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On second order finite-volume approximations for 3D mixed boundary value problems

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The main topic of the paper is to present a way of constructing the second order finite-volume approximations on nonuniform grids to solve 3D mixed boundary value problems for diffusion equation with piecewise constant coefficients. For obtaining the difference equations, a linear combination of the balance relations for the normal flow densities over two boxes is approximated. A set of 19- and 27-point schemes is described and investigated. Representation of the entries of the local balance matrix and assembling of the global balance matrix are given. Monotonicity conditions are obtained in the form of inequalities for meshsteps. The numerical solution error is estimated in the uniform and weighted Eucledian norms. The theoretical approach is confirmed by the results of computational experiments.

1. Introduction

There are several approaches to obtain the finite difference equations. One way to do this is to approximate the integral balance relations. The well-known methods of such a kind include the integral identities by G.I. Marchuk [1] and the integro-interpolated method by A.A. Samarski [2]. These methods were further developed by V.K. Sauliev [3] and I.V. Fryazinov [4]. In recent years a new type of similar approximations is under investigation: it is the so-called finite-volume (or "box") method (see [5], for example).

The finite-volume technology including the construction of high accuracy order finite-volume approximations has not been thoroughly investigated yet. The results concerning such approximations are obtained for two-dimensional Dirichlet boundary value problem on a nonuniform rectangular grid [6]. The basic constructing principle is in the following: to raise up the order of approximation of integro-balanced conservative law, one should use a linear combination of the integral relations over two cells of different size around the node under consideration. Optimization of the weight parameters of the combination and application of special quadrature formulas provide the second and even the third accuracy order on a nonuniform grid for the diffusion equation with piecewise constant coefficients.

The aim of this paper is to obtain and generalize 2D results [7] for three-dimensional mixed boundary value problems with piecewise constant coeffi-

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cients of the differential equation: to obtain the second order approximations for different grid stencils and variable grid steps, to investigate algebraic properties of the resulting linear system and to estimate the truncation errors. The construction of 19- and 27-point approximations providing the symmetricity of the matrix of the final system of linear equations and the truncation error $O(h^2)$ on a nonuniform grid is described in Section 2. In particular, for the uniform grid and constant coefficients one of the suggested 19-point approximations coincides with the well-known Mikeladze scheme of the order $O(h^4)$. Definition of the local balance matrices, presentation of their elements, assembling of the global balance matrix and analysis of the algebraic properties such as monotonicity conditions and eigenvalue bounds of the final linear system are given in Section 3. In the next section error estimates of the finite-volume solution in the uniform and weighted Euclidean norms are presented. The numerical results for the model problems, confirming the theoretical estimates, are given in Section 5.

2. Approximations on nonuniform grids

The following differential equation is under consideration:

$$-\nabla(\lambda\nabla u) \equiv -\frac{\partial}{\partial x}\Big(\lambda\frac{\partial u}{\partial x}\Big) - \frac{\partial}{\partial y}\Big(\lambda\frac{\partial u}{\partial y}\Big) - \frac{\partial}{\partial z}\Big(\lambda\frac{\partial u}{\partial z}\Big) = f(x,y,z), \quad (x,y,z) \in \Omega, \tag{1}$$

where Ω is a bounded three-dimensional computational domain with a piecewise constant medium function λ.

The boundary conditions are given on the external surface $\Gamma = \Gamma_1 \bigcup \Gamma_2$ in the form

$$u|_{\Gamma_1} = g(x,y,z), \qquad æu + \frac{\partial u}{\partial \overline{n}}\Big|_{\Gamma_2} = \gamma, \tag{2}$$

where g, $æ$, γ are some known functions and $\overline{n}$ is an outerward normal to the boundary Γ.

On the surfaces of discontinuity of λ ("internal boundaries") the conjugate conditions hold:

$$u|_{\Gamma_+} = u|_{\Gamma_-}, \qquad \lambda_+\frac{\partial u}{\partial n}\Big|_{\Gamma_+} = \lambda_-\frac{\partial u}{\partial n}\Big|_{\Gamma_-}, \tag{3}$$

where the signs "+, –" mean one-sided values of the function and its normal derivative on the different sides of Γ.

The external and internal boundaries are considered to be multi–connected, all their parts being parallel to the coordinate axes and planes.

It is assumed that the input data provide such a smoothness of solution which is required in the following.

Let a nonuniform mesh

$$x_{i+1} = x_i + h_i^x, \quad y_{j+1} = y_j + h_j^y, \quad z_{k+1} = z_k + h_k^z,$$
$$i = 0, \ldots, L+1, \quad j = 0, \ldots, M+1, \quad k = 0, \ldots, N+1,$$

be given in Ω such that the boundary crosses the grid lines in the nodes only. Later we omit the upper indexes for the meshsteps and will use the notations $h_i^x \equiv h_i$, $h_j^y \equiv h_j$, $h_k^z \equiv h_k$.

Further considerations will be carried out for the elementary volumes around one grid node. We introduce two kinds ("small" and "big ") of elementary volumes (boxes) around the node (i, j, k):

$$\begin{aligned} V_{ijk} &= \{x_i - h_{i-1}/2 \le x \le x_i + h_i/2, \; y_j - h_{j-1}/2 \le y \le y_j + h_j/2, \\ &\quad\; z_k - h_{k-1}/2 \le z \le z_k + h_k/2\}, \\ \overline{V}_{ijk} &= \{x_{i-1} \le x \le x_{i+1}, \; y_{j-1} \le y \le y_{j+1}, \; z_{k-1} \le z \le z_{k+1}\}. \end{aligned}$$

Let us denote by S_{ijk} and $\overline{S}_{ijk}$ the surfaces of the small and big volumes. Each of the volumes V_{ijk} and $\overline{V}_{ijk}$ is divided by the coordinate planes into eight subvolumes V_{ijk}^n and $\overline{V}_{ijk}^n$, $n = 1, \ldots, 8$. We denote the medium constants in these subvolumes by λ_n and call the mutual boundaries of the subvolumes around the node (i, j, k) the "inner" boundaries. Then the union of all outer boundaries of the small subvolume gives the surface S_{ijk} and that of the big subvolume gives $\overline{S}_{ijk}$.

Later indices (i, j, k) are omitted for the sake of brevity.

To construct the difference equations, we approximate a linear combination over small and big boxes with a weight parameter p of two balance relations which give the link between the surface integral of the normal flow density and the volume integral of the source function:

$$J \equiv p \int_S J^n dS + (1-p) \int_{\overline{S}} J^n dS = p \int_V f \, dV + (1-p) \int_{\overline{V}} f \, dV, \qquad (4)$$

where J^n is a density flow in the direction of the outer normal with the projections onto different coordinate axes given as $J^x = -\lambda u^x$, $J^y = -\lambda u^y$, $J^z = -\lambda u^z$ and f can be regarded as a source function. Each surface integral in (4) can be presented as the sum of eight integrals corresponding to eight subvolumes:

$$J = \sum_{k=1}^{8} J_k.$$

Here J_k is a full flow through the surface of the k-th subvolume including the integrals over the inner and outer parts of its boundary: $J_k = J_k^{\text{inn}} + J_k^{\text{out}}$. Let us note that the integrals over the inner parts of all eight subvolumes mutually annulate because of the conjugate conditions (3). Particularly, for the integrals over big and small subvolumes in the upper-north-east octant of V and $\overline{V}$ (we call this octant number 3)

$$\overline{V}_3 = \{x_i \le x \le x_{i+1},\ y_j \le y \le y_{j+1},\ z_k \le z \le z_{k+1}\},$$
$$V_3 = \{x_i \le x \le x_i + h_i/2,\ y_j \le y \le y_j + h_j/2,\ z_k \le z \le z_k + h_k/2\},$$

one can write

$$J_3 \equiv pJ_3 + (1-p)\overline{J}_3 =$$
$$-\lambda_3\Big\{p\Big(\int_{y_j}^{y_{j+1/2}}\int_{z_k}^{z_{k+1/2}} u^x(x_{i+\frac{1}{2}}, y, z)\,dydz + \int_{x_i}^{x_{i+1/2}}\int_{z_k}^{z_{k+1/2}} u^y(x, y_{j+\frac{1}{2}}, z)\,dxdz +$$
$$\int_{x_i}^{x_{i+1/2}}\int_{y_j}^{y_{j+1/2}} u^z(x, y, z_{k+\frac{1}{2}})\,dxdy\Big) + (1-p)\Big(\int_{y_j}^{y_{j+1}}\int_{z_k}^{z_{k+1}} u^x(x_{i+1}, y, z)\,dydz +$$
$$\int_{x_i}^{x_{i+1}}\int_{z_k}^{z_{k+1}} u^y(x, y_{j+1}, z)\,dxdz + \int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}} u^z(x, y, z_{k+1})\,dxdy\Big)\Big\}. \tag{5}$$

To get the difference relation we need two stages of approximation: approximation of the integrals and that of the derivatives. The implementation of these procedures is done with the help of the central rectangular and the trapezoidal quadrature formulas:

$$\lambda(u_{ijk} - u_{ijk+1}) = \int_{z_k}^{z_{k+1}} J^z(x_i, y_j, z)dz = h_k J^z_{i,j,k+1/2} + \psi^1_{i,j,k+1/2}$$
$$= \frac{h_k}{2}(J^z_{i,j,k} + J^z_{i,j,k+1}) + \psi^2_{i,j,k+1/2}. \tag{6}$$

Here the remainder terms $\psi^{1,2}$ are given by the relations

$$\psi^1_{i,j,k+1/2} = \frac{h_k^3}{24}J^z_{zz}(x_i, y_j, \xi'), \quad \psi^2_{i,j,k+1/2} = -\frac{h_k^3}{12}J^z_{zz}(x_i, y_j, \xi''),$$
$$\xi', \xi'' \in [z_k, z_{k+1}].$$

For the sake of brevity, we present the formulas only for the x-projection of relation (5). We will show in detail the approximations of integrals and derivatives to give the expression of the main remainder terms.

Below the approximate values of the integrals with the main remainder terms are given:

$$\int_{y_j}^{y_{j+1/2}} \int_{z_k}^{z_{k+1/2}} u^x(x_{i+1/2}, y, z)dydz$$

$$= \frac{h_k h_j}{4} u^x_{i+1/2,j+1/4,k+1/4} + \frac{1}{384}(h_k^3 h_j u^{xzz}_{i+1/2,j+1/4,\xi_1} + h_k h_j^3 u^{xyy}_{i+1/2,\xi_2,k+1/4}),$$

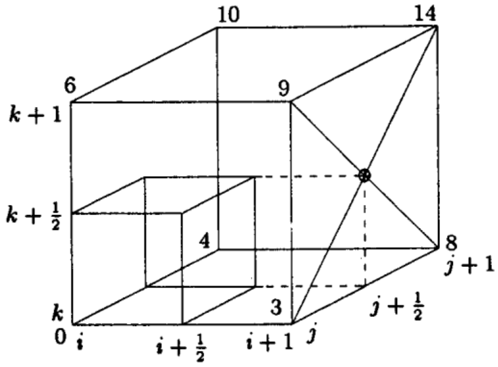
$$\xi_1 \in [z_k, z_{k+1/2}], \quad \xi_2 \in [y_j, y_{j+1/2}],$$

$$\int_{y_j}^{y_{j+1}} \int_{z_k}^{z_{k+1}} u^x(x_{i+1}, y, z)dydz$$

$$= h_k h_j u^x_{i+1,j+1/2,k+1/2} + \frac{1}{24}(h_k^3 h_j u^{xzz}_{i+1,j+1/2,\xi_3} + h_k h_j^3 u^{xyy}_{i+1,\xi_4,k+1/2}),$$

$$\xi_3 \in [z_k, z_{k+1}], \quad \xi_4 \in [y_j, y_{j+1}],$$

i.e., the value of the integral is approximately equal to function value in the middle of the face multiplied by the face area.



Local node numbering

We use different approximations of the derivatives in the middle of the faces which lead to 19- or 27-point schemes correspondingly. There are three possibilities for approximate representation of the value $u^x_{i+1,j+1/2,k+1/2}$ (see Figure):

$$(u^x_3 + u^x_{14})/2, \quad (u^x_8 + u^x_9)/2, \quad (u^x_3 + u^x_{14} + u^x_8 + u^x_9)/4,$$

and seven possibilities for $u^x_{i+1/2,j+1/4,k+1/4}$ to represent them through the values of u^x in different vertices of subvolume $\overline{V_3}$, i.e., there are 21 difference schemes for x-projection of (5). But several schemes give the same approximation, so there are a family of 16 different schemes approximating relation (5) for one fixed parameter p. Only one of them is 19-point one. The order of approximation for all of them is $O(h^2)$ – we shall show it on the examples below. Let us note that an arbitrary linear combination of these schemes with sixteen coefficients also belongs to the family.

We will pay attention firstly to the 19-point approximation. To get it, one should not use in the quadratures the corner nodes of $\overline{V}$ which have the grid indices $(i \pm 1, j \pm 1, k \pm 1)$.

To approximate x-projections of the flow at the middle points of the faces we use their representation in the following form:

$$\begin{aligned} u^x_{i+1/2,j,k} + u^x_{i+1/2,j+1/2,k+1/2} & \\ = 2u^x_{i+1/2,j+1/4,k+1/4} &+ \frac{h_k h_j}{8} u^{xyz}_{i+1/2,j+1/4,\xi_5} + \\ & \frac{h_j^2}{32}(u^{xyy}_{i+1/2,\xi_6,k} + u^{xyy}_{i+1/2,\xi_7,k+1/2}) + \\ & \frac{h_k^2}{32}(u^{xzz}_{i+1/2,j+1/4,\xi_8} + u^{xzz}_{i+1/2,j+1/4,\xi_9}), \end{aligned}$$

$$\begin{aligned} u^x_{i+1/2,j+1,k} + u^x_{i+1/2,j,k+1} & \\ = 2u^x_{i+1/2,j+1/2,k+1/2} &- \frac{h_k h_j}{4} u^{xyz}_{i+1/2,j+1/2,\xi_{10}} + \\ & \frac{h_j^2}{8}(u^{xyy}_{i+1/2,\xi_{11},k} + u^{xyy}_{i+1/2,\xi_{12},k+1}) + \\ & \frac{h_k^2}{8}(u^{xzz}_{i+1/2,j+1/2,\xi_{13}} + u^{xzz}_{i+1/2,j+1/2,\xi_{14}}), \end{aligned}$$

$$\begin{aligned} u^x_{i+1,j+1,k} + u^x_{i+1,j,k+1} & \\ = 2u^x_{i+1,j+1/2,k+1/2} &- \frac{h_k h_j}{4} u^{xyz}_{i+1,j+1/2,\xi_{15}} + \\ & \frac{h_j^2}{8}(u^{xyy}_{i+1,\xi_{16},k} + u^{xyy}_{i+1,\xi_{17},k+1}) + \\ & \frac{h_k^2}{8}(u^{xzz}_{i+1,j+1/2,\xi_{18}} + u^{xzz}_{i+1,j+1/2,\xi_{19}}), \end{aligned}$$

$$\xi_6, \xi_7 \in [y_j, y_{j+1/2}], \quad \xi_5, \xi_8, \xi_9 \in [z_k, z_{k+1/2}],$$
$$\xi_{11}, \xi_{12}, \xi_{16}, \xi_{17} \in [y_j, y_{j+1}], \quad \xi_{10}, \xi_{13}, \xi_{14}, \xi_{15}, \xi_{18}, \xi_{19} \in [z_k, z_{k+1}].$$

So, x-projection of relation (5) with omitted remainder terms in the brackets can be written as:

$$\begin{aligned} -\frac{J_3^x}{\lambda_3} \approx \frac{p}{2}\Big(u^x_{i+1/2,j,k} &+ \frac{u^x_{i+1/2,j+1,k} + u^x_{i+1/2,j,k+1}}{2}\Big)\frac{h_j h_k}{4} + \\ & \frac{1-p}{2}(u^x_{i+1,j+1,k} + u^x_{i+1,j,k+1})h_j h_k. \end{aligned} \qquad (7)$$

To improve the error estimate when approximating, we do two special tricks: firstly, we approximate also the integrals over the inner surfaces of the small box (i.e., approximate the full flow over the small box boundary) and include the remainder terms in the full error whereas annulate the quadrature terms due to the conjugate conditions. Secondly, we change the sign of the integrals over the inner boundary parts of the big box and it does not influence equation (4) because their sum is equal to zero also due to the conjugate conditions, but we use the corresponding remainder terms from the neighbouring box (they have the opposite sign) to improve the error.

The truncation error in (7) is $O(h^4)$: the second order of the remainder terms in the brackets is multiplied by the factor $h_j h_k$. To get the coefficients in the corresponding grid equation of order $O(h^{-2})$, what is conventional for difference schemes, one should multiply the coefficients by the factor $1/h^3$ and have the error order $O(h)$ here. To obtain here the desired order $O(h^2)$, which is $O(h^5)$ when no scaling is applied, one should recall the remark above: when dealing with the integrals over the inner surfaces of the boxes, the remainder terms will have the same values of the multipliers but with the opposite signs and the values of different derivatives of the third order will have index i instead of $i+1/2$ or $i+1$.

To approximate the derivatives, we add to the big box part the terms from the inner boundary $u^x_{i,j+1,k}$ and $u^x_{i,j,k+1}$ and approximate the whole sum with the help of the trapezium formula (6) (we use ξ in relevant the sence like above):

$$
\begin{aligned}
-(u^x_{i+1,j+1,k} + u^x_{i+1,j,k+1} + u^x_{i,j+1,k} + u^x_{i,j,k+1})
&= 2\frac{u_{i,j+1,k} - u_{i+1,j+1,k} + u_{i,j,k+1} - u_{i+1,j,k+1}}{h_i} + \\
&\quad \frac{h_i^2}{6}(u^{xxx}_{\xi,j+1,k} + u^{xxx}_{\xi,j,k+1}). \qquad (7.1)
\end{aligned}
$$

To approximate the derivatives for the small box, we use the central rectangular formula for all three derivatives. Doing so, one can write

$$
\begin{aligned}
-\Big(u^x_{i+1/2,j,k} + \frac{u^x_{i+1/2,j+1,k} + u^x_{i+1/2,j,k+1}}{2}\Big)
&= \frac{1}{h_i}\Big(u_{i,j,k} - u_{i+1,j,k} + \frac{1}{2}(u_{i,j+1,k} - u_{i+1,j+1,k} + u_{i,j,k+1} - u_{i+1,j,k+1})\Big) - \\
&\quad \frac{h_i^2}{24}\Big(u^{xxx}_{\xi,j,k} + \frac{1}{2}(u^{xxx}_{\xi,j+1,k} + u^{xxx}_{\xi,j,k+1})\Big). \qquad (7.2)
\end{aligned}
$$

Now we need to attain one higher order of accuracy. Writing out the sum of the coefficients for different derivatives of the remainder terms after their substituting into (7) and setting the sum equal to zero to annulate these terms, one can get the following equation for the weight parameter p:

$$-p/96 + (1-p)/6 = 0.$$

From it follows that the desired order $O(h^3)$ in the brackets of (7) is obtained for $p = 16/17$.

So, relation (7) takes after the approximation the final difference form:

$$\frac{J_3^x}{\lambda_3} \approx \frac{p}{2h_i}\Big(u_{i,j,k}-u_{i+1,j,k}+\frac{1}{2}(u_{i,j+1,k}-u_{i+1,j+1,k}+u_{i,j,k+1}-u_{i+1,j,k+1})\Big)\frac{h_j h_k}{4} + \frac{1-p}{h_i}(u_{i,j+1,k} - u_{i+1,j+1,k} + u_{i,j,k+1} - u_{i+1,j,k+1})h_j h_k, \quad (8)$$

and after the similar computations for y- and z-projections of the flow the full left-hand side of the balance for the subvolume number 3 is given by the relation

$$\begin{aligned} J_3 \approx\ & \frac{p}{8}(r_{jki}(u_{i,j,k}-u_{i+1,j,k}) + r_{ikj}(u_{i,j,k}-u_{i,j+1,k}) + r_{ijk}(u_{i,j,k}-u_{i,j,k+1})) + \\ & \Big(1-\frac{15}{16}p\Big)\Big(r_{jki}(u_{i,j+1,k} - u_{i+1,j+1,k} + u_{i,j,k+1} - u_{i+1,j,k+1}) + \\ & r_{ikj}(u_{i+1,j,k} - u_{i+1,j+1,k} + u_{i,j,k+1} - u_{i,j+1,k+1}) + \\ & r_{ijk}(u_{i+1,j,k} - u_{i+1,j,k+1} + u_{i,j+1,k} - u_{i,j+1,k+1})\Big), \end{aligned} \quad (9)$$

$$r_{nml} = \lambda_3 \frac{h_n h_m}{h_l}.$$

After similar transformations with the balances for the rest subvolumes of V and $\overline{V}$ and after gathering together the coefficients of the stencil points, one can write the matrix form of the balance relation:

$$(Au_h)_0 \equiv a_0 u_0 - \sum_{s\in S} a_s u_s = f_0, \quad (10)$$

where $u_0 \equiv u_{ijk}$ and S means the 19-point stencil (it is full 27-point stencil without eight corner nodes) around the node (i,j,k). As for the approximation of the volume integrals in the right-hand side of (4), by applying the necessary order of interpolation of the function f, the corresponding error can be done arbitrarily small without any changes in the matrix of system (10). Here we use the following notation for the approximate value of the right-hand side integrals:

$$p\int\limits_{V_{ijk}} f dV + (1-p)\int\limits_{\overline{V}_{ijk}} f dV = f_{ijk} + \psi_{ijk}^f, \quad (11)$$

$$f_{ijk} \equiv f_{ijk}^h v_{ijk}, \quad v_{ijk} = (h_i + h_{i-1})(h_j + h_{j-1})(h_k + h_{k-1})/8.$$

We consider also one 27-point scheme, where the analogues of (7), (8) and (9) are the following:

$$\begin{aligned}\frac{J_3^x}{\lambda_3} \approx\ & \frac{p}{4}(u^x_{i+1/2,j,k}+u^x_{i+1/2,j,k+1/2}+u^x_{i+1/2,j+1/2,k+1/2}+u^x_{i+1/2,j+1/2,k})\frac{h_j h_k}{4} + \\ & \frac{1-p}{4}(u^x_{i+1,j,k} + u^x_{i+1,j+1,k} + u^x_{i+1,j+1,k+1} + u^x_{i+1,j,k+1})h_j h_k, \end{aligned} \qquad (7a)$$

$$\begin{aligned}\frac{J_3^x}{\lambda_3} \approx\ & \frac{p}{4h_i}\Big(u_{i,j,k} - u_{i+1,j,k} + \frac{1}{2}(u_{i,j,k} - u_{i+1,j,k} + u_{i,j,k+1} - u_{i+1,j,k+1}) + \\ & \quad \frac{1}{4}(u_{i,j,k} - u_{i+1,j,k} + u_{i,j,k+1} - u_{i+1,j,k+1} + \\ & \quad\quad u_{i,j+1,k+1} - u_{i+1,j+1,k+1} + u_{i,j+1,k} - u_{i+1,j+1,k}) + \\ & \quad \frac{1}{2}(u_{i,j+1,k} - u_{i+1,j+1,k} + u_{i,j,k} - u_{i+1,j,k})\Big)\frac{h_k h_j}{4} + \\ & \frac{1-p}{2h_i}(u_{i,j,k} - u_{i+1,j,k} + u_{i,j+1,k} - u_{i+1,j+1,k} + \\ & \quad\quad u_{i,j+1,k+1} - u_{i+1,j+1,k+1} + u_{i,j,k+1} - u_{i+1,j,k+1})h_k h_j, \end{aligned} \qquad (8a)$$

$$\begin{aligned}\frac{J_3}{\lambda_3} \approx\ & r_{jki}\Big(B(u_{i,j,k} - u_{i+1,j,k}) + C(u_{i,j,k+1} - u_{i+1,j,k+1} + \\ & \quad\quad u_{i,j+1,k} - u_{i+1,j+1,k}) + D(u_{i,j+1,k+1} - u_{i+1,j+1,k+1})\Big) + \\ & r_{ikj}\Big(B(u_{i,j,k} - u_{i,j+1,k}) + C(u_{i,j,k+1} - u_{i,j+1,k+1} + \\ & \quad\quad u_{i+1,j,k} - u_{i+1,j+1,k}) + D(u_{i+1,j,k+1} - u_{i+1,j+1,k+1})\Big) + \\ & r_{ijk}\Big(B(u_{i,j,k} - u_{i,j,k+1}) + C(u_{i,j+1,k} - u_{i,j+1,k+1} + \\ & \quad\quad u_{i+1,j,k} - u_{i+1,j,k+1}) + D(u_{i+1,j+1,k} - u_{i+1,j+1,k+1})\Big). \end{aligned} \qquad (9a)$$

Here we use the following notations:

$$B = \frac{32-23p}{64}, \quad C = \frac{32-29p}{64}, \quad D = \frac{32-31p}{64}. \qquad (12)$$

The analogue of the matrix form of the difference equation differs from (10) only in the set S: in this case S is full 27-point stencil around the node (i,j,k).

It follows from (12) that for $p = 32/31$ the constant $D = 0$, i.e., 27-point scheme turns into 19-point one.

3. Balance matrices

In this section we consider the procedure of computing the coefficients of system (10) using the volume-by-volume approach. Let us define in the subvolume $\overline{V}_3$ the local solution and the local flow vectors (see Figure for the node numbering)

$$u_3 = \{u_3^m\}, \quad J_3 = \{J_3^m\}, \quad m \in M = \{0,3,4,6,8,9,10,14\}.$$

Here J_3^m is the flow around the m-th corner of the subvolume and J_3^0 is defined with the help of (5).

Each of J_3^m is a linear form of u_3^m:

$$J_3^m = \sum_{l \in M} a_{ml} u_3^l.$$

Then we can write down the vector equation for the volume in the following form:

$$J_3 = A_3 u_3, \tag{13}$$

where A_3 is a square matrix of the eighth order

$$A_3 = \begin{pmatrix} a_{0,0} & a_{0,3} & \cdots & a_{0,10} & a_{0,14} \\ a_{3,0} & a_{3,3} & \cdots & a_{3,10} & a_{3,14} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{10,0} & a_{10,3} & \cdots & a_{10,10} & a_{10,14} \\ a_{14,0} & a_{14,3} & \cdots & a_{14,10} & a_{14,14} \end{pmatrix}$$

with its entries given by the following formulas (with taking into account the symmetricity of the matrix) in the case of the 27-point approximation:

$$\begin{aligned}
a_{0,0} &= B(r_{jki} + r_{ikj} + r_{ijk}) = a_{ii}, \\
a_{0,3} &= Br_{jki} - C(r_{ikj} + r_{ijk}) = a_{4,8} = a_{6,9} = a_{10,14}, \\
a_{0,4} &= Br_{ikj} - C(r_{jki} + r_{ijk}) = a_{3,8} = a_{6,10} = a_{9,14}, \\
a_{0,6} &= Br_{ijk} - C(r_{jki} + r_{ikj}) = a_{3,9} = a_{4,10} = a_{8,14}, \\
a_{0,8} &= -Dr_{ijk} + C(r_{jki} + r_{ikj}) = a_{3,4} = a_{6,14} = a_{9,10}, \\
a_{0,9} &= -Dr_{ikj} + C(r_{jki} + r_{ijk}) = a_{3,6} = a_{8,10} = a_{4,14}, \\
a_{0,10} &= -Dr_{jki} + C(r_{ikj} + r_{ijk}) = a_{3,14} = a_{8,9} = a_{4,6}, \\
a_{0,14} &= D(r_{ijk} + r_{jki} + r_{ikj}) = a_{3,10} = a_{6,8} = a_{4,9},
\end{aligned}$$

where r_{nml} is defined by (9) and the values of B, C, D – by (12). In the case of the 19-point approximation the matrix entries are calculated via the same formulas but with

$$B = p/8, \quad C = 1 - 15p/16, \quad D = 0.$$

From here it follows that for $p = 16/15$ this scheme turns into 7-point one since $C = 0$.

Till now the boundary conditions have not been taken into account yet. The example below will clear the situation with the third type boundary conditions. Let the left face of the volume $\overline{V}_3$ have the boundary conditions of the third type (2) with $æ \neq 0$. Then the integrals in (4) over this face do not vanish and equation (5) has additional integrals

$$p \int\limits_{y_j}^{y_{j+1/2}} \int\limits_{z_k}^{z_{k+1/2}} u^x(x_i, y, z)\, dydz + (1-p) \int\limits_{y_j}^{y_{j+1}} \int\limits_{z_k}^{z_{k+1}} u^x(x_i, y, z)\, dydz.$$

We approximate these integrals similarly to the approximation of the rest integrals in (5) without increasing of the total truncation error. In doing so, relation (7) will have the following additive:

$$-\frac{p}{2}\left(u^x_{i,j,k} + \frac{u^x_{i,j+1,k} + u^x_{i,j,k+1}}{2}\right)\frac{h_j h_k}{4} - \frac{1-p}{2}(u^x_{i,j+1,k} + u^x_{i,j,k+1})h_j h_k,$$

and (7a) will have

$$-\frac{p}{4}(u^x_{i,j,k} + u^x_{i,j,k+1/2} + u^x_{i,j+1/2,k+1/2} + u^x_{i,j+1/2,k})\frac{h_j h_k}{4} - \\ \frac{1-p}{4}(u^x_{i,j,k} + u^x_{i,j+1,k} + u^x_{i,j+1,k+1} + u^x_{i,j,k+1})h_j h_k.$$

In order to get the sum of x-derivatives of u for the use of the trapezium formula to approximate the big box derivatives here, we add the inner boundary derivatives $u^x_{i+1,j+1,k}$ and $u^x_{i+1,j,k+1}$ and the terms which are equal to zero due to the boundary conditions and have:

$$u^x_{i+1,j+1,k} + u^x_{i+1,j,k+1} - u^x_{i,j+1,k} - u^x_{i,j,k+1} + \\ 2(u^x_{i,j+1,k} - æ u_{i,j+1,k} + \gamma_{i,j+1,k}) + 2(u^x_{i,j,k+1} - æ u_{i,j,k+1} + \gamma_{i,j,k+1}).$$

After the approximations we have the difference equation (8) and its addition:

$$\frac{p h_j h_k}{8}\Big(-æ u_{i,j,k} + \gamma_{i,j,k} + \frac{1}{2}(-æ u_{i,j+1,k} + \gamma_{i,j+1,k} - æ u_{i,j,k+1} + \gamma_{i,j,k+1})\Big) + \\ (1-p)h_j h_k(-æ u_{i,j+1,k} + \gamma_{i,j+1,k} - æ u_{i,j,k+1} + \gamma_{i,j,k+1}),$$

and in the 27-point case

$$\frac{ph_jh_k}{16}\Big(-æu_{i,j,k}+\gamma_{i,j,k}+\frac{1}{2}(-æu_{i,j,k}+\gamma_{i,j,k}-æu_{i,j,k+1}+\gamma_{i,j,k+1})+$$
$$\frac{1}{4}\Big(-æu_{i,j,k}+\gamma_{i,j,k}-æu_{i,j,k+1}+\gamma_{i,j,k+1}-$$
$$æu_{i,j+1,k+1}+\gamma_{i,j+1,k+1}-æu_{i,j+1,k}+\gamma_{i,j+1,k}\Big)+$$
$$\frac{1}{2}(-æu_{i,j,k}+\gamma_{i,j,k}-æu_{i,j+1,k}+\gamma_{i,j+1,k})\Big)+$$
$$\frac{(1-p)h_jh_k}{2}\Big(-æu_{i,j,k}+\gamma_{i,j,k}-æu_{i,j+1,k+1}+\gamma_{i,j+1,k+1}-$$
$$æu_{i,j+1,k}+\gamma_{i,j+1,k}-æu_{i,j,k+1}+\gamma_{i,j,k+1}\Big).$$

This gives the additions into the corresponding matrix entries in (13) and the corresponding right-hand sides of equation (10).

Similarly, one can introduce for each subvolume V_l corresponding vectors $u_l = \{u_l^m\}, J_l = \{J_l^m\}$ and write down the relation $J_l = A_l u_l$, where the matrix A_l can be called a local balance matrix.

Just as in the finite element technology, the final matrix of the system of linear equations (the global balance matrix) is assembled from eight local matrices:

$$A \equiv \sum_l \overline{A_l} = \sum_l P_l^{-1} A_l P_l.$$

Each A_l is a square 8×8-matrix, $\overline{A_l}$ is an "extended" local $N \times N$-matrix with the same nonzero entries as in A_l, N is the total number of nodes and P_l is extending $8 \times N$-matrix with only eight nonzero unit entries. The matrix P_l is defined by the following transformation: $u = P_l u_l$ and is an orthogonal one, i.e., $P_l^{-1} = P_l^t$.

The final system of equations has the form

$$Au = f. \tag{14}$$

The Dirichlet conditions are taken into account by the direct elimination of the unknowns, corresponding to the Dirichlet nodes, from the equations for the neighbours of such nodes.

The coefficients $(a_m)^{i,j,k}$ from system (10) for the node (i,j,k), i.e., the entries of the global balance matrix A, are calculated via the entries a_{ml} of the eight local balance matrices A_l of the subvolumes, which have the node (i,j,k) as their vertex, in the case of 27-point scheme by the following formulas:

$$
\begin{aligned}
a_0^{ijk} &= a_{0,0}^{ijk} + a_{3,3}^{i-1jk} + a_{4,4}^{ij-1k} + a_{6,6}^{ijk-1} + \\
&\quad a_{8,8}^{i-1j-1k} + a_{9,9}^{i-1jk-1} + a_{10,10}^{ij-1k-1} + a_{14,14}^{i-1j-1k-1}, \\
a_3^{ijk} &= a_{0,3}^{ijk} + a_{4,8}^{ij-1k} + a_{6,9}^{ijk-1} + a_{10,14}^{ij-1k-1}, \\
a_4^{ijk} &= a_{0,4}^{ijk} + a_{3,8}^{i-1jk} + a_{6,10}^{ijk-1} + a_{9,14}^{i-1jk-1}, \\
a_6^{ijk} &= a_{0,6}^{ijk} + a_{4,10}^{ij-1k} + a_{8,14}^{i-1j-1k} + a_{3,9}^{i-1jk}, \\
a_7^{ijk} &= a_{3,4}^{i-1jk} + a_{9,10}^{i-1jk-1}, \\
a_8^{ijk} &= a_{0,8}^{ijk} + a_{6,14}^{ijk-1}, \\
a_9^{ijk} &= a_{0,9}^{ijk} + a_{4,14}^{ij-1k}, \\
a_{10}^{ijk} &= a_{0,10}^{ijk} + a_{3,14}^{i-1jk}, \\
a_{11}^{ijk} &= a_{3,6}^{i-1jk} + a_{8,10}^{i-1j-1k}, \\
a_{12}^{ijk} &= a_{4,6}^{ij-1k} + a_{8,9}^{i-1j-1k}, \\
a_{13}^{ijk} &= a_{6,8}^{i-1j-1k}, \\
a_{14}^{ijk} &= a_{0,14}^{ijk}, \\
a_{15}^{ijk} &= a_{3,10}^{i-1jk}, \\
a_{16}^{ijk} &= a_{4,9}^{ij-1k}.
\end{aligned} \tag{15}
$$

Here i, j, k are the indices of the low-left node of the subvolume number 3, i.e., the coefficients a_{pq}^{ijk} are these of the local balance matrix A_3. The rest coefficients are calculated via the same formulas but with replacing $\lambda_3 \equiv \lambda_{ijk}$ by the medium constant from the corresponding subvolume.

As for the 19-point scheme, the assembling formulas are the same except of the last line: in this case $a_{14}^{ijk} = 0$.

Let us remark that for the uniform grid, $\lambda_{ijk} \equiv 1$ and $p = 32/31$, the coefficients of equation (14) are proportional to these of the famous Mikeladze scheme [8] of the fourth order of accuracy.

One can easily check that under the conditions

$$
\begin{aligned}
&\alpha \le h_k h_{k-1}\Big(\frac{1}{h_j^2} + \frac{1}{h_i^2}\Big), \quad h_k^2\Big(\frac{1}{h_i h_{i-1}} + \frac{1}{h_j h_{j-1}}\Big) \le \beta, \\
&\alpha \le h_i h_{i-1}\Big(\frac{1}{h_k^2} + \frac{1}{h_j^2}\Big), \quad h_i^2\Big(\frac{1}{h_k h_{k-1}} + \frac{1}{h_j h_{j-1}}\Big) \le \beta, \\
&\alpha \le h_j h_{j-1}\Big(\frac{1}{h_i^2} + \frac{1}{h_k^2}\Big), \quad h_j^2\Big(\frac{1}{h_i h_{i-1}} + \frac{1}{h_k h_{k-1}}\Big) \le \beta,
\end{aligned} \tag{16}
$$

the matrix of L_h in the 27-point scheme is a Stieltjes one, i.e., it is symmetric and has the diagonal dominance and nonnegative values of a_k. Here the values of α and β are the following:

$$\alpha = \frac{32 - 31p}{32 - 29p}, \quad \beta = \frac{32 - 23p}{32 - 29p}.$$

As for such conditions in 19-point case, they can be written out from the conditions that off-diagonal matrix entries are greater than zero, i.e., $a_3 > 0$, $a_4 > 0$, $a_6 > 0$ but their form is too cumbersome to give it here. It is easy to check that for the optimal parameter $p = 16/17$ there does not exist any monotone scheme of the second order.

4. Error estimates

In this section the error of numerical solution

$$z_h \equiv \{z_{ijk} = u(x_i, y_j, z_k) - u_{ijk}\}$$

is investigated in the uniform and weighted Eucledian norms

$$\|z_h\|_\infty = \max_{ijk}\{|z_{ijk}|\}, \quad \|z_h\|_H = (Hz_h, z_h)^{1/2} = \Big(\sum_{ijk} w_{ijk} z_{ijk}^2\Big)^{1/2},$$
$$w_{ijk} = (h_i + h_{i-1})(h_j + h_{j-1})(h_k + h_{k-1})/8, \quad H = \mathrm{diag}\{w_{ijk}\},$$

where indexes (i, j, k) correspond to all the grid points in $\Omega \bigcup \Gamma_2$, i.e., only the Dirichlet boundary nodes are excluded.

The error vector satisfies the equation

$$Az_h = \psi_h, \tag{17}$$

where A is a matrix representing the discrete approximation (10) of the original operator of the boundary value problem (1), (2), (3) with its coefficients given by (15). Total error of approximation ψ_h consists of the errors of the approximations of the flows (see (7.1) and (7.2)) and the approximation of the right-hand side of (4) (see (11)).

To estimate the uniform norm $\|z_h\|_\infty$, we suppose that there exists a majorant function $\overline{u}(x, y, z)$ with the following properties [9]:

a) the relation
$$-\nabla(\lambda\nabla\overline{u}) = \overline{f}(x, y, z) \geq 1 \tag{18}$$
is satisfied in Ω except the points on the "inner boundary", where the conjugate condition for $\overline{u}$ is valid;

b) the boundary conditions

$$\overline{u}|_{\Gamma_1} = \overline{g}(x,y,z) > 0, \quad æ\overline{u} + \frac{\partial \overline{u}}{\partial \overline{n}}\Big|_{\Gamma_2} = \overline{\gamma} \geq 0, \tag{19}$$

are satisfied on the external boundary of the computational domain Ω;

c) the function $\overline{u}$ is positive and bounded in $\overline{\Omega} = \Omega \bigcup \Gamma$ and has some smoothness properties at the internal points of Ω which are sufficient for validity of the following relation:

$$A_h \overline{u}_h = \overline{f}_h + \overline{\psi}_h, \quad \overline{\psi}_h = \{\overline{\psi}_{ijk} = O(h^\theta), \theta > 0\}. \tag{20}$$

Remark. The latter condition means, firstly, that the function $\overline{u}$ can have weaker smoothness properties than u: (e.g., $\theta < 2$ in (20)) and, secondly, that for a small enough meshsize $h < h_0$) the following inequality is true:

$$A_h \overline{u}_h \geq \alpha e_h, \quad e_h = \{1\}, \quad \alpha > 0. \tag{21}$$

Theorem 1. *Let the solution u of the boundary value problem* (1), (2), (3) *and the function f have bounded mixed derivatives of the fourth and the second order correspondingly at the internal points of the computational domain Ω. Let there exist a bounded function $\overline{u}$ with properties* (18)–(21) *and entries of the matrix A_h satisfy the monotonicity condition* (16). *Then the error vector z_h for $p = 16/17$ has the uniform norm*

$$\|z_h\|_\infty \leq \frac{\|\psi_h\|_\infty \|\overline{u}_h\|_\infty}{\alpha} = O(h^2).$$

The proof of this result can be easily established with the help of the inequality $\|z_h\|_\infty \leq \|A^{-1}\|_\infty \|\psi_h\|_\infty$ and the consequence of the inequality (21) for the monotone matrices (see [10], for example): $\|A^{-1}\|_\infty \leq \|\overline{u}_h\|_\infty / \alpha$. This theorem is nontrivial, if a set of the majorant functions $\overline{u}$ is not empty.

Following the results for two-dimensional case presented in [9], we construct below two examples of the majorant functions for 3D Dirichlet boundary value problems.

Let a domain Ω consist of the strips

$$\Omega_k = x_{k-1} < x < x_k, \quad k = 1, 2, \ldots, m,$$

in which the coefficient λ from the original equation (1) has the constant value λ_k. Let

$$D = \{x, y, z : a_1 < x < b_1, \quad a_2 < y < b_2, \quad a_3 < z < b_3\}$$

be a described around Ω parallelepiped. Then a function is defined

$$\overline{u}(x,y,z) = \frac{1}{2}\Big[\Big(\frac{b_2 - a_2}{2}\Big)^2 - \Big(y - \frac{b_2 + a_2}{2}\Big)^2 + v(x)\Big],$$

where $v(x)$ is a piecewise linear function which in the subdomain Ω_k has the form

$$v(x) \equiv v_k(x) = (x - c_k)/\lambda_k, \quad x \in \Omega_k, \quad k = 1, \ldots, m.$$

The values of c_k are defined from the continuity conditions by the formulas

$$c_1 = x_0 = a_1, \quad c_k = x_{k-1} - (x_{k-1} - c_{k-1})\lambda_k/\lambda_{k-1}, \quad k = 2, \ldots, m,$$

which provide the conjugate conditions (3) for the function $\overline{u}$.

The second example presents the boundary value problem with the self-intersected inner plane boundaries $x = x_0$, $y = y_0$, $z = z_0$ which divide the computational domain Ω into its subdomains Ω_k with the constant values of the coefficient $\lambda = \lambda_k$. Here (x_0, y_0, z_0) is some internal point of Ω. In this case the majorant function $\overline{u}$ with necessary properties has the form

$$\overline{u} = \frac{1}{6}\Big[R^2\Big(1 + \frac{R}{\lambda_0}\Big) - (x - x_0)^2 - (y - y_0)^2 - (z - z_0)^2 + (x - x_0)(y - y_0)(z - z_0)/\lambda_k\Big], \quad (x,y,z) \in \Omega_k,$$

where $\lambda_0 = \min\limits_k \lambda_k$ and R is maximum distance between the point (x_0, y_0, z_0) and the external boundary of Ω.

The estimation of $\|z_h\|_H$ can be carried out by analysing the spectral properties of the symmetrized matrix

$$\overline{A} = HA_h = \overline{A}^t,$$

under the assumption that the monotone conditions hold. Since $\overline{A}$ is a Stiltjies matrix in this case, it can be presented by the sum

$$\overline{A} = A_0 + A_1,$$

where A_1 is some symmetric positive semi-definite matrix and A_0 corresponds to the "usual" 7-point approximation of the original mixed boundary value problem on the same nonuniform grid but for the constant coefficient $\lambda \equiv 1$:

$$\lambda_0 = \min_{\Omega} \lambda(x,y,z), \quad A_0 = \min_{\substack{ijk \in \Omega \\ m \in M}} a_m^{ijk} \lambda_0 \overline{A}_0,$$

$$(\overline{A}_0 u)_{ijk} = -a_{ijk}u_{i-1,j,k} - a_{i+1jk}u_{i+1,j,k} - b_{ijk}u_{i,j-1,k} - b_{ij+1k}u_{i,j+1,k} - c_{ijk}u_{i,j,k-1} - c_{ijk+1}u_{i,j,k+1} + d_{ijk}u_{i,j,k},$$

$$a_{ijk} = \frac{h_j h_k}{4h_{i-1}}, \quad b_{ijk} = \frac{h_i h_k}{4h_{j-1}}, \quad c_{ijk} = \frac{h_i h_j}{4h_{k-1}},$$
$$d_{ijk} = a_{ijk} + a_{i+1jk} + b_{ijk} + b_{ij+1k} + c_{ijk} + c_{ijk+1}.$$

In this case the additional conditions are imposed on the coefficients of the system to isolate them from zero:

$$\min_{\substack{ijk\in\Omega \\ m\in M}} a_m^{ijk} \geq \delta > 0,$$

where, for example,

$$\delta = (1-\gamma)B(h_j + h_{j-1})(h_k + h_{k-1})/h_i, \quad 0 < \gamma < 1,$$

i.e., conditions (16) turn into (16′) by changing β to $\beta\gamma$.

The estimations for the eigenvalues μ of such a matrix are well-known (see [9], for example), so we obtain the inequalities

$$\mu(\overline{A}) \geq \mu(A_0) \geq C\gamma \frac{\check{h}^3}{\hat{h}},$$

where $\hat{h}$ and $\check{h}$ mean the maximal and minimal meshsizes correspondingly, and constant C does not depend on h. From equation (17) and by simple transformations we get

$$\|z_h\|_H = (HA_h^{-1}\psi_h, A_h^{-1}\psi_h)^{1/2} \leq \lambda_{max}(\overline{A}^{-1})\lambda_{\max}(H)\|\psi_h\|_H.$$

Now the spectral properties of the matrices $\overline{A}$, H permit us to establish the following

Theorem 2. *Let a solution of the boundary value problem* (1), (2), (3) *and function f have bounded mixed derivatives at the internal points of the computational domain Ω of the fourth and the second order, correspondingly. Let monotonicity conditions* (16′) *be satisfied. Then the inequality*

$$\|z_h\|_H \leq \frac{\overline{C}}{\gamma\lambda_0}\Big(\frac{\hat{h}}{\check{h}}\Big)^4 (\hat{h})^2,$$

where $\overline{C}$ does not depend on h, is valid.

Let us note that to establish the estimate of the weighted Euclidean norm of the error, even the existence of any majorant function is not required.

5. Numerical experiments

In order to demonstrate the validity of the given estimations and efficiency of the constructed approximations, we present the results of numerical experiments for two model problems.

Solution of the algebraic systems was made by the iterative incomplete factorization method [10] with conjugate gradient acceleration and stop criterion

$$\|r^n\|_2/\|r^0\|_2 \le \epsilon = 10^{-8},$$

where n is the number of iteration.

Problem 1. The Laplace equation (1) with $\lambda \equiv 1, f \equiv 0$ is solved in the unit square with the Dirichlet boundary conditions which correspond to two kinds of harmonic polynomials:

$$u(x,y,z) = x^3 - 3xy^2, \quad u(x,y,z) = x^4 - 6x^2y^2 + y^4.$$

The numerical solutions were obtained using 27-point stensil on two sets of embedded grids: uniform and nonuniform. The uniform grids were cubic ones with the number of nodes $L = 11, 21, 41$ in each coordinate direction. The second grid set has the same constant meshsteps in z-direction and $h_i^x = h_i^y = h_0 q^{i-1}$, $i = 1, \dots, L$ with $q = 0.95$.

Tables 1, 2 give the mean square errors $\delta = \|u - u^n\|_2/L^3$ of numerical solutions for different weight parameters.

In Table 1 we present the errors δ for the problem with the exact solution in the form of the fourth order polynomial for the uniform grid.

Table 1. Error δ on the uniform grids

p \ L	11	21	41
1	0.00044	0.00012	0.00030
16/17	0.00088	0.00023	0.000060
32/31	$2.8 \cdot 10^{-9}$	$1.5 \cdot 10^{-9}$	$1.6 \cdot 10^{-9}$

It is easy to see that two approximations have the error $O(h^2)$. For this test with the Mikeladze parameter $p = 32/31$ we get the exact solution, i.e., $\delta \approx \epsilon$. And for the exact solution in the form of the cubic harmonic polynomial all these schemes provide $\delta \approx \epsilon$ in accordance with theoretical estimates. It is interesting to note that in this example the scheme with the optimal parameter $p = 16/17$ (in the sense of the estimate on nonuniform grid) is the worst scheme experimentally.

Table 2 presents the similar tests but on the nonuniform grids. The upper error value in each cell of the table corresponds to the cubic and the low value – to the fourth order polynomial.

Table 2. Error δ on the nonuniform grids

p \ L	11	21	41
1	0.000057 0.00067	0.000031 0.00025	0.000017 0.00011
16/17	10^{-8} 0.00091	10^{-8} 0.00027	10^{-8} 0.000092
32/31	0.00011 0.00044	0.000061 0.00023	0.000033 0.00012

One can make several conclusions from these data.

The unique scheme of the second order on the nonuniform grid is that corresponding to $p = 16/17$. Firstly, the errors for the test with the cubic solution only in this case give $\delta \approx \epsilon$. Secondly, for harmonic polynomial of the fourth order the error dependence on the meshsize is close to $O(h^2)$.

The error dependence on h both for $p = 1$ and $p = 32/31$ is close to the linear one. It is intertesting to mention that the best scheme on the uniform grid – the scheme with $p = 32/31$ – is the worst in this case.

Problem 2. The Poisson equation (1) with $\lambda \equiv 1$, and two right-hand sides

$$f \equiv 0, \quad f = 3\pi^2 \sin(\pi x)\sin(\pi y)\sin(\pi z)$$

is solved in the unit square on the unifor·n grids with Dirichlet boundary conditions which provide the exact solution in the form

$$u(x, y, z) = \sin(\pi x)\sin(\pi y)[\mathrm{sh}(\pi\sqrt{2}z)/\,\mathrm{sh}(\pi\sqrt{2}) + \rho],$$

where $\rho = 0$ for $f = 0$ and $\rho = \sin(\pi z)$ for $f \neq 0$.

Table 3 presents the errors of the test results for 27-point scheme. The upper value in each cell corresponds to $f = 0$ and the low value – to $f \neq 0$.

Table 3. Error on the uniform grids

p \ L	11^3	21^3	41^3
1	0.00062 0.0016	0.00016 0.00047	0.000043 0.00013
16/17	0.0012 0.0035	0.00033 0.00097	0.000085 0.00025
32/31	$1.6 \cdot 10^{-4}$ $1.4 \cdot 10^{-4}$	$1.0 \cdot 10^{-6}$ $9.9 \cdot 10^{-6}$	$9.4 \cdot 10^{-8}$ $6.2 \cdot 10^{-7}$

So, the error is $O(h^2)$ for the first two schemes and $O(h^4)$ for parameter $p = 32/31$.

The experiments have shown that the monotonicity conditions (16) have appeared to be a too strong restriction to use the incomplete factorization method (e.g., for the mesh parameter $q = 0.9$ the iterative process diverges), and the question of solvability of the system in this case is still open.

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