Bull. Nov. Comp. Center, Num. Anal., 7 (1996), 37–49

On one domain decomposition method with nonmatching grids for solving parabolic equations*

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In the article we propose and study a noniterative domain decomposition algorithm without overlapping subdomains and with the use of the penalty functionals on the interface between subdomains. Such type of algorithms were considered in [1–5]. In all these works the error estimate for optimal penalty parameter is $O(\sqrt{\tau})$. In this work the error estimate is improved and our approach is similar to [6], where the alternating direction type splitting was used.

1. The original boundary value problems

Let Ω be a bounded open connected polytop in R^m, $m = 2,3$, and let Ω_1 and Ω_2 be subdomains of Ω such that

$$\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2, \quad \Omega_1 \cap \Omega_2 = \emptyset.$$

Then, let

$$a_p(u,v) = \int_{\Omega_p} \sum_{i,j=1}^{m} \lambda_{i,j}(\bar{x}) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j}\, d\bar{x}, \quad p = 1,2. \tag{1.1}$$

Here $\bar{x} = (x_1,\dots,x_m)$ denotes a point in R^m. The bilinear forms are symmetric, continuous in $H_0^1(\Omega_p) \times H_0^1(\Omega_p)$, and $H_0^1(\Omega_p)$–elliptic, i.e., there are positive numbers λ_1 and λ_2 such that

$$|a_p(u,v)| \le \lambda_1 |u|_{H^1(\Omega_p)} |v|_{H^1(\Omega_p)}, \quad |a_p(u,u)| \ge \lambda_2 |u|^2_{H^1(\Omega_p)}. \tag{1.2}$$

Here $H_0^1(\Omega_p)$ is the subspace of $H^1(\Omega_p)$ obtained by the taking of closure, in the norm of $H^1(\Omega_p)$, of the set of infinitely differentiable functions with compact support in Ω_p. Next, we introduce one-parameter families of continuous linear functionals on $H^1(\Omega_p)$ by using the duality pairing on $H^{-1}(\Omega_p) \times H^1(\Omega_p)$; i.e., $l_p(t;v_p) = (f_p(t), v_p)_p$, $p = 1,2$, where $(\cdot,\cdot)_p$ is the scalar product in $L_2(\Omega_p)$, $t \in [t_0, t_*]$. Here and in what follows $u(t)$ is the value of a function $u : [t_0,t_*] \to X$ and $\frac{du}{dt}(t)$ is the strong limit in X of the elements $[u(t)]_\tau \equiv (u(t+\tau) - u(t))/\tau$ as $\tau \to 0$ (if it exists).

*Supported by the Russian Foundation of Basic Research under Grant 95–01–01530.

We denote by H_p^k the space of the functions in $H^k(\Omega_p)$ extended to Ω by zero beyond $\overline{\Omega}_p$. Now, we may introduce the space $\widehat{H}^k = H_1^k \times H_2^k$ of vector-functions with a norm $\|\cdot\|_{\widehat{H}^k}$. The scalar product in $\widehat{L}_2$ is

$$(u, v) = (u_1, v_1)_1 + (u_2, v_2)_2.$$

In the space $\widehat{H}^1$, we introduce the bilinear form

$$a(u, v) = a_1(u_1, v_1) + a_2(u_2, v_2).$$

We distinguish the following subspace in $\widehat{H}^1$:

$$\widehat{H}^{1,0} = \{v \in \widehat{H}^1 \mid v_1(\bar{x}) = v_2(\bar{x}),\ \bar{x} \in S = \overline{\Omega}_1 \cap \overline{\Omega}_2\}.$$

Now, we formulate the parabolic Neumann problem as some problem in the subspace $\widehat{H}^{1,0}$. Assume $u_0 \in \widehat{L}_2$ and $f \in L_2(t_0, t_*; \widehat{H}^{-1})$. The problem is to find a vector-function $u \in L_2(t_0, t_*; \widehat{H}^{1,0})$, $\frac{du}{dt} \in L_2(t_0, t_*; \widehat{H}^{-1})$ and the equalities

$$\Big(\frac{du}{dt}(t), v\Big) + a(u(t), v) = (f(t), v), \tag{1.3}$$

$$(u(t_0), v) = (u_0, v) \tag{1.4}$$

hold for every $v \in \widehat{H}^{1,0}$ and almost every $t \in (t_0, t_*)$. It is easy to see that problem (1.3), (1.4) is equivalent to the conventional Neumann problem in the space $H^1(\Omega)$. We point out that the Neumann problem is considered exclusively for notational simplicity. All results of the present article remain valid for other boundary value problems.

Following [1], we formulate the Neumann problem with conditions of a nonideal contact on the interface: given the same initial data as in problem (1.3), (1.4), find the function $u^\rho \in L_2(t_0, t_*; \widehat{H}^1)$, $\frac{du^\rho}{dt} \in L_2(t_0, t_*; \widehat{H}^{-1})$ and, for every $v \in \widehat{H}^1$ and almost every $t \in (t_0, t_*)$, the equalities

$$\Big(\frac{du^\rho}{dt}(t), v\Big) + a(u^\rho(t), v) + \frac{1}{\rho}\int_S (u_1^\rho(t) - u_2^\rho(t))(v_1 - v_2)\, ds = (f(t), v), \tag{1.5}$$

$$(u^\rho(t_0), v) = (u_0, v) \tag{1.6}$$

hold, where $\rho > 0$.

It is shown in [1] that if a solution to problem (1.3), (1.4) is sufficiently smooth in subdomains, then the following inequalities hold:

$$\|u^\rho - u\|_{C(t_0,t_*;\widehat{L}_2)} \le c_1\rho\|u\|_{H^1(t_0,t_*;\widehat{H}^2)}, \quad \|u^\rho\|_X \le c_2\|u\|_X, \tag{1.7}$$

where X is an arbitrary subspace in $L_2(t_0, t_*; \widehat{H}^1)$, $\rho \le \rho_0$, and the numbers c_1 and c_2 are independent of the parameter ρ and the vector-functions u and

u^ρ. These equalities justify the use of the penalty method for solving problem (1.3), (1.4) on applying the domain decomposition method to problem (1.5), (1.6).

All notations connected with the approximation of problem (1.5), (1.6) by the finite element method adhere to [5].

2. Alternating direction type decomposition

In this section we describe the domain decomposition method in vector-matrix form. The splitting method is based on the following additive presentation:

$$A_\rho = A + \frac{1}{\rho}B. \tag{2.1}$$

Let N be a natural number, $\tau = (t_* - t_0)/N$ and $t_n = t_0 + n\tau$, $n = 1, \ldots, N$. We write down the method as follows:

$$\frac{\bar{u}^{n+\frac{1}{2}} - \bar{u}^n}{\tau/2} + A\bar{u}^n + \frac{1}{\rho}B\bar{u}^{n+\frac{1}{2}} = \bar{f}^{n+\frac{1}{2}}, \tag{2.2}$$

$$\frac{\bar{u}^{n+1} - \bar{u}^{n+\frac{1}{2}}}{\tau/2} + A\bar{u}^{n+1} + \frac{1}{\rho}B\bar{u}^{n+\frac{1}{2}} = \bar{f}^{n+\frac{1}{2}}, \tag{2.3}$$

where $\bar{f}^{n+\frac{1}{2}} = \frac{1}{2}(\bar{f}^n + \bar{f}^{n+1})$ and $(\bar{u}^0_p)_i = \rho_{p,i}u_{0,p}(\bar{x}_{p,i})$, $p = 1, 2$ (see [5]). Here we assume $u_{0,p} \in H^2(\Omega_p)$.

Let $\bar{w}^n$ be the vector, corresponding to the Ritz projection of $u^\rho(t_n)$ (solution to problem (1.5), (1.6)) on space of piecewise linear functions. Introduce the following sequence of vectors: $\bar{\xi}^n = \bar{u}^n - \bar{w}^n$, $\bar{\xi}^{n+\frac{1}{2}} = \bar{u}^{n+\frac{1}{2}} - \frac{1}{2}(\bar{w}^n + \bar{w}^{n+1})$, $n = 0, \ldots, N-1$. Then from (2.2), (2.3) it follows:

$$\frac{\bar{\xi}^{n+\frac{1}{2}} - \bar{\xi}^n}{\tau/2} + A\bar{\xi}^n + \frac{1}{\rho}B\bar{\xi}^{n+\frac{1}{2}} = \bar{g}^{n+\frac{1}{2}}, \tag{2.4}$$

$$\frac{\bar{\xi}^{n+1} - \bar{\xi}^{n+\frac{1}{2}}}{\tau/2} + A\bar{\xi}^{n+1} + \frac{1}{\rho}B\bar{\xi}^{n+\frac{1}{2}} = \bar{g}^{n+1}, \tag{2.5}$$

where

$$\bar{g}^{n+\frac{1}{2}} = \bar{\alpha}^n + \frac{\tau}{2}\bar{\beta}^n, \quad \bar{g}^{n+1} = \bar{\alpha}^n - \frac{\tau}{2}\bar{\beta}^n, \tag{2.6}$$

and

$$\bar{\alpha}^n = (\bar{\alpha}^n_1, \bar{\alpha}^n_2)^T,$$

$$(\bar{\alpha}^n_p)_i = \frac{1}{\rho_{p,i}}\Big\{\frac{1}{2}\Big(\frac{du^\rho_p}{dt}(t_n) + \frac{du^\rho_p}{dt}(t_{n+1}), \varphi_{p,i}\Big)_p - d_{h,p}([w^n_p]_\tau, \varphi_{p,i})\Big\},$$

$$\bar{\beta}^n = A\bar{w}^n_\tau, \quad \bar{w}^n_\tau = \frac{\bar{w}^{n+1} - \bar{w}^n}{\tau},$$

vector $\bar{w}^n_\tau$ corresponds to vector-function $[w^n]_\tau$.

3. Convergence theorems

Rewrite system (2.5), (2.6) in the form

$$\Big(E+\frac{\tau}{2\rho}B\Big)\bar{\xi}^{n+\frac{1}{2}}=\Big(E-\frac{\tau}{2}A\Big)\bar{\xi}^{n}+\frac{\tau}{2}\bar{\alpha}^{n}+\frac{\tau^{2}}{4}\bar{\beta}^{n}, \tag{3.1}$$

$$\Big(E+\frac{\tau}{2}A\Big)\bar{\xi}^{n+1}=\Big(E-\frac{\tau}{2\rho}B\Big)\bar{\xi}^{n+\frac{1}{2}}+\frac{\tau}{2}\bar{\alpha}^{n}-\frac{\tau^{2}}{4}\bar{\beta}^{n}. \tag{3.2}$$

Let us denote $\bar{\varphi}^n=(E+\frac{\tau}{2}A)\bar{\xi}^n$, $R_A=(E+\frac{\tau}{2}A)^{-1}(E-\frac{\tau}{2}A)$. Then, it is not difficult to receive from (3.1), (3.2) the following equations

$$\bar{\varphi}^{n+1}+\frac{\tau}{\rho}B\bar{\xi}^{n+\frac{1}{2}}-R_A\bar{\varphi}^{n}-\tau\bar{\alpha}^{n}=0, \tag{3.3}$$

$$\bar{\varphi}^{n+1}+R_A\bar{\varphi}^{n}-2\bar{\xi}^{n+\frac{1}{2}}+\frac{\tau^{2}}{2}\bar{\beta}^{n}=0. \tag{3.4}$$

From (3.4) it follows that $\bar{\xi}^{n+\frac{1}{2}}=\frac{\tau^2}{2}\bar{\beta}^n+\frac{\bar{\varphi}^{n+1}+R\bar{\varphi}^n}{2}$. Substitute this equality into (3.3):

$$\bar{\varphi}^{n+1}-R_A\bar{\varphi}^{n}+\frac{\tau}{2\rho}B(\bar{\varphi}^{n+1}+R_A\bar{\varphi}^{n})=\tau\bar{\alpha}^{n}-\frac{\tau^{3}}{4\rho}B\bar{\beta}^{n}. \tag{3.5}$$

Then, let us multiply (3.5) by $\bar{\varphi}^{n+1}+R_A\bar{\varphi}^n$ and use ε–inequality. As a result we will obtain

$$\Big(1-\frac{\tau\varepsilon_1}{2}\Big)\|\bar{\varphi}^{n+1}\|^2\le\Big(1+\frac{\tau\varepsilon_2}{2}\Big)\|R_A\bar{\varphi}^{n}\|^2+\frac{\tau}{2}\Big(\frac{1}{\varepsilon_1}+\frac{1}{\varepsilon_2}\Big)\|\bar{\alpha}^{n}\|^2+\frac{\tau^5}{32\rho}\|\bar{\beta}^{n}\|^2_B.$$

Let $\tau\le 1$, $\varepsilon_1=\frac{1}{2}$, $\varepsilon_2=1$. Then,

$$\|\bar{\varphi}^{n+1}\|^2\le(1+\tau)\|R_A\bar{\varphi}^{n}\|^2+3\tau\|\bar{\alpha}^{n}\|^2+\frac{\tau^5}{16\rho}\|\bar{\beta}^{n}\|^2_B.$$

Using mesh Gronwall's lemma and taking into account that $\|R_A\|\le 1$ (according to (1.2) matrix A is positive semi-definite), we will receive

$$\|\bar{\varphi}^{n}\|^2\le c_3\Big\{\|\bar{\varphi}^{0}\|^2+\tau\sum_{k=0}^{n-1}\|\bar{\alpha}^{k}\|^2+\frac{\tau^5}{\rho}\sum_{k=0}^{n-1}\|\bar{\beta}^{k}\|^2_B\Big\}. \tag{3.6}$$

Let us estimate the norms in right-hand side of (3.6). Firstly, with the use of the estimate for $\|u^\rho(t)-w(t)\|_{\widehat{H}^1}$ from [5] we have

$$\|\bar{\varphi}^{0}\|^2\le c_4\Big(1+\frac{h}{\rho}\Big)(\tau^2+h^2)\|u^\rho\|^2_{L_2(t_0,t_*;\widehat{H}^2)}. \tag{3.7}$$

Let us consider the presentation

$$\bar{\alpha}^k = \bar{\alpha}^{k,1} + \bar{\alpha}^{k,2} + \bar{\alpha}^{k,3}, \tag{3.8}$$

where

$$(\bar{\alpha}_p^{k,1})_i = \frac{1}{\rho_{p,i}}\Big\{\frac{1}{2}\Big(\frac{du_p^\rho}{dt}(t_k) + \frac{du_p^\rho}{dt}(t_{k+1}), \varphi_{p,i}\Big)_p - ([u_p^\rho(t_k)]_\tau, \varphi_{p,i})_p\Big\},$$
$$(\bar{\alpha}_p^{k,2})_i = \frac{1}{\rho_{p,i}}\Big\{([u_p^\rho(t_k)]_\tau, \varphi_{p,i})_p - d_{h,p}(\Pi_{h,p}[u_p^\rho(t_k)]_\tau, \varphi_{p,i})\Big\},$$
$$(\bar{\alpha}_p^{k,3})_i = \frac{1}{\rho_{p,i}} d_{h,p}\Big(\Pi_{h,p}[u_p^\rho(t_k)]_\tau - [w_p^k]_\tau, \varphi_{p,i}\Big).$$

Then the use of standard technique results in the following estimates:

$$\|\bar{\alpha}^{k,1}\|^2 \le c_5 \tau^3 \Big\|\frac{d^3u^\rho}{dt^3}\Big\|^2_{L_2(t_k,t_{k+1};\widehat{L}_2)}, \tag{3.9}$$

$$\|\bar{\alpha}^{k,2}\|^2 \le c_6 \frac{h^2}{\tau}\Big(\Big\|\frac{du^\rho}{dt}\Big\|^2_{L_2(t_k,t_{k+1};\widehat{H}^1)} + h^2\Big\|\frac{du^\rho}{dt}\Big\|^2_{L_2(t_k,t_{k+1};\widehat{H}^2)}\Big), \tag{3.10}$$

$$\|\bar{\alpha}^{k,3}\|^2 \le c_7 \frac{h^2}{\tau}\Big(\Big(1+\frac{h}{\rho}\Big)\Big\|\frac{du^\rho}{dt}\Big\|^2_{L_2(t_k,t_{k+1};\widehat{H}^2)}\Big). \tag{3.11}$$

Presentation (3.8) and inequalities (3.9)–(3.11) give the estimate for $\|\bar{\alpha}^k\|$. To estimate $\|\bar{\beta}^k\|_B^2$ let us note that $(\bar{\beta}_p^k)_i = \frac{1}{\rho_{p,i}} a_p([w_p^k]_\tau, \varphi_{p,i})$, and it is clear that we have to estimate

$$a_p([u_p^\rho(t_k)]_\tau, \varphi_{p,i}) = \int_S \frac{\partial [u_p^\rho(t_k)]_\tau}{\partial n_p}\varphi_{p,i}\, ds - \int_{\Omega_p} \sum_{j,l=1}^m \lambda_{j,l} \frac{\partial [u_p^\rho(t_k)]_\tau}{\partial x_j \partial x_l}\varphi_{p,i}\, d\bar{x}.$$

With the use of the trace theorem and taking into account that $\|B\| \le \frac{c'}{h}$, we obtain:

$$\|\bar{\beta}^k\|_B^2 \le c_8 \frac{1}{\tau h^2}\Big(1+\frac{h}{\rho}\Big)\Big\|\frac{du^\rho}{dt}\Big\|^2_{L_2(t_k,t_{k+1};\widehat{H}^2)}. \tag{3.12}$$

Substitute inequalities (3.7), (3.9)–(3.12) into (3.6). As $\|\bar{\xi}^n\| \le \|\bar{\varphi}^n\|$ we have:

$$\|\bar{\xi}^n\|^2 \le c_9\Big\{M_{\rho,\tau}^2\tau^4 + M_{\rho,h}^2\Big(1+\frac{h}{\rho}\Big)\Big(\tau^2 + h^2 + \frac{\tau^4}{h^2\rho}\Big)\Big\}, \tag{3.13}$$

where $c_9 = c_9(c_3, \dots, c_8)$ and

$$M_{\rho,\tau} = \Big\|\frac{d^3u^\rho}{dt^3}\Big\|_{L_2(t_0,t_*;\widehat{L}_2)}, \quad M_{\rho,h} = \|u^\rho\|_{H^1(t_0,t_*;\widehat{H}^2)}. \tag{3.14}$$

The existence of $M_{\rho,\tau}$ and $M_{\rho,h}$ are provided by corresponding *a priori* smothness of the solution of the problem (1.5), (1.6). We have proved the following

Theorem 3.1. *Let the smothness of the solution of* (1.5), (1.6) *correspond to* (3.14). *Then for the solution of* (2.2), (2.3) *at* $\tau \le \tau_0$ *and* $h \le h_0$ *the following estimate is valid:*

$$\max_{1\le n\le N} \|u^n - u^\rho(t_n)\|_{\widehat{L}_2} \le cM(u^\rho)\Big\{\tau + h + \frac{1}{\sqrt{\rho}}\Big(\frac{\tau^2}{h} + h\sqrt{h}\Big) + \frac{\tau^2}{\rho\sqrt{h}}\Big\}, \quad (3.15)$$

where τ_0, h_0, c *do not depend on* τ, h, ρ *and functional* $M(u^\rho)$ *is defineded by* $M_{\rho,\tau}$ *and* $M_{\rho,h}$.

Inequality (3.15) means that for fixed ρ we have error estimate $O(\tau + h + \frac{\tau^2}{h})$. For small ρ we will use (1.7) and will receive the error estimate for the solution of original problem (1.3), (1.4). According to (1.7) we have

$$\max_{1\le n\le N} \|u^n - u(t_n)\|_{\widehat{L}_2} \le c'M(u)\Big\{\rho + \tau + h + \frac{1}{\sqrt{\rho}}\Big(\frac{\tau^2}{h} + h\sqrt{h}\Big) + \frac{\tau^2}{\rho\sqrt{h}}\Big\}.$$

It is a difficult problem to find optimal ρ, which minimizes the right-hand side of the last inequality. Instead of the one we will find minimum of two following expressions appart: $\rho + \frac{1}{\sqrt{\rho}}(\frac{\tau^2}{h} + h\sqrt{h})$ and $\rho + \frac{\tau^2}{\rho\sqrt{h}}$. Then we will obtain corresponding: $\rho_1 = [\frac{1}{2}(\frac{\tau^2}{h} + h\sqrt{h})]^{2/3} = O(h + \tau^{\frac{4}{3}}h^{-\frac{2}{3}})$ and $\rho_2 = \tau h^{-\frac{1}{4}}$. The best estimate is realized at $\rho = \rho_1$. Then the following theorem is proved:

Theorem 3.2. *Let the smothness of the solution of* (1.3), (1.4) *provide the existence of functional* $M(u)$. *Then, for* $\tau = h^\alpha$ *and* $\rho = c'(h^{\frac{3}{2}} + h^{2\alpha-1})^{\frac{2}{3}}$ *at* $h \le h_0$ *the following inequality*

$$\max_{1\le n\le N} \|u^n - u(t_n)\|_{\widehat{L}_2} \le cM(u)\begin{cases} h^{\frac{4}{3}\alpha-\frac{2}{3}}, & \alpha < \frac{5}{4}, \\ h, & \alpha \ge \frac{5}{4}, \end{cases} \quad (3.16)$$

holds, and numbers h_0, c, c' *do not depend on* h *and vector-function* u.

4. Convergence for one-dimensional problem

The estimate from Theorem 3.2 may be improved for one-dimensional problem or for the problems with dividing variables. For simplicity we will consider convergence for one-dimensional problem, but all analysis will be realized for multidimensional case and specific of one-dimensionalness will be used only in the proof of Lemma 4.1. For this aim we will use the technique from [6]. This technique requires an existence of A_ρ^{-1}, and, therefore, in this section we will consider the Dirichlet problem. The Neumann problem may be considered too, but it requires some modifications (see [6]). Let us present the solution of (3.1), (3.2) as

$$\bar{\xi}^n = \bar{\eta}^n + \bar{\zeta}^n, \tag{4.1}$$

where $\bar{\eta}^n$ and $\bar{\zeta}^n$ are the solutions of the following systems:

$$\begin{aligned}
&\Big(E + \frac{\tau}{2\rho}B\Big)\bar{\eta}^{n+\frac{1}{2}} = \Big(E - \frac{\tau}{2}A\Big)\bar{\eta}^n + \frac{\tau}{2}\bar{\alpha}^n,\\
&\Big(E + \frac{\tau}{2}A\Big)\bar{\eta}^{n+1} = \Big(E - \frac{\tau}{2\rho}B\Big)\bar{\eta}^{n+\frac{1}{2}} + \frac{\tau}{2}\bar{\alpha}^n,\\
&\bar{\eta}^n = \bar{\xi}^n,
\end{aligned} \tag{4.2}$$

and

$$\begin{aligned}
&\Big(E + \frac{\tau}{2\rho}B\Big)\bar{\zeta}^{n+\frac{1}{2}} = \Big(E - \frac{\tau}{2}A\Big)\bar{\zeta}^n + \frac{\tau^2}{4}\bar{\beta}^n,\\
&\Big(E + \frac{\tau}{2}A\Big)\bar{\zeta}^{n+1} = \Big(E - \frac{\tau}{2\rho}B\Big)\bar{\zeta}^{n+\frac{1}{2}} - \frac{\tau^2}{4}\bar{\beta}^n,\\
&\bar{\zeta}^n = 0.
\end{aligned} \tag{4.3}$$

Similar to previous analysis (inequality (3.13)) it is easy to receive the estimate for the solutin of the first system:

$$\|\bar{\eta}^n\|^2 \le c_{10}\Big\{M^2_{\rho,\tau}\tau^4 + M^2_{\rho,h}\Big(1 + \frac{h}{\rho}\Big)(\tau^2 + h^2)\Big\}. \tag{4.4}$$

Now let us estimate the solution of (4.3). Let

$$\bar{\varphi}^n = \Big(E + \frac{\tau}{2}A\Big)\bar{\zeta}^n, \quad \bar{\varphi}^{n+\frac{1}{2}} = \Big(E + \frac{\tau}{2\rho}B\Big)\bar{\zeta}^{n+\frac{1}{2}}.$$

Then equations (4.3) will transform to

$$\bar{\varphi}^{n+\frac{1}{2}} = R_A\bar{\varphi}^n + \frac{\tau^2}{4}\bar{\beta}^n, \quad \bar{\varphi}^{n+1} = R_B\bar{\varphi}^{n+\frac{1}{2}} - \frac{\tau^2}{4}\bar{\beta}^n, \tag{4.5}$$

where $R_B = (E + \frac{\tau}{2\rho}B)^{-1}(E - \frac{\tau}{2\rho}B)$ and $\bar{\varphi}^0 = 0$. Denote $\bar{\psi}^n = \bar{\varphi}^n$, $\bar{\psi}^{n+\frac{1}{2}} = \bar{\varphi}^{n+\frac{1}{2}} - \frac{\tau^2}{4}\bar{\omega}^n$, where $\bar{\omega}^n$ will be defined later. Then from (4.5) it follows:

$$\bar{\psi}^{n+\frac{1}{2}} = R_A\bar{\psi}^n + \frac{\tau^2}{4}(\bar{\beta}^n - \bar{\omega}^n), \quad \bar{\psi}^{n+1} = R_B\bar{\psi}^{n+\frac{1}{2}} - \frac{\tau^2}{4}(\bar{\beta}^n - R_B\bar{\omega}^n). \tag{4.6}$$

Let us define vecror $\bar{\omega}^n$ by the following equality:

$$\Big(E + \frac{\tau}{2}A\Big)(\bar{\beta}^n - \bar{\omega}^n) = \Big(E + \frac{\tau}{2\rho}B\Big)(\bar{\beta}^n - R_B\bar{\omega}^n).$$

Then, $\bar{\omega}^n = A_\rho^{-1}(A - \frac{1}{\rho}B)\bar{\beta}^n$. Let $\bar{\delta}^n = (E + \frac{\tau}{2}A)(\bar{\beta}^n - \bar{\omega}^n)$. Then,

$$\bar{\delta}^n = \frac{2}{\rho}\Big(E + \frac{\tau}{2}A\Big)A_\rho^{-1}B\bar{\beta}^n. \tag{4.7}$$

Let us multiply the first equation from (4.6) by $E + \frac{\tau}{2}A$ and the second one by $E + \frac{\tau}{2\rho}B$. As a result we will receive

$$\bar{\psi}^{n+\frac{1}{2}} - \bar{\psi}^n + \frac{\tau}{2}A(\bar{\psi}^{n+\frac{1}{2}} + \bar{\psi}^n) = \frac{\tau^2}{4}\bar{\delta}^n,$$

$$\bar{\psi}^{n+1} - \bar{\psi}^{n+\frac{1}{2}} + \frac{\tau}{2\rho}B(\bar{\psi}^{n+1} + \bar{\psi}^{n+\frac{1}{2}}) = -\frac{\tau^2}{4}\bar{\delta}^n.$$

Taking into account positive semi-definitness of the matrices A and B, from these equations it is not difficult to obtain the inequality

$$\|\bar{\varphi}^{n+1}\|^2 \le \|\bar{\varphi}^n\|^2 + \frac{\tau^3}{4}(\langle\bar{\delta}_\tau^n, \bar{\varphi}^{n+1}\rangle - \langle\bar{\delta}^n, \bar{\varphi}^n\rangle_\tau),$$

and after summing:

$$\|\bar{\varphi}^n\|^2 \le \frac{\tau^2}{4}|\langle\bar{\delta}^n, \bar{\varphi}^n\rangle| + \frac{\tau^3}{4}\sum_{k=0}^{n-1}|\langle\bar{\delta}_\tau^k, \bar{\varphi}^{k+1}\rangle|.$$

With the use of the Cauchy-Buniakovsky and ε - inequalities, of the mesh Gronwall's lemma and assuming $\tau \le 1$, we will obtain

$$\|\bar{\varphi}^n\|^2 \le c_{11}\tau^4\Big\{\|\bar{\delta}^n\|^2 + \tau\sum_{k=1}^{n-1}\|\bar{\delta}_\tau^k\|^2\Big\}. \tag{4.8}$$

Remark 4.1. The latter form of our equations, which allows us to present the right-hand side as

$$\langle\bar{\delta}^n, \bar{\varphi}^n - \bar{\varphi}^{n+1}\rangle = \tau(\langle\bar{\delta}_\tau^n, \bar{\varphi}^{n+1}\rangle - \langle\bar{\delta}^n, \bar{\varphi}^n\rangle_\tau),$$

is the central moment of the analysis in this section. The necessary order of convergence is provided by not local estimate on one time step, but after the use of the mesh Gronwall's lemma.

The following lemma allows us to estimate $\|\bar{\delta}^n\|$ and $\|\bar{\delta}_\tau^k\|$.

Lemma 4.1. *For arbitrary vector $\bar{v}$ the following estimates are valid:*

$$\frac{1}{\rho}\|A_\rho^{-1}B\bar{v}\| \le \frac{\nu'}{\sqrt{h}}\|\bar{v}\|, \quad \frac{1}{\rho}\|AA_\rho^{-1}B\bar{v}\| \le \frac{\nu''}{h}\|\bar{v}\|, \tag{4.9}$$

where the numbers ν' and ν'' do not depend on h, ρ and vector $\bar{v}$.

Proof. For one-dimensional problem the proof may be received from direct inversion of matrix A_ρ. But to show difficulties for multidimensional problem we will give the proof in general case and will indicate the moment, where we use one-dimensionalness. The structure of A and B is

$$A = \begin{pmatrix} A_{11} & A_{1S} & 0 & 0 \\ A_{1S}^T & A_{SS}^{(1)} & 0 & 0 \\ 0 & 0 & A_{SS}^{(2)} & A_{S2} \\ 0 & 0 & A_{S2}^T & A_{22} \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & B_{11} & B_{12} & 0 \\ 0 & B_{12}^T & B_{22} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

In accordance with it $\bar{v} = (\bar{v}_1, \bar{v}_S^{(1)}, \bar{v}_S^{(2)}, \bar{v}_2)^T$, and let $\bar{v}_S = (\bar{v}_S^{(1)}, \bar{v}_S^{(2)})^T$. Let $\bar{w}$ be the solution of the system $A_\rho \bar{w} = \frac{1}{\rho} B\bar{v}$. Let us denote $\Lambda_{\rho,S} = \Lambda_S + \frac{1}{\rho} B_S$, where

$$\Lambda_S = \begin{pmatrix} \Lambda_S^{(1)} & 0 \\ 0 & \Lambda_S^{(2)} \end{pmatrix}, \quad B_S = \begin{pmatrix} B_{11} & B_{12} \\ B_{12}^T & B_{22} \end{pmatrix},$$

$\Lambda_S^{(1)} = A_{SS}^{(1)} - A_{1S}^T A_{11}^{-1} A_{1S}$, $\Lambda_S^{(2)} = A_{SS}^{(2)} - A_{S2} A_{22}^{-1} A_{S2}^T$ are positive semi-definite matrices, but $\Lambda_{\rho,S}$ is a positive definite matrix. Then,

$$\Lambda_{\rho,S}(\bar{v}_S - \bar{w}_S) = \Lambda_S \bar{v}_S. \tag{4.10}$$

From (4.10) we obtain

$$\|\bar{w}_S\|_{\Lambda_S} \le 2\|\bar{v}_S\|_{\Lambda_S}. \tag{4.11}$$

In one-dimensional case $\Lambda_S^{(p)}$ are the numbers of the same order in h and from (4.11) immediately follows that

$$\|\bar{w}_S\| \le c'\|\bar{v}_S\| \le c'\|\bar{v}\|. \tag{4.12}$$

Yet it is not true in multidimensional case. As it is shown in [7] (Lemma 6.1) $\|\bar{w}\| \le \frac{c''}{\sqrt{h}}\|\bar{w}_S\|$, and according to (4.12) we have the first inequality from (4.9) with $\nu' = c'c''$.

Let $\bar{z} = \frac{1}{\rho} A A_\rho^{-1} B\bar{v} = A\bar{w}$. It is esy to show that $\bar{z} = (0, \bar{z}_S^{(1)}, \bar{z}_S^{(2)}, 0)^T$ and $\bar{z}_S = (\bar{z}_S^{(1)}, \bar{z}_S^{(2)})^T = \Lambda_S \bar{w}_S$. Then,

$$\|\bar{z}_S\| \le \|\Lambda_S\|\,\|\bar{w}_S\| \le c'\|\Lambda_S\|\,\|\bar{v}\|.$$

For one-dimensional problem $\|\Lambda_S\| \le \frac{\lambda}{h}$, and we obtain the second inequality from (4.9) with $\nu'' = c'\lambda$. □

According to (4.7) and Lemma 4.1 we have

$$\|\bar{\delta}^n\| \le \Big(\frac{2\nu'}{\sqrt{h}} + \frac{\tau\nu''}{h}\Big)\|\bar{\beta}^n\|. \tag{4.12}$$

The estimate for $\|\bar{\beta}^n\|^2$ is

$$\|\bar{\beta}^n\|^2 \le c_{12}\Big(1 + \frac{h}{\rho}\Big)\frac{1}{\tau h}\Big\|\frac{du^\rho}{dt}\Big\|^2_{L_2(t_n,t_{n+1};\widehat{H}^2)}, \tag{4.13}$$

(see previous section). From (4.12), (4.13) and a continuity of an imbedding $H^1(t_0, t_*; \widehat{H}^2)$ into $C(t_0, t_*; \widehat{H}^2)$ it follows:

$$\|\delta^n\|^2 \le c_{13}\Big(1+\frac{h}{\rho}\Big)\Big(1+\frac{\tau^2}{h}\Big)\frac{1}{h^2}\Big\|\frac{du^\rho}{dt}\Big\|^2_{H^1(t_0,t_*;\widehat{H}^2)}. \qquad (4.14)$$

Analogously we have:

$$\|\bar{\delta}^k_\tau\|^2 \le c_{14}\Big(1+\frac{h}{\rho}\Big)\Big(1+\frac{\tau^2}{h}\Big)\frac{1}{\tau h^2}\Big\|\frac{d^2u^\rho}{dt^2}\Big\|^2_{L_2(t_k,t_{k+1};\widehat{H}^2)}. \qquad (4.15)$$

Substitute estimates (4.14), (4.15) into (4.8), and as $\|\bar{\zeta}^n\| \le \|\bar{\varphi}^n\|$ we will receive

$$\|\bar{\zeta}^n\|^2 \le c_{15}\Big(1+\frac{h}{\rho}\Big)\Big(1+\frac{\tau^2}{h}\Big)\frac{\tau^4}{h^2}\|u^\rho\|^2_{H^2(t_0,t_*;\widehat{H}^2)}. \qquad (4.16)$$

From (4.4) and (4.16) immediately follows

Theorem 4.1. *For the conditions of Theorem 3.1 and* $\|u^\rho\|_{H^2(t_0,t_*;\widehat{H}^2)} < \infty$ *the following estimate is valid:*

$$\max_{1\le n\le N}\|u^n - u^\rho(t_n)\|_{\widehat{L}_2} \le cM'(u^\rho)\sigma(\tau,h)\sqrt{1+\frac{h}{\rho}},$$

where $\sigma(\tau,h) = \tau + h + \frac{\tau^2}{h}\sqrt{1+\frac{\tau^2}{h}}$ *and functional* $M'(u^\rho)$ *is defineded by* $M_{\rho,\tau}$, $M_{\rho,h}$ *and* $\|u^\rho\|_{H^2(t_0,t_*;\widehat{H}^2)}$.

Then for the solution of the original problem it follows from (1.7):

$$\max_{1\le n\le N}\|u^n - u(t_n)\|_{\widehat{L}_2} \le cM'(u)\|\Big(\rho + \sigma(\tau,h)\sqrt{1+\frac{h}{\rho}}\Big).$$

As easy to see at $\rho = c'\sqrt{\tau^2+h^2}$ we have error estimate $O(\sigma(\tau,h))$, and optimal ρ does not improve this estimate.

Theorem 4.2. *Let* $M'(u) < \infty$. *Then for* $\rho = c'\sqrt{\tau^2+h^2}$ *at* $\tau \le \tau_0$, $h \le h_0$ *the following estimate is valid:*

$$\max_{1\le n\le N}\|u^n - u(t_n)\|_{\widehat{L}_2} \le cM'(u)\Big(\tau + h + \frac{\tau^2}{h}\sqrt{1+\frac{\tau^2}{h}}\Big).$$

5. Numerical experiments

Make some tests according to the method under consideration. Firstly, we will consider one-dimensional problem. Thus, we can confirm the fact that the estimate obtained in Theorem 4.2 can not be improved in regard to the power of τ. As for the power of h, the estimate is not optimal. Then we will demonstrate two-dimensional example with the use of nonmatching grid.

Let consider the Dirichlet problem for the heat equation in interval $\Omega = (0,1)$:

$$\frac{\partial u}{\partial t} = \lambda_0 \frac{\partial^2 u}{\partial x^2}, \quad (t,x) \in (0,1) \times \Omega,$$
$$u(t,0) = u(t,1) = 0, \quad t \in (0,1),$$
$$u(0,x) = \sin \pi x, \quad x \in [0,1].$$

The solution of this problem is

$$u(t,x) = e^{-\lambda_0 \pi^2 t} \sin \pi x.$$

Put $\lambda_0 = 0.1$. In this assumption the $\widehat{L}_2$-norm of the solution decrease approximately in e times to the time $t = 1$.

The domain decomposition is $\Omega_1 = (0, \frac{3}{8})$, $\Omega_2 = (\frac{3}{8}, 1)$. In the following tables we use the notations: $\varepsilon = (\sum_{p=1}^{2} \sum_i h(u_p(1,x_i) - u_p^N(1,x_i))^2)^{1/2}$ is the $\widehat{L}_2$-norm of the error for time $t = 1$; $\rho_{\tau,h} = \sqrt{\tau^2 + h^2}$ and ρ_{opt} is the value of parameter ρ for which error ε takes minimum. In Table 1 the results are presented at $\tau = h$ and $\rho = 0.7\rho_{\tau,h}$, $0.65\rho_{\tau,h}$, $0.75\rho_{\tau,h}$. The constant 0.7 approximately gives the equality $0.7\rho_{\tau,h} = \rho_{\text{opt}}$ at $\tau = h = 2^{-8}$.

Table 1

$\tau = h$	2^{-4}	2^{-5}	2^{-6}	2^{-7}	2^{-8}	2^{-9}
ρ_{opt}	$3.8_{10^{-2}}$	$2.5_{10^{-2}}$	$1.4_{10^{-2}}$	$7.5_{10^{-3}}$	$3.8_{10^{-3}}$	$1.9_{10^{-3}}$
ε	$6.58_{10^{-4}}$	$1.64_{10^{-4}}$	$4.10_{10^{-5}}$	$1.06_{10^{-5}}$	$2.63_{10^{-6}}$	$6.44_{10^{-7}}$
$0.7\rho_{\tau,h}$	$6.19_{10^{-2}}$	$3.09_{10^{-2}}$	$1.55_{10^{-2}}$	$7.73_{10^{-3}}$	$3.87_{10^{-3}}$	$1.93_{10^{-3}}$
ε	$7.57_{10^{-4}}$	$1.89_{10^{-4}}$	$4.75_{10^{-5}}$	$1.20_{10^{-5}}$	$3.04_{10^{-6}}$	$7.84_{10^{-7}}$
$0.65\rho_{\tau,h}$	$5.75_{10^{-2}}$	$2.87_{10^{-2}}$	$1.44_{10^{-2}}$	$7.18_{10^{-3}}$	$3.59_{10^{-3}}$	$1.80_{10^{-3}}$
ε	$7.26_{10^{-4}}$	$1.75_{10^{-4}}$	$4.16_{10^{-5}}$	$1.05_{10^{-5}}$	$3.68_{10^{-6}}$	$1.80_{10^{-6}}$
$0.75\rho_{\tau,h}$	$6.63_{10^{-2}}$	$3.31_{10^{-2}}$	$1.66_{10^{-2}}$	$8.29_{10^{-3}}$	$4.14_{10^{-3}}$	$2.07_{10^{-3}}$
ε	$7.93_{10^{-4}}$	$2.08_{10^{-4}}$	$5.80_{10^{-5}}$	$1.79_{10^{-5}}$	$6.44_{10^{-6}}$	$2.67_{10^{-6}}$

Table 2

h	2^{-4}	2^{-5}	2^{-6}	2^{-7}	2^{-8}	2^{-9}
ε	$1.27_{10^{-3}}$	$4.00_{10^{-4}}$	$4.75_{10^{-5}}$	$2.85_{10^{-4}}$	$7.86_{10^{-4}}$	$1.77_{10^{-3}}$

It is interesting to note that at $\rho = \rho_{\text{opt}}$ and $\rho = 0.7\rho_{\tau,h}$ assimptoticaly the error is propotional to τ^2, nevertheless, ρ is propotional to τ. But the first estimate from (1.7) gives the optimal error, and presented fact is provided by compensating of the mesh error and the error from (1.7) (both these errors have different sighns). For other constants in $\rho = c'\rho_{\tau,h}$ (in Table 1 these are 0.65 and 0.75) we have a convergence which is provided by Theorem 4.2 (ε is propotional to τ). Then Table 2 illustrates conditionally convergence (the third term in the estimate of Theorem 4.2) – at fixed $\tau = 2^{-6}$ the error increases with decreasing of the spatial step h.

Now let us consider two-dimensional example with the use of nonmatching grids. Let Ω be the square $(0,1) \times (0,1)$. Treat in Ω the Dirichlet problem

$$\frac{\partial u}{\partial t} = \lambda_0 \Big(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} \Big), \quad (t, \bar{x}) \in (0,1) \times \Omega,$$

$$u(t, \bar{x}) = 0, \quad (t, \bar{x}) \in (0,1) \times \partial\Omega,$$

$$u(0, \bar{x}) = \sin \pi x_1 \sin \pi x_2, \quad \bar{x} \in \overline{\Omega}$$

with the exact solution

$$u(t, \bar{x}) = e^{-2\lambda_0 \pi^2 t} \sin \pi x_1 \sin \pi x_2.$$

Here we put $\lambda_0 = 0.05$ which gives the decreasing of $\widehat{L}_2$-norm of the solution approximately in e times to the time $t = 1$.

Here the domain decomposition is $\Omega_1 = (0, \frac{3}{8}) \times (0,1)$, $\Omega_2 = (\frac{3}{8}, 1) \times (0,1)$. In Ω_1 we use the grid with the step h and in Ω_2 with the step $2h$ in both variables. In this example we use the formular for parameter ρ not from Theorem 3.2, but from the one-dimensional example. We think that the error estimate from Theorem 3.2 is not optimal and in fact a convergence in two-dimensional problem is the same as in one-dimensional case. Let us note that the mesh problem with nonmatching grid is "bad" – the variables are not divided. Here $\varepsilon = (\sum_{p=1}^{2} \sum_i h^2 (u_p(1, \bar{x}_i) - u_p^N(1, \bar{x}_i))^2)^{1/2}$. In Table 3 the constant 14 approximately gives equality $14\rho_{\tau,h} = \rho_{\text{opt}}$ at $\tau = h = 2^{-8}$. Before to present the numerical results make a shot remark on the realization scheme (2.2), (2.3) in two-dimensional case. For the inverse matrix $(E + \frac{\tau}{2\rho} B)$ we use the lumping procedure and note that matrix $B_{2,2} - B_{1,2}^T((E_{1,1} + \frac{\tau}{2\rho} B_{1,1})^{-1} B_{1,2}$ has three-diagonal form (digonal form for matching grids).

Table 3

$\tau = h$	2^{-4}	2^{-5}	2^{-6}	2^{-7}	2^{-8}
ρ_{opt}	1.2_{10^0}	$6.2_{10^{-1}}$	$3.1_{10^{-1}}$	$1.6_{10^{-1}}$	$7.9_{10^{-2}}$
ε	$7.02_{10^{-3}}$	$3.65_{10^{-3}}$	$1.87_{10^{-3}}$	$9.44_{10^{-4}}$	$4.75_{10^{-4}}$
$14\rho_{\tau,h}$	1.24_{10^0}	$6.19_{10^{-1}}$	$3.09_{10^{-1}}$	$1.55_{10^{-1}}$	$7.73_{10^{-2}}$
ε	$7.02_{10^{-3}}$	$3.65_{10^{-3}}$	$1.87_{10^{-3}}$	$9.44_{10^{-4}}$	$4.75_{10^{-4}}$
$13\rho_{\tau,h}$	1.15_{10^0}	$5.75_{10^{-1}}$	$2.87_{10^{-1}}$	$1.44_{10^{-1}}$	$7.18_{10^{-2}}$
ε	$7.03_{10^{-3}}$	$3.67_{10^{-3}}$	$1.88_{10^{-3}}$	$9.52_{10^{-4}}$	$4.79_{10^{-4}}$
$15\rho_{\tau,h}$	1.33_{10^0}	$6.63_{10^{-1}}$	$3.31_{10^{-1}}$	$1.66_{10^{-1}}$	$8.29_{10^{-2}}$
ε	$7.09_{10^{-3}}$	$3.67_{10^{-3}}$	$1.87_{10^{-3}}$	$9.47_{10^{-4}}$	$4.76_{10^{-4}}$

The notations which we use here were introduced in the proof of Lemma 4.1. On the second step instead (2.3) we consider the equation with factorized matrix:

$$\Big(E+\frac{\tau}{2}A_1\Big)\Big(E+\frac{\tau}{2}A_2\Big)\bar{u}^{n+1}=\Big(E-\frac{\tau}{2\rho}B\Big)\bar{u}^{n+\frac{1}{2}}+\frac{\tau}{2}\bar{f}^{n+\frac{1}{2}},$$

where A_1 and A_2 are the matrices corresponding to the second derivatives on every variables.

As we see for optimal ρ and for $c'\rho_{\tau,h}$ with different constants the error decreases proportionally to τ (here the previous effect of decreasing of the error in four times is absent).

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