Bull. Nov. Comp. Center, Comp. Science, 28 (2008), 55–66

Categorical modelling of trace equivalence for timed automata models with invariants

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Abstract.

Formal models for real-time systems have been actively studied over the past several years. Much of the theory of untimed systems has been lifted to the real-time setting. An example is the notion of trace equivalence applied to timed transition systems with invariants which is studied here within the general categorical framework of open maps. In particular, we demonstrate how to characterize standard timed trace equivalence in terms of spans of open maps with a natural choice of a path category.

1. Introduction

During the last years various timed extensions of concurrent models have been actively studied in order to handle quantitative aspects of the behaviour of concurrent and real-time systems. Much of the theory of untimed systems has been lifted successfully to these real-time models. For example, many results from automata theory have been transferred to timed automata models [3, 4, 1]. The real-time models often include timed versions of equivalences on concurrent processes, e.g., timed bisimilarity equivalence, timed testing equivalence and timed trace equivalence.

In this paper we focus on a timed variant of trace equivalence which has already been treated for real-time models by many researchers (see, for instance, [2, 21, 22]). Our aim here is to apply the general categorical framework of open maps [14] to an extension of timed transition systems by invariants which have to be true in all reachable states of the model. Such invariants represent global properties of timed transition systems and may be used for specification and verification of systems and processes.

The general idea of the open maps approach is to formalize that one categorical model of computation is more expressive than another in terms of embeddings. This approach also provides a general concept of abstract bisimilarity for any categorical model in terms of spans of so-called open maps, which are those morphisms which, roughly speaking, reflect and preserve behavior. Formally, the definition of open maps is parameterized not just on a categorical presentation of a model (i.e., on the choice of morphisms), but also on the notion of a computation path and what does it mean to extend a computation path by another one. This abstract definition of bisimilarity makes possible a uniform definition of an equivalence over different models (see [14, 18, 9, 10]) and allows one to apply general results from the categorical setting (e.g., the existence of canonical models and characteristic games and logics) to concrete behavioural equivalences.

The outline of the paper is as follows. The basic notions and notations related to the open maps approach are defined in Section 2. In Section 3, we introduced a model of timed transition systems with invariants, given a category of timed transition systems with invariants and its subcategory from [13], and represent some properties of this category. Next, in Section 4, we show how timed trace equivalence can be captured by the open maps approach. Finally, Section 5 contains conclusions and remarks on the future work.

2. Introduction to open maps

In this section we briefly recall the basic definitions from the category theory.

The notion of a category was introduced by S. Eilenberg and S. Mac Lane in 1944 in connection with the problem of axiomatization of the group theory of homologies and cohomologies of the topological spaces. This notion gradually found its use in the applied fields of mathematics as well.

Definition 1. *We say that the category* $\mathbf{M}$ *is assigned, if we have:*

- *a set* $\mathbf{M}$*, the elements of which we call the* objects of the category*,*
- *a set* $\mathbf{M}(X, Y)$*, the elements of which we call the* morphisms *from* X *to* Y*,*
- *a rule of composition:* $\circ : \mathbf{M}(X, Y) \times \mathbf{M}(Y, Z) \longrightarrow \mathbf{M}(X, Z)$*,*
- *the morphism* $1_X \in \mathbf{M}(X, X)$*, called* identity*,*

satisfied the following axioms:

- *for all* $f \in \mathbf{M}(X, Y)$*,* $g \in \mathbf{M}(Y, Z)$ *and* $h \in \mathbf{M}(Z, V)$*:* $h \circ (g \circ f) = (h \circ g) \circ f$*,*
- *for all* $f \in \mathbf{M}(X, Y)$ *and* $g \in \mathbf{M}(Y, Z)$ *it holds:* $1_Y \circ f = f$ *and* $g \circ 1_Y = g$*.*

One of the most important terms of the category theory is the notion of *open morphisms*. For a given category the open morphisms are defined by a specific subcategory and represent simulations between objects of the category. Let $\mathbf{M}$ be a category and $\mathbf{P} \hookrightarrow \mathbf{M}$ be some subcategory of the category $\mathbf{M}$. Let us give the formal definition of $\mathbf{P}$-open morphisms.

Definition 2. *A morphism $f : X \to Y$ in the category* **M** *is called* **P**-open*, if for any morphism $m : P \to Q$ in the subcategory* **P** *and all morphisms $p : P \to X$, $q : Q \to Y$, $f : X \to Y$ such that $q \circ m = f \circ p$, there exists a morphism $p' : Q \to X$ such that $p' \circ m = p$ and $f \circ p' = q$.*

Note that objects of the category **M** and **P**-open morphisms form a subcategory in the category **M**, because identity morphisms and compositions of **P**-open morphisms are obviously **P**-open.

In [14], the notion of **P**-open morphism was used to define the abstract **P**-bisimilarity in the setting of objects of **M**.

Definition 3. *Two objects X and X' of* **M** *are called* **P**-bisimular *iff there is a span of* **P***-open morphisms $X \xleftarrow{f} X_0 \xrightarrow{f'} X'$.*

To extract a subclass of categories, for which a specific bisimilarity is indeed an equivalence relation, let us define a category with pullbacks.

Definition 4. *A category $\mathcal{M}$ has* pullbacks*, iff for any two morphisms $T_1 \stackrel{\mu_1}{\to} T_0 \stackrel{\mu_2}{\leftarrow} T_2$ there exist T and two morphisms $T_1 \stackrel{\pi_1}{\leftarrow} T \stackrel{\pi_2}{\to} T_2$ such that*

- *$\mu_1 \circ \pi_1 = \mu_2 \circ \pi_2$,*
- *for any other T' and morphisms $T_1 \stackrel{\phi_1}{\leftarrow} T' \stackrel{\phi_2}{\to} T_2$ such that $\mu_1 \circ \phi_1 = \mu_2 \circ \phi_2$, there exists a unique morphism $\xi : T' \to T$ such that $\phi_i = \pi_i \circ \xi$ ($i = 1, 2$).*

The following proposition shows that if the category **M** has pullbacks, then **P**-bisimilarity is always an equivalence relation.

Proposition 1. [14] *Pullbacks of* **P***-open morphisms are* **P***-open.*

3. Timed transition systems with invariants

In this section we describe a model of timed transition systems with invariants, a category for this model proposed in the paper [13], and consider the properties of the category. Furthermore, this section contains a number of useful definitions and notations.

As a model for real time systems, we use timed transition systems with invariants [16] which are an extension of timed transition systems and are a basic interleaving model for concurrent and real-time systems.

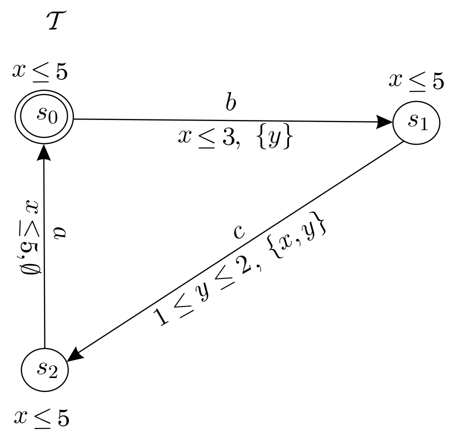

Figure 1. Timed transition system with invariants $\mathcal{T}$

Definition 5. A timed system $\mathcal{T}$ *is a six-tuple* $(S, \Sigma_{\mathcal{T}}, s_0, X, T, I)$*, where*

- S *is a set of states, and* s_0 *is the initial state,*
- Σ *is a finite alphabet of actions,*
- X *is a set of clock variables (clocks),*
- T *is a set of transitions such that* $T \subseteq S \times\ \Sigma_{\mathcal{T}} \times \Delta \times 2^X \times S$*. Here* Δ *is a clock constraint generated by the following grammar* $\Delta ::= c\#x \mid x + c\ \#y \mid \Delta \wedge \Delta$*, where* $\# \in \{\leq, <, \geq, >, =\}$*,* c *is a real valued constant, and* x, y *are clock variables,*
- I *assigns an invariant that is given by the same syntax as clock constraints to each state; thus the invariant for a state* s*,* ι_s*, can be generated by the grammar* Δ*.*

A transition $(s, \sigma, \delta, \lambda, s')$ *is denoted by* $s \underset{\delta,\ \lambda}{\overset{\sigma}{\rightarrow}} s'$*.*

Example 1. *For the timed transition system* $\mathcal{T}$ *depicted in Figure 1, we have the following: the alphabet of actions* Σ^1 *consists of three actions* a*,* b*, and* c*, and the set of clocks* X_1 *includes two clocks* x *and* y*.* $\diamondsuit$

In order to explain the behavior of a timed transition system with invariants, we define some useful notions and notations. Let $\mathbf{R}^+$ be the set of real nonnegative numbers and $\mathbf{N}$ be the set of natural numbers. For all $n \in \mathbf{N}$ we define the set $\mathbf{R}^n$ as the Cartesian product of n sets $\mathbf{R}^+$.

Definition 6. A timed word *over an alphabet* Σ *is a finite sequence of pairs* $\alpha = (\sigma_1, d_1)\ (\sigma_2, d_2)\ (\sigma_3, d_3)\ \dots\ (\sigma_n, d_n)$*, where for all* $1 \le i \le n$ $\sigma_i \in \Sigma$, $d_i \in \mathbf{R}^+$ *and for all* $1 \le i \le n-1$ $d_i \le d_{i+1}$.

A pair (σ, d) represents an occurrence of action σ at time d relative to starting time 0.

Definition 7. A clock evaluation *is a function* $\nu : X \to \mathbf{R}^+$ *which assigns times to the clock variables of a system. We define* $(\nu + c)(x) := \nu(x) + c$ *for all clock variables* $x \in X$*. If* λ *is a set of clocks, then*

$$\nu[\lambda \to 0](x) = \begin{cases} 0 & \text{if } x \in \lambda, \\ \nu(x) & \text{otherwise.} \end{cases}$$

A constraint δ is satisfied by a clock evaluation ν iff the expression $\delta[\nu(x)/x]$ evaluates to true. Here $\delta[y/x]$ is a syntactic substitution of y for x in δ. A constraint δ defines a subset of $\mathbf{R}^m$ (m is the number of clocks in the set X). We will call this subset the meaning of δ and denote it by $\|\delta\|_X$. The meaning of an invariant ι_s, $\|\iota_s\|_X$, is defined in the same way as the meaning of a constraint δ. A clock evaluation ν defines a point in $\mathbf{R}^m$ (denoted by $\|\nu\|_X$). Thus, the constraint δ is satisfied by the clock evaluation ν iff $\|\nu\|_X \in \|\delta\|_X$.

Definition 8. *Let* $\mathcal{T} = (S, \Sigma_\tau, s_0, X, T, I)$ *be a timed transition system with invariants.* A configuration *of* $\mathcal{T}$ *is a pair* $\langle s, \nu\rangle$*, where* s *is a state and* ν *is a clock evaluation. The configuration* $C_0(\mathcal{T}) = \langle s_0, \nu_0\rangle$*, where* ν_0 *is the constant* 0 *function, is called* the initial configuration*. We define the set of all configurations of* $\mathcal{T}$ *as* $Conf(\mathcal{T})$.
We say that $\mathcal{T}$ *can make a run* $\langle s_0, \nu_0\rangle \overset{\sigma_1}{\underset{d_1}{\to}} \langle s_1, \nu_1\rangle \overset{\sigma_2}{\underset{d_2}{\to}} \dots \overset{\sigma_n}{\underset{d_n}{\to}} \langle s_n, \nu_n\rangle$ *iff for all* $i > 0$ *there exists a transition* $s_{i-1} \overset{\sigma_i}{\underset{\delta_i,\ \lambda_i}{\to}} s_i$ *such that* $\|\nu_{i-1} + (d_i - d_{i-1})\|_X \in \|\delta_i\|_X$*, for all* $\tau \in [0,\ (d_i - d_{i-1}))$ *it holds that* $\|\nu_{i-1} + \tau\|_X \in \|\iota_{s_{i-1}}\|_X$ *and* $\nu_i = (\nu_{i-1} + (d_i - d_{i-1}))[\lambda_i \to 0]$ *(here* s_0 *is the initial state,* ν_0 *is the constant* 0 *function and* $d_0 = 0$*) and for the last state* $\|\nu_n\|_X \in \|\iota_{s_n}\|_X$*. The timed word* $\alpha = (\sigma_1, d_1)\ (\sigma_2, d_2)\ \dots (\sigma_n, d_n)$ *is generated by this run.*

In the paper [13] the authors constructed the category of the timed transition systems with invariants, $\mathcal{CTTS}^{\iota}_{\Sigma}$, which consists of the timed transition systems with invariants and morphisms between them. Let us give the definition of a morphism between two timed transition systems with invariants.

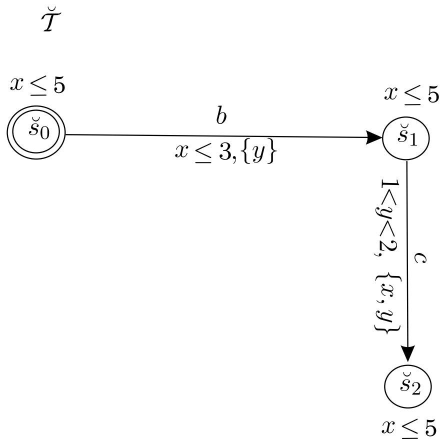

Figure 2. The timed transition system with invariants $\breve{\mathcal{T}}$

Definition 9. [13] *A morphism (μ, η) between timed transition systems with invariants $\mathcal{T}_1 = (S_1, \Sigma, s_1^0, X_1, T_1, I_1)$ and $\mathcal{T}_2 = (S_2, \Sigma, s_2^0, X_2, T_2, I_2)$ consists of two components: a map $\mu : S_1 \to S_2$ between the states and a map $\eta : X_2 \to X_1$ between the clocks. These maps must satisfy $\mu(s_1^0) = s_2^0$, for all $s_1 \in S_1$ $\|\iota_{s_1}\|_{X_1} \subseteq \|\iota_{\mu(s_1)}[\eta(x)/x]\|_{X_1}$, and whenever there is a transition in $\mathcal{T}_1$ of the form $s_1 \underset{\delta_1,\ \lambda_1}{\overset{\sigma}{\to}} s_1'$, there is a transition $\mu(s_1) \underset{\delta_2,\ \lambda_2}{\overset{\sigma}{\to}} \mu(s_1')$ in T_2 satisfying the following two constraints:*

1. $\lambda_2 = \eta^{-1}(\lambda_1)$, *where* $\eta^{-1}(\lambda_1) = \{x \in X_2 \mid \eta(x) \in \lambda_1\}$,
2. $\|\delta_1\|_{X_1} \subseteq \|\delta_2[\eta(x)/x]\|_{X_1}$.

Example 2. *It is easy to check that the pair of maps (μ, η) such that $\mu_1(\breve{s}_i) = s_i$, $(0 \leq i \leq 2)$ and $\eta_1(x) = x$, $\eta_1(y) = y$ is a morphism from the timed transition system with invariants $\breve{\mathcal{T}}$ in Figure 2 to the timed transition system with invariants $\mathcal{T}$ in Figure 1.* ♢

Let us introduce an auxiliary notation. For a function $\eta : X' \to X$ and a clock valuation $\nu : X \to \mathbf{R}^+$, we define $\eta^{-1}(\nu) : X' \to \mathbf{R}^+$ as follows: $\eta^{-1}(\nu)(x') := \nu(\eta(x'))$.

Consider a useful simulation property of a morphism, proved in [13].

Theorem 1. [13] *Given timed transition systems with invariants $\mathcal{T}$ and $\mathcal{T}'$ with alphabet Σ and a morphism (μ, η) between $\mathcal{T}$ and $\mathcal{T}'$, whenever $\langle s_0, \nu_0 \rangle \underset{\tau_1}{\overset{\sigma_1}{\to}} \langle s_1, \nu_1 \rangle \ldots \langle s_{n-1}, \nu_{n-1} \rangle \underset{\tau_n}{\overset{\sigma_n}{\to}} \langle s_n, \nu_n \rangle$ is a run of $\mathcal{T}$ generating the* *timed word* $(\sigma_1, \tau_1) \ldots (\sigma_n, \tau_n)$*, then* $\langle \mu(s_0), \eta^{-1}(\nu_0) \rangle \overset{\sigma_1}{\underset{\tau_1}{\rightarrow}} \langle \mu(s_1), \eta^{-1}(\nu_1) \rangle \ldots \langle \mu(s_{n-1}), \eta^{-1}(\nu_{n-1}) \rangle \overset{\sigma_n}{\underset{\tau_n}{\rightarrow}} \langle \mu(s_n), \eta^{-1}(\nu_n) \rangle$ *is a run of* $\mathcal{T}'$ *generating the same word.*

Timed transition systems with invariants under an alphabet Σ and morphisms between them form a category of timed transition systems with invariants, $\mathcal{CTTS}^{\iota}_{\Sigma}$, in which the composition of two morphisms $(\mu, \eta) : \mathcal{T} \longrightarrow \mathcal{T}'$ and $(\mu', \eta') : \mathcal{T}' \longrightarrow \mathcal{T}''$ is defined as $(\mu', \eta') \circ (\mu, \eta) := (\mu' \circ \mu, \eta \circ \eta')$, and the identity morphism is the morphism where both μ and η are the identity functions. As proved in [13], $\mathcal{CTTS}^{\iota}_{\Sigma}$ is indeed a category which has pullbacks and binary products.

4. Timed trace equivalence and open maps

In this section we first introduce the notion of timed trace equivalence in the setting of timed transition systems with invariants and then show how the equivalence can be captured by open maps.

Definition 10. *For a timed transition system with invariants* $\mathcal{T}$*, the set* $L(\mathcal{T}) = \{ \alpha = (\sigma_1, d_1) \ldots (\sigma_n, d_n) \mid \mathcal{T}$ *can make a run* $\langle s_0, \nu_0 \rangle \overset{\sigma_1}{\underset{d_1}{\rightarrow}} \langle s_1, \nu_1 \rangle \overset{\sigma_2}{\underset{d_2}{\rightarrow}} \ldots \overset{\sigma_n}{\underset{d_n}{\rightarrow}} \langle s_n, \nu_n \rangle \}$ *is called the* language *of the timed transition system* $\mathcal{T}$*.*

Example 3. *The set* $L(\breve{\mathcal{T}}) = \{\epsilon, (b, t_1), (b, t_1)(c, t_2) \mid t_1 \leq 3,\ 1 < t_2 - t_1 < 2\}$ *is the language of the timed transition system* $\breve{\mathcal{T}}$*, depicted in Figure 2.* $\diamondsuit$

Now we define the notion of a trace equivalence for timed transition systems with invariants.

Definition 11. *Timed transition systems with invariants* $\mathcal{T}$ *and* $\mathcal{T}'$ *are called* timed trace equivalent *iff* $L(\mathcal{T}) = L(\mathcal{T}')$*.*

Example 4. *Consider the timed transition systems with invariants shown in Figure 1, 2 and 3. The timed transition systems with invariants* $\widehat{\mathcal{T}}$ *and* $\bar{\mathcal{T}}$ *are timed trace equivalent, while the timed transition systems with invariants* $\mathcal{T}$ *and* $\breve{\mathcal{T}}$ *are not, because, for example, we have* $(b, 1)(c, 2) \in L(\mathcal{T})$ *but this is not the case for the timed transition system with invariants* $\breve{\mathcal{T}}$*.* $\diamondsuit$

Next, following the paper [14], we construct a subcategory of observations of the category $\mathcal{CTTS}^{\iota}_{\Sigma}$.

We choose timed words over Σ as "observation objects".

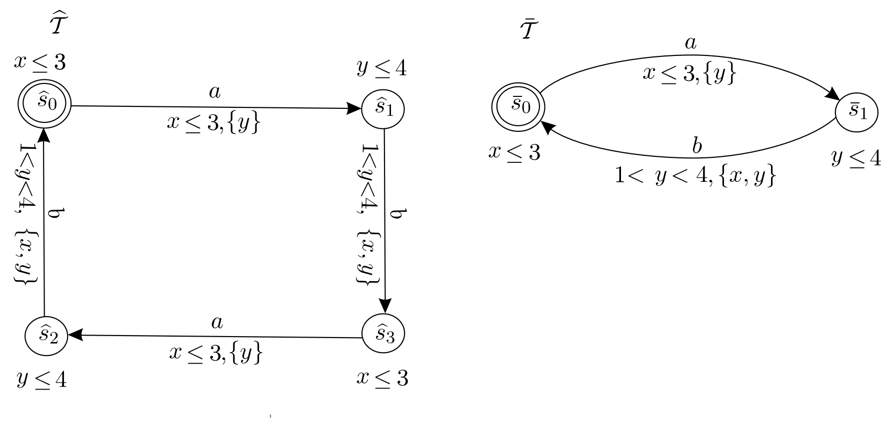

Figure 3. The trace equivalent timed transition systems with invariants

Definition 12. [13] *Given a timed word $\alpha = (\sigma_1, \tau_1) \ldots (\sigma_n, \tau_n)$, we define a timed transition system with invariants $\mathcal{T}^\alpha = (S^\alpha, 0, \Sigma, V^\alpha, T^\alpha, I^\alpha)$ corresponding to α as follows:*

$$0 \xrightarrow[\delta_1, \lambda_1]{\sigma_1} 1 \ldots (n-1) \xrightarrow[\delta_n, \lambda_n]{\sigma_n} n,$$

i.e. the states are the integers $0, 1, \ldots, (n-1), n$, with 0 as the initial state, and the set of clock variables, V^α, consists of the 2^n subsets of states $\{1, 2, \ldots, n\}$. In addition, we define $\lambda_i = \{x \mid i \in x\}$ and $\delta_i = \bigwedge_{x \in V^\alpha} (x = \tau_i - \tau_{I(i,x)})$, where $I(i,x) := \max\{k \in x \cup \{0\} \mid k < i\}$ and $\tau_0 := 0$. The index returned by $I(i,x)$ is the index of the last state at which x was reset. The invariants are defined inductively to be of the form $\bigwedge_{x \in X} (c_x \leq x < c'_x)$. The initial invariant is $\bigwedge_{x \in X} (0 \leq x < \tau_1)$. Assume that the invariant on the state $(i-1)$ is $\bigwedge_{x \in X} (c_x^{i-1} \leq x < \widetilde{c}_x^{i-1})$, then the invariant on state i is $\bigwedge_{x \in X} (\text{ if } x \in \lambda_i \text{ then } (0 \leq x < \widetilde{\tau}_i) \text{ else } (\widetilde{c}_x^{i-1} \leq x < \widetilde{c}_x^{i-1} + \widetilde{\tau}_i))$, where $\widetilde{\tau}_i = \tau_i - \tau_{i-1}$. The constraint on the final state is $\bigwedge_{x \in X} (\text{ if } x \in \lambda_i \text{ then } (x = 0) \text{ else } (x = \widetilde{c}_x^{i-1}))$.
The class of timed transition systems of the form $\mathcal{T}^\alpha$ is denoted as $\mathcal{TTS}_\Sigma^\alpha$.

With respect to the set of actions Σ, let $\mathcal{P}_\Sigma^*$ denote the full subcategory of the category $\mathcal{CTTS}_\Sigma^\iota$ with objects from $\mathcal{TTS}_\Sigma^\alpha$ and with identity morphisms and morphisms with $\mathcal{T}_\epsilon$[1] as a domain.

Given the category $\mathcal{CTTS}_\Sigma^\iota$ and subcategory $\mathcal{P}_\Sigma^*$, we can now apply the general framework from [14] to define the alternative notion of open maps.

[1] $\mathcal{T}_\epsilon$ denotes the timed transition system corresponding to the empty timed word ϵ.

Lemma 1. *A morphism* $(\mu,\eta):\mathcal{T}\to\mathcal{T}'$ *is* $\mathcal{P}^*_\Sigma$-open *iff for all morphisms* $(\mu_1,\eta_1):\mathcal{O}\to\mathcal{T}'$ *with* $\mathcal{O}\in\mathcal{P}^*_\Sigma$ *there exists a morphism* $(\mu',\eta'):\mathcal{O}\to\mathcal{T}$ *such that* $(\mu_1,\eta_1)=(\mu,\eta)\circ(\mu',\eta')$.

Our next aim is to characterize $\mathcal{P}^*_\Sigma$-openness of a morphism relative to the corresponding subcategory of observations defined as above.

Theorem 2. *A morphism between* $\mathcal{T}$ *and* $\mathcal{T}'$ *is* $\mathcal{P}^*_\Sigma$*-open iff whenever* $\langle s'_0,\nu'_0\rangle \overset{\sigma_1}{\underset{\tau_1}{\to}} \langle s'_1,\nu'_1\rangle \ldots \langle s'_{n-1},\nu'_{n-1}\rangle \overset{\sigma_n}{\underset{\tau_n}{\to}} \langle s'_n,\nu'_n\rangle$ *is a run of* $\mathcal{T}'$ *generating the timed word* $(\sigma_1,\tau_1)\ldots(\sigma_n,\tau_n)$*, there exists a run of* $\mathcal{T}$ $\langle s_0,\nu_0\rangle \overset{\sigma_1}{\underset{\tau_1}{\to}} \langle s_1,\nu_1\rangle \ldots \langle s_{n-1},\nu_{n-1}\rangle \overset{\sigma_n}{\underset{\tau_n}{\to}} \langle s_n,\nu_n\rangle$ *generating the same word and satisfying the following conditions:* $s'_i=\mu(s_i)$ *and* $\nu'_i=\eta^{-1}(\nu_i)$ *for all* $0\le i\le n$.

Proof. ($\Rightarrow$) Let $(\mu,\eta):\mathcal{T}\to\mathcal{T}'$ be a $\mathcal{P}^*_\Sigma$-open morphism. Assume that $\langle s'_0,\nu'_0\rangle \overset{\sigma_1}{\underset{\tau_1}{\to}} \langle s'_1,\nu'_1\rangle \ldots \langle s'_{n-1},\nu'_{n-1}\rangle \overset{\sigma_n}{\underset{\tau_n}{\to}} \langle s'_n,\nu'_n\rangle$ is a run of $\mathcal{T}'$ generating the timed word $\alpha=(\sigma_1,\tau_1)\ldots(\sigma_n,\tau_n)$. According to Theorem 4 from [13], for this run there exists a morphism $(\mu_1,\eta_1):\mathcal{T}_\alpha\to\mathcal{T}'$ in $\mathcal{CTTS}^\iota_\Sigma$ such that $\mu_1(i)=s'_i$ $(i=0..n)$ and $\eta_1(x)=\{i\mid(1\le i\le n)\wedge(\nu'_i(x)=0)\}$. By Lemma 1 we have a morphism $(\mu_2,\eta_2):\mathcal{T}_\alpha\to\mathcal{T}$ in $\mathcal{CTTS}^\iota_\Sigma$ such that $(\mu_1,\eta_1)=(\mu,\eta)\circ(\mu_2,\eta_2)$, because (μ,η) is a- $\mathcal{P}^*_\Sigma$-open morphism. Now, by Theorem 4 from [13], we can find a run of $\mathcal{T}$ $\langle s_0,\nu_0\rangle \overset{\sigma_1}{\underset{\tau_1}{\to}} \langle s_1,\nu_1\rangle \ldots \langle s_{n-1},\nu_{n-1}\rangle \overset{\sigma_n}{\underset{\tau_n}{\to}} \langle s_n,\nu_n\rangle$ for morphism (μ_2,η_2) such that $\mu_2(i)=s_i$ $(i=0..n)$ and $\eta_2(x)=\{i\mid(1\le i\le n)\wedge(\nu_i(x)=0)\}$. It means that $s'_i=\mu(s_i)$ and $\nu'_i=\eta^{-1}(\nu_i)$ for all $0\le i\le n$.

($\Leftarrow$) For the if part of the theorem assume that $(\mu_1,\eta_1):\mathcal{T}_\alpha\to\mathcal{T}'$ is a morphism from the category $\mathcal{CTTS}^\iota_\Sigma$. By Theorem 4 from [13], we can find a run of $\mathcal{T}'$ $\langle s'_0,\nu'_0\rangle \overset{\sigma_1}{\underset{\tau_1}{\to}} \langle s'_1,\nu'_1\rangle \ldots \langle s'_{n-1},\nu'_{n-1}\rangle \overset{\sigma_n}{\underset{\tau_n}{\to}} \langle s'_n,\nu'_n\rangle$ generating the timed word $(\sigma_1,\tau_1)\ldots(\sigma_n,\tau_n)$ such that $\mu_1(i)=s'_i$ $(i=0,\ldots,n)$ and $\eta_1(x)=\{i\mid(1\le i\le n)\wedge(\nu'_i(x)=0)\}$. Now, from the assumptions of the theorem, there exists a run of $\mathcal{T}$ $\langle s_0,\nu_0\rangle \overset{\sigma_1}{\underset{\tau_1}{\to}} \langle s_1,\nu_1\rangle \ldots \langle s_{n-1},\nu_{n-1}\rangle \overset{\sigma_n}{\underset{\tau_n}{\to}} \langle s_n,\nu_n\rangle$ generating the same timed word such that $s'_i=\mu(s_i)$ and $\nu'_i=\eta^{-1}(\nu_i)$ for all $0\le i\le n$. By Theorem 4 from [13], this implies an existence of a morphism $(\mu_2,\eta_2):\mathcal{T}_\alpha\to\mathcal{T}$ in $\mathcal{CTTS}^\iota_\Sigma$ such that $\mu_2(i)=s_i$ $(i=0,\ldots,n)$ and $\eta_2(x)=\{i\mid(1\le i\le n)\wedge(\nu_i(x)=0)\}$. Thus, we have $(\mu_1,\eta_1)=(\mu,\eta)\circ(\mu_2,\eta_2)$. Now, from Lemma 1, we conclude that (μ,η) is $\mathcal{P}^*_\Sigma$-open. □

Example 5. *According to Theorem 2, the morphism* (μ,η) *defined in Example 2 is not* $\mathcal{P}^*_\Sigma$*-open, because for the run of* $\mathcal{T}$ $\langle s_0,\nu_0\rangle \overset{b}{\underset{1}{\to}} \langle s_1,\nu_1\rangle \overset{c}{\underset{2}{\to}} \langle s_2,\nu_2\rangle$ *(where* $\nu_0(x)=\nu_0(y)=\nu_1(y)=\nu_2(x)=\nu_2(y)=0$ *and* $\nu_1(x)=1$*),* *generating the timed word* $(b,1)(c,2)$*, there is no run of* $\breve{\mathcal{T}}$*, generating the same timed word.* $\diamondsuit$

From [14] and by the fact that $\mathcal{CTTS}^{\iota}_{\Sigma}$ has pullbacks [13], we can conclude that $\mathcal{P}^{*}_{\Sigma}$-bisimilarity is an equivalence relation generated by open maps.

Now we can present our main result.

Theorem 3. *Two timed transition systems* $\mathcal{T}$ *and* $\mathcal{T}'$ *are* $\mathcal{P}^{*}_{\Sigma}$*-bisimilar iff they are timed trace equivalent.*

Proof. ($\Rightarrow$) Let $\mathcal{T}$ and $\mathcal{T}'$ be $\mathcal{P}^{*}_{\Sigma}$-bisimular timed transition systems with invariants. This means that there exists a span of $\mathcal{P}^{*}_{\Sigma}$-open morphisms $\mathcal{T} \overset{(\mu,\eta)}{\leftarrow} \mathcal{T}^{*} \overset{(\mu',\eta')}{\rightarrow} \mathcal{T}'$ with a vertex $\mathcal{T}^{*}$. By Theorems 1 and 2, it is easy to check that $L(\mathcal{T}) = L(\mathcal{T}^{*})$ and $L(\mathcal{T}') = L(\mathcal{T}^{*})$. Thus, we have $L(\mathcal{T}) = L(\mathcal{T}')$.

($\Leftarrow$) Let $\mathcal{T}$ and $\mathcal{T}'$ be timed trace equivalent timed transition systems with invariants. According to [13], $\mathcal{CTTS}^{\iota}_{\Sigma}$ has binary products. Thus, we have the timed transition system with invariants $\mathcal{T} \times \mathcal{T}'$, that is a binary product of $\mathcal{T}$ and $\mathcal{T}'$, and two projecting morphisms $(\mu,\eta) : \mathcal{T} \times \mathcal{T}' \to \mathcal{T}$ and $(\mu',\eta') : \mathcal{T} \times \mathcal{T}' \to \mathcal{T}'$. In order to complete this proof, we need to demonstrate that (μ,η) and (μ',η') are $\mathcal{P}^{*}_{\Sigma}$-open morphisms. Assume $\langle s_0,\nu_0\rangle \underset{\tau_1}{\overset{\sigma_1}{\rightarrow}} \langle s_1,\nu_1\rangle \underset{\tau_2}{\overset{\sigma_2}{\rightarrow}} \ldots \underset{\tau_n}{\overset{\sigma_n}{\rightarrow}} \langle s_n,\nu_n\rangle$ is a run of $\mathcal{T}$, generating the timed word $\alpha = (\sigma_1,\tau_1)\ldots(\sigma_n,\tau_n)$. Since $L(\mathcal{T}) = L(\mathcal{T}')$, there exists a run $\langle s'_0,\nu'_0\rangle \underset{\tau_1}{\overset{\sigma_1}{\rightarrow}} \langle s'_1,\nu'_1\rangle \underset{\tau_2}{\overset{\sigma_2}{\rightarrow}} \ldots \underset{\tau_n}{\overset{\sigma_n}{\rightarrow}} \langle s'_n,\nu'_n\rangle$ of $\mathcal{T}'$ generating the same timed word α. By Theorem 4 from [13], this implies existence of morphisms $(\mu_1,\eta_1) : \mathcal{T}_{\alpha} \to \mathcal{T}$ and $(\mu_2,\eta_2) : \mathcal{T}_{\alpha} \to \mathcal{T}'$. From the definition of binary products (see [20]), we get a morphism $(\mu_3,\eta_3) : \mathcal{T}_{\alpha} \to \mathcal{T} \times \mathcal{T}'$ such that $(\mu_1,\eta_1) = (\mu,\eta) \circ (\mu_3,\eta_3)$ and $(\mu_2,\eta_2) = (\mu',\eta') \circ (\mu_3,\eta_3)$. From this diagram and by Theorems 1 and 2, it follows that (μ,η) is a $\mathcal{P}^{*}_{\Sigma}$-open morphism. In a similar way we can show that (μ',η') is a $\mathcal{P}^{*}_{\Sigma}$-open morphism as well. $\square$

5. Conclusion

In this paper, we have tried to review Joyal, Nielsen, and Winskel's theory of open maps [14] to provide a timed variant of a well-known trace equivalence in the category of timed transition systems with invariants. In particular, we have developed a categorical characterization of the timed trace equivalence. In the future, we hope to extend the obtained results to other observational equivalences (e.g., equivalences taking into account internal actions, etc.) and to other classes of timed models (e.g., time Petri nets, networks of timed automata, etc.). In particular, relying on the paper [14], we contemplate to adapt the unfolding methods for time Petri nets from [6] and open maps based characterizations for timed event structures from [22] to transfer the general concept of bisimilarity to the timed models.

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