

Differentiating the kernel $K(t,z)$ with respect to the variable t, we obtain

$$K_t(t,z) = e^{A(t,z)} s_x(t, x(t,z)) f(x(t,z)) + e^{A(t,z)} f'(x(t,z)) s(t, x(t,z)).$$

Next we introduce the function

$$H(t) = \int_0^t h(\tau)\, d\tau.$$

And after this we arrive at the homogeneous Volterra equation of the second kind

$$K(z,z)H(z) - \int_0^z K_t(t,z)H(t)\, dt = 0.$$

It follows that $H(t) \equiv 0$, and therefore $w \equiv 0$. □

The authors express their sincere gratitude to A.N. Artyushin for useful advice.

References

[1] Chernyi G.G. Introduction to Hypersonic Flow. — New York–London: Academic Press, 1961.

[2] Sedov L.I. Similarity and Dimensional Methods in Mechanics. — New York–London: Academic Press, 1959.

[3] Stanyukovich K.P. The Unsteady Motion of a Continuous Medium. — Moscow: Nauka, 1971 (In Russian).

[4] Kozhanov A.I. Nonlinear loaded equations and inverse problems Computational Mathematics and Mathematical Physics. — 2004. — Vol. 44, No. 4. — P. 722–744 (In Russian).

[5] Kantorovich L.V., Akilov G.P. Functional Analysis. — Moscow: Nauka, 1977 (In Russian).

[6] Dunford N., Schwartz J.T. Linear Operators: General Theory. — New York: Interscience, 1958.