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About one form of the dynamic equation of porous media in terms of velocities, stresses, and pressure

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Abstract. The form of the equation of motion of porous media in terms of velocities, stresses and pressure as a symmetric t-hyperbolic system has been obtained.

1. Introduction

The poroelasticity theory is widely used in geomechanics, biophysics and other areas of science and technology.

In 1989, V.N. Dorovsky [1] constructed a nonlinear mathematical model of fluid motion through an elastic deformable porous medium (the poroelasticity theory) on the basis of the laws of conservation, their invariance with respect to Galilean transform, and the quasi-linear equation of fluid motion consistent with thermodynamic equilibrium conditions. In other words, the structure of the equations is such that, with an arbitrary character of interaction of subsystems, the equation of motion is quasi-linear and common conservation laws are fulfilled when the basic thermodynamic identity is identically kept. The existence of four types of sound oscillations is shown—two transversal (in an isotropic medium their properties coincide) and two longitudinal. The key difference of a linearized Dorovsky model from well-known models of the Frenkel–Biot type [2, 3] is in that the Dorovsky model in the isotropic case is described by three elastic constants [4–6]. These elastic parameters are one-to-one expressed by three oscillations velocities [6]. This circumstance is important for the numerical modeling of elastic waves in a porous medium when distributions of velocities of acoustic waves and physical densities of the matrix and of saturating fluids as well as porosity are known.

The symmetric hyperbolic form of representing the model allows obtaining a simple proof of uniqueness of the continuous dependence of a solution to the Cauchy problem and boundary conditions with dissipative boundary conditions on initial data. It also gives a possibility to apply well developed numerical methods [7, 8].

In this paper, the form of a dynamic equation of porous media in terms of velocities, stresses and pressure as a symmetric t-hyperbolic system has been obtained.

2. About one form of Dorovsky's equations for porous media as symmetric t-hyperbolic system

The linearized system of the Dorovsky's equations is the following [4, 5]:

$$\begin{gathered}
\rho_s \frac{\partial u_i}{\partial t} + \partial_k h_{ik} + \frac{\rho_s}{\rho} \partial_i p = 0, \\
\rho_l \frac{\partial v_i}{\partial t} + \frac{\rho_l}{\rho} \partial_i p = 0, \\
\frac{\partial h_{ik}}{\partial t} + \mu \left(\partial_i u_k + \partial_k u_i \right) + \left(\lambda - \frac{\rho_s}{\rho} K \right) \delta_{ik} \operatorname{div} \boldsymbol{u} - \frac{\rho_l}{\rho} K \delta_{ik} \operatorname{div} \boldsymbol{v} = 0, \\
\frac{\partial p}{\partial t} - (K - \alpha \rho \rho_s) \operatorname{div} \boldsymbol{u} + \alpha \rho \rho_l \operatorname{div} \boldsymbol{v} = 0.
\end{gathered} \tag{1}$$

Here $\boldsymbol{u} = (u_1, u_2, u_3)$ and $\boldsymbol{v} = (v_1, v_2, v_3)$ are the velocity vectors of the solid matrix with the partial density $\rho_s = \rho_s^f (1 - d_0)$ and a liquid with partial density $\rho_l = \rho_l^f d_0$, respectively, d_0 is the porosity, p is the pore pressure, h_{ik} is the stress tensor, ρ_s^f and ρ_l^f are the physical densities of the solid matrix and fluid, respectively, $\lambda > 0$, $\mu > 0$ are the Lame constants, $\alpha = \rho \alpha_3 + K/\rho^2$, $K = \lambda + \frac{2}{3}\mu$, $\rho = \rho_l + \rho_s$, $\rho^3 \alpha_3 > 0$ is the module of volumetric compression of a fluid component of a heterophase medium, δ_{ik} is the Kronecker delta, $\partial_i = \frac{\partial}{\partial x_i}$.

Let us introduce the new unknowns of the function

$$\tilde{\sigma}_{ij} = -h_{ij} - \frac{\rho_s}{\rho} p \delta_{ij}. \tag{2}$$

After elimination of $\operatorname{div} \boldsymbol{u}$, system (1) will be rewritten in terms of $\boldsymbol{u}$, $\boldsymbol{v}$, $\tilde{\sigma}_{ik}$ and p as

$$\rho_s \frac{\partial u_i}{\partial t} - \partial_k \tilde{\sigma}_{ik} = 0, \qquad \rho_l \frac{\partial v_i}{\partial t} + \frac{\rho_l}{\rho} \partial_i p = 0, \tag{3}$$

$$\frac{1}{2\mu} \frac{\partial \tilde{\sigma}_{ik}}{\partial t} - \frac{\Lambda}{2\mu\Delta} \delta_{ik} \frac{\partial \tilde{\sigma}_{mm}}{\partial t} + \frac{\tilde{\alpha}}{\Delta} \delta_{ik} \frac{\partial p}{\partial t} - \frac{1}{2} \left(\partial_i u_k + \partial_k u_i \right) = 0, \tag{4}$$

$$\frac{\tilde{\alpha}}{\Delta} \frac{\partial \tilde{\sigma}_{mm}}{\partial t} + \frac{3\rho_s}{\rho} \frac{K \rho_l / \rho_s + \tilde{\alpha}}{\Delta} \frac{\partial p}{\partial t} + \frac{\rho_l}{\rho} \operatorname{div} \mathbf{v} = 0, \tag{5}$$

Here $\Lambda = \lambda \alpha \rho^2 - K^2$, $\tilde{\alpha} = \alpha \rho \rho_s - K$, $\Delta = 3K(\alpha \rho^2 - K)$.

By the direct calculation, it is easy to verify that

$$K \frac{\rho_l}{\rho_s} + \tilde{\alpha} = \alpha_3 \rho^2 \rho_s + K \frac{\rho_l^2}{\rho \rho_s} > 0, \qquad \Delta = 3K \alpha_3 \rho^3 > 0.$$

Let us denote by λ_s, μ_s the Lame coefficients of a homogeneous isotropic body and $K_s = \lambda_s + \dfrac{2}{3}\mu_s$. Using the formula $\lim\limits_{d_0 \to 0} \rho^2 \alpha_3 = K_s/\rho_s^f$ [5], we obtain in (5) as porosity d_0 tends to zero that

$$\frac{\partial p}{\partial t} = -\frac{1}{3}\frac{\partial \tilde{\sigma}_{mm}}{\partial t}. \tag{6}$$

Taking into account (6), we obtain in (4) as porosity tends to zero that

$$\frac{1}{\mu_s}\frac{\partial \tilde{\sigma}_{ik}}{\partial t} - \frac{\lambda_s}{(3\lambda_s + 2\mu_s)\mu_s}\delta_{ik}\frac{\partial \tilde{\sigma}_{mm}}{\partial t} = \partial_i u_k + \partial_k u_i,$$

which coincides with the elasticity theory formulas (see, e.g. [9]) differentiated with respect to time.

Introducing the vector

$$\boldsymbol{w} = (u_1, u_2, u_3, v_1, v_2, v_3, \tilde{\sigma}_{12}, \tilde{\sigma}_{13}, \tilde{\sigma}_{23}, \tilde{\sigma}_{11}, \tilde{\sigma}_{22}, \tilde{\sigma}_{33}, p)^T,$$

we rewrite system (3)–(5) in the vector form

$$A\frac{\partial \boldsymbol{w}}{\partial t} + B_k \partial_k \boldsymbol{w} = 0. \tag{7}$$

Here $A = (a_{i,j})$, $i,j = \overline{1,13}$, is a symmetric matrix with the following entries: $a_{i,i} = \rho_s$, $a_{i+3,i+3} = \rho_l$, $a_{i+6,i+6} = 1/\mu$, $a_{i+9,i+9} = \Big(1 - \dfrac{\Lambda}{\Delta}\Big)/(2\mu)$, $i = \overline{1,3}$; $a_{10,11} = a_{11,10} = a_{10,12} = a_{12,10} = a_{11,12} = a_{12,11} = -\Lambda/(2\mu\Delta)$, $a_{10,13} = a_{13,10} = a_{11,13} = a_{13,11} = a_{12,13} = a_{13,12} = \tilde{\alpha}/\Delta$; $a_{13,13} = \frac{3\rho_s}{\rho}\frac{K\rho_l/\rho_s + \tilde{\alpha}}{\Delta}$; and other entries are zeroes. Matrices $B_k = (b^k_{i,j})$, $i,j = \overline{1,13}$, $k = \overline{1,3}$, are symmetric with entries $b^1_{1,10} = b^1_{10,1} = b^1_{2,7} = b^1_{7,2} = b^1_{3,8} = b^1_{8,3} = b^2_{1,7} = b^2_{7,1} = b^2_{2,11} = b^2_{11,2} = b^2_{3,9} = b^2_{9,3} = b^3_{1,8} = b^3_{8,1} = b^3_{2,9} = b^3_{9,2} = b^3_{3,12} = b^3_{12,3} = -1$, $b^1_{4,13} = b^1_{13,4} = b^2_{5,13} = b^2_{13,5} = b^3_{6,13} = b^3_{13,6} = \rho_l/\rho$ and with zeroes at other places.

If we show that the matrix A is positive definite, system (7) will be symmetric t-hyperbolic (according to Friedrichs). Taking into account the positiveness of partial densities of the matrix ρ_s and fluids ρ_l, as well as the shear modulus μ, it is sufficient to show positive definiteness of the matrix

$$\hat{A} = \begin{pmatrix} a_{10,10} & a_{10,11} & a_{10,12} & a_{10,13} \\ a_{10,11} & a_{11,11} & a_{11,12} & a_{11,13} \\ a_{10,12} & a_{11,12} & a_{12,12} & a_{12,13} \\ a_{10,13} & a_{11,13} & a_{12,13} & a_{13,13} \end{pmatrix}.$$

The direct calculations show that

$$a_{10,10} = \frac{1 - \dfrac{\Lambda}{\Delta}}{2\mu} \geq \frac{1}{3\mu} > 0,$$

$$\begin{vmatrix} a_{10,10} & a_{10,11} \\ a_{10,11} & a_{11,11} \end{vmatrix} \geq \frac{1}{12\mu^2} > 0,$$

$$\begin{vmatrix} a_{10,10} & a_{10,11} & a_{10,12} \\ a_{10,11} & a_{11,11} & a_{11,12} \\ a_{10,12} & a_{11,12} & a_{12,12} \end{vmatrix} = \frac{\alpha\rho^2}{4\mu^2\Delta} > 0,$$

$$\begin{vmatrix} a_{10,10} & a_{10,11} & a_{10,12} & a_{10,13} \\ a_{10,11} & a_{11,11} & a_{11,12} & a_{11,13} \\ a_{10,12} & a_{11,12} & a_{12,12} & a_{12,13} \\ a_{10,13} & a_{11,13} & a_{12,13} & a_{13,13} \end{vmatrix} = \frac{1}{4\mu^2\Delta} > 0.$$

Thus, system (7) is symmetric t-hyperbolic.

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