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On the solvability of the inverse problem for the parabolic equation

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1. Introduction

The given paper considers the problem of determination of the right hand side of the high-order parabolic equation with the variable coefficients

$$L(x', \partial_t, \partial_x)\, u(t,x) = f(x_n) \cdot \lambda(t, x'), \qquad x = (x', x_n) \in R_n, \quad t > 0,$$

using the information given on the hyperplane $x_n = 0$:

$$u|_{x_n=0} = \psi(t, x'),$$

where $f(x_n)$ is the known function.

Solvability in the Sobolev weight space is proved.

A sufficiently complete bibliography on the theory of inverse problems can be found in [1–5].

2. Statement of the problem

Let us consider, in the half-space $R^+_{n+1} = \{(t,x) \mid t > 0,\ x \in R_n\}$, the parabolic high-order equation:

$$L(x', \partial_t, \partial_x)\, u(t,x) = f(x_n) \cdot \lambda(t, x'), \tag{1}$$

where $u = u(t,x)$, $x = (x', x_n) = (x_1, \dots, x_{n-1}, x_n)$,

$$L(x', \partial_t, \partial_x) = \partial_t + \sum_{|\alpha|=2m} a_\alpha(x')\, \partial_x^\alpha,$$

$$\partial_x^\alpha = \partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n}, \quad \partial_t = \frac{\partial}{\partial t}, \quad \partial_{x_j} = \frac{\partial}{\partial x_j},$$

$$\alpha = (\alpha_1, \dots, \alpha_n), \quad \alpha_j \ge 0, \quad j = 1, 2, \dots, n.$$

We introduce the following notations: $r = (1, 2m, \dots, 2m, 2p)$ – $(n+1)$-dimensional vector; $s = (1, 2m, \dots, 2m)$ – n-dimensional vector; $2p > 4m + 1$.

Problem 1. *It is necessary to find the functions*

$$(u(t,x), \lambda(t,x')) \in W^r_{2,\gamma}(R^+_{n+1}) \times L_{2,\gamma}(R^+_n)$$

(the rest functions are known) from equation (1), *if the functions*

$$u|_{t=0} = 0, \quad x \in R_n, \tag{2}$$

$$u|_{x_n=0} = \psi(t,x), \quad t > 0, \quad x' \in R_{n-1}, \tag{3}$$

are known.

Here $W^r_{2,\gamma}(R^+_{n+1})$ denotes the weight Sobolev spaces [6].
Let γ_0 be such a positive number that the inequality

$$\frac{C}{\gamma_0}\|f, W^{2p}_2(R_1)\| \leq \frac{1}{2}, \quad 2p > 4m+1, \tag{4}$$

$$C = \frac{|a_{0,\dots,0,2m}(x'_0)|}{|f(0)|}\int_{R_1}\frac{\eta^{4m}}{1+\eta^{2p}}d\eta,$$

$$\psi(0,x') = 0, \quad x' \in R_{n-1}, \tag{5}$$

and the condition A

$$\psi \in W^s_{2,\gamma}(R^+_n), \quad f \in W^{2p}_2(R_1), \quad f(0) \neq 0, \quad \gamma > \gamma_0$$

are fulfilled.

Theorem 1. *Let the coefficients of operator $L(x', \partial_t, \partial_x)$ be constant. If the conditions A,* (4), (5) *are fulfilled, then, with $\gamma > \gamma_0$, Problem 1 has the unique solution $u(t,x) \in W^r_{2,\gamma}(R^+_{n+1})$, $\lambda(t,x') \in L_{2,\gamma}(R^+_n)$.*

Proof. Let $x'_0 \in R_{n-1}$ be fixed. Consider equation (1) when the coefficients at the point x'_0 are frozen. We reduce Problem 1 to the linear integrodifferential equation by the method of work [1]. We obtain the Cauchy problem from (1)–(3), in the Fouerier transform (with respect to the variable x), of the function $u(t,x)$ after excluding of the unknown $\lambda(t,x')$. The solution is given by the formula

$$v(t,\xi) = \int_0^t e^{-(t-s)L_{2m}(i\xi)} g(t,\xi)ds, \tag{6}$$

where

$$v(t,\xi) = F_{x\to\xi}[u(t,x)] = \frac{1}{(2\pi)^{n/2}} \int_{R_n} e^{-ix.\xi}\, u(t,x)dx,$$

$$g(t,\xi) = G(t,\xi) + \frac{\hat f(\xi_n)}{f(0)}\, a_{0,\dots,0,2m}(x_0') \int_{R_1} (-i\xi_n)^{2m}\, v\, d\xi_n,$$

$$G(t,\xi) = \Big(\partial_t\hat\psi + \sum_{|\alpha'|=2m} a_{\alpha'}(x_0')(-i\xi')^{\alpha'}\,\hat\psi\Big)\,\frac{\hat f(\xi_n)}{f(0)},$$

$$\hat f(\xi_n) = F_{x_n\to\xi_n}[f(x_n)], \quad \hat\psi(t,\xi') = F_{x'\to\xi'}[\psi(t,x')],$$

$$L_{2m}(i\xi) = \sum_{|\alpha|=2m} a_\alpha(x_0')(-i\xi)^\alpha.$$

Write formula (6) in the equivalent form using the function $\theta(t)$,

$$v(t,\xi) = \int_{R_1} \theta(t-s)\, e^{-(t-s)\,L_{2m}(i\xi)}\, \theta(s)\, g(s,\xi)ds. \tag{7}$$

Estimate the norm of the function $v(t,\xi)$ in $L_{2,\gamma}(R_1^+)$. We have

$$\|v, L_{2,\gamma}(R_1^+)\| \le \|v, L_{2,\gamma}(R_1)\| \le \frac{C_1}{\gamma + |\xi|^{2m}} \|g, L_{2,\gamma}(R_1^+)\|. \tag{8}$$

Here we have used the Young inequiality and

$$C_1(\gamma + |\xi|^{2m}) \le |\gamma + L_{2m}(i\xi)| \le C_2(\gamma + |\xi|^{2m}).$$

Estimate the norm $\|\frac{1+|\xi_n|^{2p}}{\gamma+|\xi|^{2m}}\, g, L_{2,\gamma}(R_2^+)\|$, using the explicit form of the function $g(t,\xi)$:

$$\Big\|\frac{1+|\xi_n|^{2p}}{\gamma + |\xi|^{2m}}\, g, L_{2,\gamma}(R_2^+)\Big\| \le \frac{1}{\gamma + |\xi'|^{2m}} \|(1+|\xi_n|^{2p})\, G, L_{2,\gamma}(R_2^+)\| +$$
$$\frac{C}{\gamma_0}\|(1+|\xi_n|^{2p})\, v, L_{2,\gamma}(R_2^+)\| \cdot \|(1+|\xi_n|^{2p})\,\hat f, L_2(R_1)\|. \tag{9}$$

From estimaties (8), (9) with allowance for (4), we obtain

$$\|(1+|\xi_n|^{2p} + |\xi'|^{2m})\, v, L_{2,\gamma}(R_{n+1}^+)\| \le 2\|(1+|\xi_n|^{2p})\, G, L_{2,\gamma}(R_{n+1}^+)\|.$$

Hence, by the Plancherel theorem it follows that

$$u(t,x) = F^{-1}_{\xi\to x}[v(t,\xi)] \in W^r_{2,\gamma}(R_{n+1}^+).$$

The uniqueness is proved in a standard way. □

Corollary 1. *If $\psi \in W^s_{2,\gamma}(R_n^+)$, $f \in W_2^{2p}(R_1)$, $\gamma > \gamma_0$, then the solution to Problem 1 is given by the formulas*

$$u(t,x) = T_1(f, G_1), \quad G_1(t,x') = \partial_t\psi + \sum_{|\alpha'|=2m} a_{\alpha'}(x'_0)\,\partial_{x'}^{\alpha'}\psi,$$

$$\lambda(t,x') = \Big(G_1(t,x') + a_{0,\dots,0,2m}(x'_0)\int_{R_1}(-ixi_n)^{2m}F_{x_n\to\xi_n}[u(t,x)]\,d\xi_n\Big)/f(0),$$

where $T_1 : W_2^{2p}(R_1)\times W_{2,\gamma}^{s}(R_n^+) \to W_{2,\gamma}^{r}(R_{n+1}^+)$.

Remark 1. In fulfilling the conditions of Theorem 1 we estimate the norm

$$\|\partial_x^\alpha T_1(f,G_1), L_{2,\gamma}(R_{n+1}^+)\| \le \frac{C\,\|f, W_2^{2p}(R_1)\|}{\gamma^{(2m-|\alpha'|)/2m}}\|G_1, L_{2,\gamma}(R_n^+)\|. \quad (10)$$

Choose $\varepsilon > 0$ such that the following inequalety is fulfilled

$$\frac{\varepsilon}{|f(0)|}\|f, W_2^{2p}(R_1)\|\Big(C + \frac{C_1}{\gamma_0}\Big) = q_1 < 1. \quad (11)$$

Theorem 2. *Suppose the conditions A,* (4), (5), (11) *be fulfilled, and the coefficients* $a_\alpha(x')$ *satisfy the conditions*

$$a_\alpha(x') \equiv a_\alpha, \quad |x'| > r_1 > 0, \quad \sum_{|\alpha|=2m}\sup_{x'}|a_\alpha(x') - a_\alpha(x'_0)| \le \varepsilon.$$

Then the statement of Theorem 1 is true.

Proof. After the exclusion of the unknown function $\lambda(t,x')$ from (1)–(3) we obtain the Chauchy problem for the integrodifferential equation

$$\begin{aligned} L^0(x',\partial_t,\partial_x)\,u(t,x) &= \partial_t u(t,x) + \sum_{|\alpha|=2m} a_\alpha(x')\,\partial_x^\alpha u(t,x) + \\ &\quad \frac{f(x_n)}{f(0)}a_{0,\dots,0,2m}(x'_0)\int_{R_1}(-ixi_n)^{2m}F_{x_n\to\xi_n}[u(t,x)]\,d\xi_n \\ &= \mu(t,x')f(x_n), \quad (12)\\ u|_{t=0} &= 0, \quad x\in R_n. \quad (13)\end{aligned}$$

Here $\mu(t,x') = (\frac{\partial}{\partial t}\psi + \sum_{|\alpha'|=2m} a_{\alpha'}(x')\,\partial_{x'}^{\alpha'}\psi)/f(0)$.

The solution to the Chaucy problem (12), (13) is sought for in the form

$$u(t,x) = T_1(f, G_1), \quad (14)$$

where $\mu_1(t,x') \in L_{2,\gamma}(R_n^+)$ - the unknown function, the operator T_1 is defined in Corollary 1.

Apply the operator $L^0(x'_0,\partial_t,\partial_x) - L^0(x',\partial_t,\partial_x)$ to the function $u(t,x)$ from (14) and after simple transformations we obtain for $\mu_1(t,x')$ the equation

$$\mu_1(t,x') = S_1(\mu_1) + \mu(t,x'). \tag{15}$$

Here $S_1(\mu_1) = f^{-1}(0)\,(L^0(x'_0,\partial_t,\partial_x) - L^0(x',\partial_t,\partial_x))\,T_1(f,\mu_1)\,|_{x_n=0}$.

We shall show that the norm of the oerator $S_1(\mu_1)$ is small in the norm $L_{2,\gamma}(R_n^+)$, if the coefficients little differ from constants. Really

$$\begin{aligned}\|S_1(\mu_1), L_{2,\gamma}(R_n^+)\| \le &\sum_{|\alpha|=2m} \|(a_\alpha(x') - a_\alpha(x'_0))\,\partial_x^\alpha T_1(f,\mu_1), L_{2,\gamma}(R_n^+)\| + \\ &\Big\|(a_{0,\dots,0,2m}(x') - a_{0,\dots,0,2m}(x'_0)) \times \\ &\int_{R_1} (-i\xi_n)^{2m} F_{x_n\to\xi_n}[T_1(f,\mu_1)]d\xi_n, L_{2,\gamma}(R_n^+)\Big\| \\ \le\ &\varepsilon\Big(C + \frac{C_1}{\gamma_0}\Big)\,\|f, W_2^{2p}(R_1)\|\cdot\|\mu_1, L_{2,\gamma}(R_n^+)\|.\end{aligned}$$

Here we have used inequiality (10). Hence, with allowance for (11), from equation (15), we obtain the estimate

$$\|\mu_1, L_{2,\gamma}(R_n^+)\| \le \frac{1}{1-q_1}\|\mu, L_{2,\gamma}(R_n^+)\|. \tag{16}$$

By the method of successive approximations from equation (15), we find $\mu_1(t,x')$. Substituting $\mu_1(t,x')$ into (14), we obtain the solutions of the Cauchy problem (12), (13). The function $\lambda(t,x')$ is calculated by the formula

$$\begin{aligned}\lambda(t,x') = \Big(&\frac{\partial}{\partial t}\psi + \sum_{|\alpha'|=2m} a_{\alpha'}(x'_0)\,\partial_{x'}^{\alpha'}\psi + \\ &a_{0,\dots,0,2m}(x')\int_{R_1}(-ixi_n)^{2m}F_{x_n\to\xi_n}[u(t,x)]\,d\xi_n\Big)/f(0).\end{aligned} \tag{17}$$

The uniqueness is proved in a standard way. □

Corollary 2. *The solutions to the Cauchy problem* (12), (13) *can be written in the operator form*

$$u(t,x) = T_2(f,\mu),$$

where $T_2 : W_2^{2p}(R_1)\times W_{2,\gamma}^s(R_n^+) \to W_{2,\gamma}^r(R_{n+1}^+)$.

Remark 2. $\forall\,\mu \in L_{2,\gamma}(R_n^+)$, $f \in W_2^{2p}(R_1)$, $\gamma > \gamma_0$ the following estimate is valid

$$\|\partial_x^\alpha T_2(f,\mu), L_{2,\gamma}(R_{n+1}^+)\| \le \frac{C\,\|f, W_2^{2p}(R_1)\|}{(1-q_1)\gamma^{(2m-|\alpha'|)/2m}}\|\mu, L_{2,\gamma}(R_n^+)\|. \tag{18}$$

Lemma. *Let $G \equiv B_{R+1}(0) = \{x' | |x'| < R+1\}$ be a ball in R_{n-1}. Then $\forall \delta > 0$, $\exists y^1, \dots, y^N \in \bar{G}$, $\exists \varphi_0(y), \dots, \varphi_N(y)$, such that $\varphi_0(y) \in C^\infty(R_{n-1})$, $\varphi_0(y) \geq 0$, $\varphi_j(y) \in C_0^\infty(B_\delta(y^j))$, $j = 1, 2, \dots, N$. These functions satisfy the conditions*

$$\varphi_j(y) \geq 0, \quad j = 1, 2, \dots, N,$$
$$\sum_{j=0}^{N} \varphi_j(y) \equiv 1 \quad \textit{everywhere in } R_{n-1}.$$

The proof of the lemma follows from the theorem on unit partitions [7]. Choose $\gamma_1 \geq \gamma_0$, such that the following inequality is fulfilled

$$\frac{C C_4 \| f, W_2^{2p}(R_1) \|}{(1 - q_1)\gamma_1^{1/2m}} = q_2 < 1. \tag{19}$$

Here C_4 is a positive number.

Theorem 3. *Under assumptions about infinitely differentiability of the coefficients of the operator $L(x', \partial_t, \partial_x)$, and if they are constants out of the ball $B_R(0)$ and, moreover, conditions A, (4), (5), (19) hold, then for $\gamma \geq \gamma_1$ Theorem 1 is valid.*

Proof. Let $\delta > 0$. Using the functions $\varphi_0(x'), \dots, \varphi_N(x')$ from the lemma, we constract the following functions $\tilde{\varphi}_0(y), \dots, \tilde{\varphi}_N(y)$, $y = x'$:

(a) $\tilde{\varphi}_k(y) \in C^\infty(R_{n-1})$, $0 \leq \tilde{\varphi}_k(y) \leq 1$;

(b) $\tilde{\varphi}_0(y) \equiv 1$, $y \in \operatorname{supp} \varphi_0(y)$; $\tilde{\varphi}_0(y) \equiv 0$, $|y| \leq R + 1 - \delta$;

(c) $\tilde{\varphi}_k(y) \equiv 1$, $|y - y^k| \leq \delta$; $\tilde{\varphi}_k(y) \equiv 0$, $|y - y^k| \geq 2\delta$; $k = 1, 2, \dots, N$.

Also introduce the functions $\beta_1(y), \dots, \beta_N(y)$:

$$\beta_j(y) \in C_0^\infty(R_{n-1}), \quad 0 \leq \beta_j(y) \leq 1,$$
$$\beta_j(y) = \begin{cases} 1, & |y - y^j| \leq 2\delta, \\ 0, & |y - y^j| \geq 3\delta, \end{cases}$$

and the integrodifferential operators

$$L_0(x', \partial_t, \partial_x)u \equiv \partial_t u + \sum_{|\alpha|=2m} a_\alpha \partial_x^\alpha u +$$
$$\frac{f(x_n)}{f(0)} a_{0,\dots,0,2m} \int_{R_1} (-ixi_n)^{2m} F_{x_n \to \xi_n}[u(t,x)]\, d\xi_n,$$

where $a_\alpha = a_\alpha(x')$, $|x'| \geq R$,

$$L_k(x',\partial_t,\partial_x)u \equiv \partial_t u + \sum_{|\alpha|=2m} a_\alpha^k(x')\,\partial_x^\alpha u +$$
$$\frac{f(x_n)}{f(0)} a_{0,\dots,0,2m}^k(x') \int_{R_1} (-ixi_n)^{2m} F_{x_n\to\xi_n}[u(t,x)]\,d\xi_n,$$

where

$$\begin{aligned} a_\alpha^k(x') &= \beta_k(x')\,a_\alpha(x') + (1-\beta_k(x'))\,a_\alpha(y^k); \\ a_\alpha^k(x') &\equiv a_\alpha(x'), \quad x' \in B_{2\delta}(y^k), \\ a_\alpha^k(x') &\equiv a_\alpha(y^k), \quad x' \notin B_{3\delta}(y^k), \quad k=1,2,\dots,N. \end{aligned}$$

By the choice of δ we can assume that coefficients of the operator L_k are almost constants.

We consider a series of the Cauchy problem:

$$\begin{aligned} &L_k(x',\partial_t,\partial_x)u_k = \mu_k(t,x')\,f(x_n), \quad t>0, \ x\in R_n, \\ &u_k|_{t=0} = 0, \quad k=0,1,\dots,N. \end{aligned} \tag{20}$$

The solution to the Cauchy problem (20) is written in the operator form

$$u_k(t,x) = T_2^k(f,\mu_k), \quad k=0,1,\dots,N, \tag{21}$$

where $T_2^0(f,\,\mu) = T_1(f,\mu)$, $T_2^k(f,\,\mu_k) = T_2(f,\mu_k)$, $k=1,2,\dots,N$, operators T_1, T_2 are defined in Remarks 1, 2. The solution to the Cauchy problem (12), (13) will be found as the sum

$$u(t,x) = \sum_{k=0}^{N} \tilde\varphi_k(x^k)\,T_2^k(f,\varphi_k\,g). \tag{22}$$

Here the function $g(t,x') \in L_{2,\gamma}(R_n^+)$ is an unknown. Apply the operator $L^0(x',\partial_t,\partial_x)$ to the function $u(t,x)$ from (22) and after simple transformations, we obtain for $g(t,x')$ the Fredholm equation of the second kind

$$g(t,x') = S_2(g(t,x')) + \mu(t,x'). \tag{23}$$

Here

$$S_2(g(t,x')) = f^{-1}(0) \sum_{i,k=0}^{N} \varphi_i \sum_{\substack{|s|+|\sigma|=2m \\ |\sigma|\le 2m-1}} b_{s\sigma}(x')\partial_{x'}^s\tilde\varphi_k\partial_{x'}^\sigma T_2^k(f,\varphi_k\,g)\big|_{x_n=0},$$

$b_{s\sigma}(x')$ are known functions, which are ditermined by the coefficients $a_\alpha(x')$. Estimate the norm of the operator $S_2(g(t,x'))$:

$$\|S_2(g),\, L_{2,\gamma}(R_n^+)\| \le C_4 \sum_{k=0}^{N} \sum_{|\sigma|\le 2m-1} \|\partial_{x'}^{\sigma} T_2^k(f,\varphi_k g)\,|_{x_n=0},\, L_{2,\gamma}(R_n^+)\|$$

$$\le q_2\, \|g,\, L_{2,\gamma}(R_n^+)\|, \tag{24}$$

$$C_4 = (2m-1)N \sum_{k=0}^{N} \varphi_i \sum_{\substack{|s|+|\sigma|=2m \\ |\sigma|\le 2m-1}} \sup_{x'} |b_{s\sigma}(x')| \cdot |\partial_{x'}^{s} \tilde{\varphi}_k|.$$

Here we have used the inequality (18). From equation (23) with allowance for estimaties (19), (24), we obtain

$$\|g,\, L_{2,\gamma}(R_n^+)\| \;\le\; (1-q_2)^{-1}\, \|\mu,\, L_{2,\gamma}(R_n^+)\|.$$

By applying the method of successive approximations to equation (23), we find $g(t,x')$. Substituing it into (22) we obtain the solution to the Cauchy problem (12), (13). The function $\lambda(t,x')$ is calculated by formulae (17).

The uniqueness is proved in a standard way. □

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