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On one problem for the statics equations of elastic-deformed porous media in a half-plane*

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One of the possible statements of a boundary value problem for a system of equations of porous media is considered. The Carleman formula for the solution of this problem is obtained.

1. Introduction

In [1], A.P. Soldatov has shown that the representation by A.V. Bitsadze [2] for the solutions of elliptic systems of second order with the constant coefficients

$$A\partial_{xx}\boldsymbol{w}(x,y) + B\partial_{xy}\boldsymbol{w}(x,y) + C\partial_{yy}\boldsymbol{w}(x,y) = 0,$$

can be written down in the form

$$\boldsymbol{w}(x,y) = \operatorname{Re}\Theta\boldsymbol{u}(z), \quad z = x + iy.$$

Here $\partial_{xy} = \frac{\partial^2}{\partial x \partial y}$, $\boldsymbol{w} = (w_1, \ldots, w_n)$; Θ and A are matrices expressed through coefficients of the system under study, $\boldsymbol{u}(z)$ is A-analytic function [3]:

Definition 1. A function vector $\boldsymbol{u}(z) \in C^1(\Omega, X)$ is called A-analytic in $\Omega \subset \mathbb{C}$ if

$$\partial_{\bar{z}}\boldsymbol{u}(z) - A\partial_z\boldsymbol{u}(z) = 0, \quad z \in \Omega.$$

Here $C^k(\Omega, X)$ is a class of the functions determined in the area Ω, with values in the Banach space X and k-time strongly continuously differentiated in Ω, X is a Hilbert space which is the complexification of a real Hilbert space X': $X = X' \oplus iX'$.

In [4, 5], the solutions of the Cauchy problem for a system of equations of the elasticity theory in a half-plane with the help of the Carleman functions were obtained.

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In the given paper, one of the possible statements of the direct problem for a system of equations of elastic-deformed porous media in the half-plane $\Pi = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$ is proposed. The formula of the solution of this problem is obtained, using the method of applications of a solution of the Cauchy problem for the Laplace equation and the system of equations of the elasticity theory in a half-plane proposed in [5–7].

2. Statement of the problem

Let us consider a problem of the Cauchy type for the Dorovsky system of equations in the half-plane Π: it is required to find the vector of displacement of an elastic porous body $\boldsymbol{U} \in C^3(\Pi; R^2) \cap C^2(\overline{\Pi}; R^2)$ and the pore pressure $P \in C^2(\Pi) \cap C^1(\overline{\Pi})$ from the following expressions [8, 9]:

$$\mu \Delta \boldsymbol{U} + \Big(\lambda + \mu + \alpha\rho_s^2 - 2K\frac{\rho_s}{\rho}\Big)\nabla \operatorname{div} \boldsymbol{U} - \rho_l\Big(\frac{K}{\rho} - \rho_s\alpha\Big)\nabla \operatorname{div} \boldsymbol{V} = 0, \quad (1)$$

$$\Big(\frac{K}{\rho} - \rho_s\alpha\Big)\nabla \operatorname{div} \boldsymbol{U} - \rho_l\alpha\nabla \operatorname{div} \boldsymbol{V} = 0, \quad (2)$$

$$\boldsymbol{U}|_M = \boldsymbol{U}_0(x), \quad T_\partial(\boldsymbol{U}, \boldsymbol{V})|_M = \boldsymbol{U}_1(x), \quad \frac{\rho_l}{\rho}P\Big|_M = P_0(x). \quad (3)$$

In formulas (1)–(3): $\boldsymbol{V}$ is the vector of displacement of fluid with the partial density ρ_l, $M = (t_1, t_2) \subset \partial\Pi$, λ, μ, $\alpha = \rho_0\alpha_3 + K/\rho_0^2$ are the constants from the equation of state [10], $K = \lambda + 2\mu/3$, $\rho = \rho_s + \rho_l$, ρ_s is the partial density of an elastic porous body, $T_\partial(\boldsymbol{U}, \boldsymbol{V}) = (T_{1\partial}(\boldsymbol{U}, \boldsymbol{V}), T_{2\partial}(\boldsymbol{U}, \boldsymbol{V}))$,

$$\begin{aligned} T_{1\partial}(\boldsymbol{U}, \boldsymbol{V}) &= 2\mu\Big(n_1\frac{\partial U_1}{\partial x} + n_2\frac{\partial U_1}{\partial y}\Big) - \mu n_2\Big(\frac{\partial U_1}{\partial y} - \frac{\partial U_2}{\partial x}\Big) + \\ &\quad \Big(\lambda + \rho_s^2\alpha - \frac{2\rho_s}{\rho}K\Big)n_1 \operatorname{div} \boldsymbol{U} - \frac{\rho_l}{\rho}(K - \rho\rho_s\alpha)n_1 \operatorname{div} \boldsymbol{V}, \\ T_{2\partial}(\boldsymbol{U}, \boldsymbol{V}) &= 2\mu\Big(n_1\frac{\partial U_2}{\partial x} + n_2\frac{\partial U_2}{\partial y}\Big) + \mu n_1\Big(\frac{\partial U_1}{\partial y} - \frac{\partial U_2}{\partial x}\Big) + \\ &\quad \Big(\lambda + \rho_s^2\alpha - \frac{2\rho_s}{\rho}K\Big)n_2 \operatorname{div} \boldsymbol{U} - \frac{\rho_l}{\rho}(K - \rho\rho_s\alpha)n_2 \operatorname{div} \boldsymbol{V}, \end{aligned}$$

$\boldsymbol{n} = (n_1, n_2)$ is a unit vector exterior to $\partial\Pi$-normal.

3. The formula of the solution

Let us exclude $\operatorname{div} \boldsymbol{V}$ from (1), using the pore pressure definition ($P = (K - \alpha\rho\rho_s) \operatorname{div} \boldsymbol{U} - \alpha\rho\rho_l \operatorname{div} \boldsymbol{V}$). After a simple transformation we will obtain the Cauchy problem for the system of equations of the elasticity theory with respect to $\boldsymbol{U}$:

$$\mu \Delta \boldsymbol{U} + (\tilde{\lambda} + \mu) \nabla \operatorname{div} \boldsymbol{U} = 0. \tag{4}$$

$$\boldsymbol{U}|_M = \boldsymbol{U}_0(x), \quad \tilde{T}_\partial \boldsymbol{U}|_M = \tilde{\boldsymbol{U}}_1(x) \tag{5}$$

where

$$\tilde{T}_\partial = \begin{pmatrix} (\tilde{\lambda}+2\mu)n_1 & \mu n_2 \\ \tilde{\lambda} n_2 & \mu n_1 \end{pmatrix} \frac{\partial}{\partial x} + \begin{pmatrix} \mu n_2 & \tilde{\lambda} n_1 \\ \mu n_1 & (\tilde{\lambda}+2\mu)n_2 \end{pmatrix} \frac{\partial}{\partial y},$$

$$\tilde{U}_{1j}(x) = U_{1j}(x) - n_j \frac{K - r\rho_s\alpha}{\rho_s\alpha} P_0(x), \quad j = 1,2, \quad \tilde{\lambda} = \lambda - (\rho^2\alpha)^{-1} K^2.$$

Definition 2 [5]. We say that a function vector $\boldsymbol{u}(z) : \Pi \to X$ belongs to the Hardy class $H_A^p(\Pi; X)$, $1 \le p < \infty$, if it is A-analytic in Π and there exists a constant $c < \infty$, such that for each $y > 0$ we have

$$\int_{-\infty}^{\infty} \|u(x+iy)\|_X^p \, dx \le c.$$

Let us suppose that $\boldsymbol{U}(x,0)$, $\int_{-\infty}^{x} (\tilde{T}_\partial \boldsymbol{U})(t,0)\, dt \in L_p(\partial\Pi; R^2)$, $1 \le p < \infty$. According to [5], the solution to the Cauchy problem (4), (5) is written down in the form:

$$\boldsymbol{U}(x,y) = \operatorname{Re} \Theta \boldsymbol{u}(z), \quad \Theta = \begin{pmatrix} i & i \\ -1 & 2k-1 \end{pmatrix},$$

where $k = (\tilde{\lambda} + 3\mu)/(\tilde{\lambda} + \mu)$, $i^2 = -1$, and $\boldsymbol{u}(z)$ is a function of the Hardy class $H_A^p(\Pi; R^2)$ with the matrix $A = \left(\begin{smallmatrix} 0 & -1 \\ 0 & 0 \end{smallmatrix}\right)$, accepting the values $\boldsymbol{U}|_M = \boldsymbol{f} = \boldsymbol{g} + i\boldsymbol{h}$ on the set M, where the functions $\boldsymbol{g}$ and $\boldsymbol{h}$ are defined from the expressions

$$\operatorname{Re} \Theta(\boldsymbol{g} + i\boldsymbol{h})(t) = \boldsymbol{U}(t), \quad \operatorname{Re} \Theta'(\boldsymbol{g} + i\boldsymbol{h})(t) = \int_{t_1}^{t} \tilde{\boldsymbol{U}}_1(\tau)\, d\tau,$$

$$\Theta' = \begin{pmatrix} 1 & 2-k \\ i & -ik \end{pmatrix}.$$

Values of the function $\boldsymbol{u}(z)$, $z = x + iy \in \Pi$, are found by the Carleman type formula [5]:

$$\boldsymbol{u}(z) = \frac{1}{\pi} \lim_{n\to\infty} \int_{t_1}^{t_2} \frac{e^{-n(\phi(z)-\phi(\tau))}}{(x-\tau)^2 + y^2} \begin{pmatrix} 1 & -n\psi(z) + 2\frac{(x-\tau)^2 - y^2}{(x-\tau)^2 + y^2} \\ 0 & 1 \end{pmatrix} \boldsymbol{f}(\tau)\, d\tau,$$

$$\phi(z) = \frac{1}{\pi}\left(\beta + i \ln \frac{|z-t_1|}{|z-t_2|}\right), \quad \psi(z) = -\frac{2}{\pi} e^{i(\beta_1 - \beta_2)} \sin\beta,$$

$$\beta = \beta_1 + \beta_2, \quad \beta_1 = \operatorname{arctg} \frac{x - t_1}{|z - t_1|}, \quad \beta_2 = \operatorname{arctg} \frac{t_2 - x}{|z - t_2|}.$$

It is necessary to determine the pore pressure $P(x, y)$. Applying to both parts of equation (2) the operator div and in view of the third proportion from (3), we obtain the Cauchy problem for the Laplace equation with respect to the pore pressure P:

$$\Delta P = 0, \quad P|_M = \frac{\rho}{\rho_l} P_0(x), \quad \left.\frac{\partial P}{\partial n}\right|_M = 0. \tag{6}$$

The solution to the Cauchy problem (6) is sought for in the form [6, 7]:

$$P(x, y) = \operatorname{Re} \tilde{P}(z),$$

$$\tilde{P}(z) = \frac{\rho}{2\pi\rho_l i} \lim_{n\to\infty} \int_{t_1}^{t_2} P_0(\tau) \left[\frac{(\tau - t_2)(z - t_1)}{(\tau - t_1)(z - t_2)}\right]^{\frac{n}{\pi i}} \frac{d\tau}{\tau - z}.$$

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