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Explicit–implicit domain decomposition methods based on splitting for solving parabolic equations*

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In the article we propose and study a new noniterative domain decomposition algorithm without overlapping subdomains and with the use of splitting procedure in one of subdomains for solution multidimensional boundary value parabolic problems. Noniterative domain decomposition algorithms were studied in the articles [1–12], where the methods were considered with overlapping [1, 2, 4–6, 9, 10] and without overlapping subdomains [1–3, 7, 8, 11, 12]. Although the methods with overlapping subdomains possess better properties as regards convergence, the methods without overlapping are preferable from the algorithmic point of view. The majority of methods without overlapping subdomains are merely conditionally convergent (at a fixed ratio between the mesh-widths of the space and time grids) [1–3, 7, 8]. In the article [11], some method was proposed which is free from this shortcoming and is based on an idea analogous to that of the penalty method, where a solution to the Dirichlet problem is approximated by a solution to the third boundary value problem with a small parameter ascribed to the normal derivative in the boundary condition [13]. It is this approach that we take in the present article. Approximation to a spatial operator by the finite element method (and it is the method that we employ) in a separate subdomain may be accomplished by use made of chaotic grids. This, in turn, leads to obvious difficulties in using implicit method in such a domain. Application of an explicit cyclic iterative process of Chebyshev type to solving the resulting linear algebraic system in fact generates an explicit scheme with variable interior mesh-widths corresponding to the mesh-width of an implicit scheme. The stability condition for such scheme is essentially weaker that the standard condition for a scheme with constant interior mesh-width. When such explicit algorithms are considered on the entire domain, the interior mesh-widths can be chosen in correspondence with the roots of Chebyshev polynomials [14–16]. Owing to the additional "explicitness" in the decomposition method, Chebyshev polynomial happened to be a rather weak tool for obtaining a stable scheme. In the case under consideration we made use of Lanczos polynomials. In [17], this approach only demonstrated

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one more possibility of constructing an explicit algorithm on the entire domain. In the present article, the application of Lanczos polynomials is of principal importance. Finally, it is worth noting that the arising inhomogeneous difference scheme has a certain analogy to the scheme with a weight varying over the spatial points whose stability was studied in [18].

The present article is the logical extention of [25]. In distinguish from [25] we consider only convex domains. Only in this case we may use the Nitshe trick for the domain with piece-wise smooth boundary (see the proof of Theorem 4.1).

1. The original boundary value problems

Let Ω be a convex bounded open connected polyhedron in R^m, $m = 2, 3$; let Ω_1 and Ω_2 be subdomains of Ω such that

$$\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2, \quad \Omega_1 \cap \Omega_2 = \emptyset,$$

and let $S = \overline{\Omega}_1 \cap \overline{\Omega}_2$. We also assume that Ω_2 is a parallelogram (parallelepiped) and the boundery S is othogonal to the m-coordinate axis.

We define the following bilinear forms in the space $H^1(\Omega_p) \times H^1(\Omega_p)$:

$$\begin{aligned} a_p(u,v) &= \int\limits_{\Omega_p} \sum_{i,j=1}^{m} \lambda_{i,j}(\bar{x}) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j}\, d\bar{x}, \quad p = 1, 2, \\ a_2^{(l)}(u,v) &= \int\limits_{\Omega_2} \lambda_{l,l}(\bar{x}) \frac{\partial u}{\partial x_l} \frac{\partial v}{\partial x_l}\, d\bar{x}, \quad l = 1, \ldots, m, \end{aligned} \tag{1.1}$$

here $\bar{x} = (x_1, \ldots, x_m)$ denotes a point in R^m. Concerning the functions $\lambda_{i,j}(\bar{x})$, we assume that in Ω_2 $\lambda_{i,j}(\bar{x}) \equiv 0$, when $i \neq j$ and that the bilinear forms are symmetric, continuous in $H_0^1(\Omega_p) \times H_0^1(\Omega_p)$, and $H_0^1(\Omega_p)$–elliptic; i.e., there are positive numbers λ_1 and λ_2 such that

$$|a_p(u,v)| \le \lambda_1 |u|_{H^1(\Omega_p)} |v|_{H^1(\Omega_p)}, \quad |a_p(u,u)| \ge \lambda_2 |u|^2_{H^1(\Omega_p)}. \tag{1.2}$$

Here $H_0^1(\Omega_p)$ is the subspace of $H^1(\Omega_p)$ obtained by the taking of closure, in the norm of $H^1(\Omega_p)$, of the set of infinitely differentiable functions with compact support in Ω_p. Moreover, $|u|_{H^1(\Omega_p)}$ is the norm of $H_0^1(\Omega_p)$. It is easy to see that $\sum\limits_{l=1}^{m} a_2^{(l)}(u,v) = a_2(u,v)$. Next, we introduce one-parameter families of continuous linear functionals on $H^1(\Omega_p)$ by using the duality pairing on $H^{-1}(\Omega_p) \times H^1(\Omega_p)$; i.e., $l_p(t; v_p) = (f_p(t), v_p)_p$, $p = 1, 2$, where $(,)_p$ is the scalar product in $L_2(\Omega_p)$, $t \in [t_0, t_*]$. Here and in what follows $u(t)$ is the value of a function $u : [t_0, t_*] \to X$ and $\frac{du}{dt}(t)$ is the strong limit in X of the elements $[u(t)]_\tau \equiv (u(t+\tau) - u(t))/\tau$ as $\tau \to 0$ (if it exists).

We denote by X_p the space of the functions in $X(\Omega_p)$ extended to Ω by zero beyond Ω_p. Now, we may introduce the space $\widehat{X} = X_1 \times \ldots \times X_s$ of vector-functions with a norm $\| \, \|_{\widehat{X}}$. For example, $\|u\|^2_{H^l} = \left(\sum_{p=1}^{2} \|u_p\|^2_{H^l(\Omega_p)}\right)$, where u_p are the components of a vector-function u. The scalar product in $\widehat{L}_2$ is

$$(u, v) = (u_1, v_1)_1 + (u_2, v_2)_2.$$

In the space $\widehat{H}^1 \times \widehat{H}^1$, we introduce the bilinear form

$$a(u, v) = a_1(u_1, v_1) + a_2(u_2, v_2).$$

We distinguish the following subspace in $\widehat{H}^1$:

$$\widehat{H}^{1,0} = \{v \in \widehat{H}^1 \mid v_1(\bar{x}) = v_2(\bar{x}),\ \bar{x} \in S\}.$$

Now, we formulate the parabolic Neumann problem as some problem in the subspace $\widehat{H}^{1,0}$. Assume $u_0 \in \widehat{L}_2$ and $f \in L_2(t_0, t_*; \widehat{H}^{-1})$. The problem is to find a vector-function $u \in L_2(t_0, t_*; \widehat{H}^{1,0})$ such that $\frac{du}{dt} \in L_2(t_0, t_*; \widehat{H}^{-1})$ and the equalities

$$\left(\frac{du}{dt}(t), v\right) + a(u(t), v) = (f(t), v), \tag{1.3}$$

$$(u(t_0), v) = (u_0, v) \tag{1.4}$$

hold for every $v \in \widehat{H}^{1,0}$ and almost every $t \in (t_0, t_*)$. It is easy to observe that problem (1.3), (1.4) is equivalent to the conventional Neumann problem in the space $H^1(\Omega)$. We point out that the Neumann problem is considered exclusively for notational simplicity. All results of the present article remain valid for other boundary value problems.

Following [11], we formulate the Neumann problem with conditions of a nonideal contact on the interface: given the same initial data as in problem (1.3), (1.4), find the function $u^\rho \in L_2(t_0, t_*; \widehat{H}^1)$ such that $\frac{du^\rho}{dt} \in L_2(t_0, t_*; \widehat{H}^{-1})$ and, for every $v \in \widehat{H}^1$ and almost every $t \in (t_0, t_*)$, the equalities

$$\left(\frac{du^\rho}{dt}(t), v\right) + a(u^\rho(t), v) + \frac{1}{\rho} \int\limits_S (u_1^\rho(t) - u_2^\rho(t))(v_1 - v_2)\, ds = (f(t), v), \tag{1.5}$$

$$(u^\rho(t_0), v) = (u_0, v) \tag{1.6}$$

hold, where $\rho > 0$. It is shown in [11] that if a solution to problem (1.3), (1.4) is sufficiently smooth in subdomains, then the following inequalities hold:

$$\|u^\rho - u\|_{C(t_0,t_*;\widehat{L}_2)} \le c_1\rho\|u\|_{H^1(t_0,t_*;\widehat{H}^2)}, \quad \|u^\rho\|_X \le c_2\|u\|_X, \tag{1.7}$$

where X is an arbitrary subspace in $L_2(t_0, t_*; \widehat{H}^1)$, $\rho \le \rho_0$, and the numbers c_1 and c_2 are independent of the parameter ρ and the vector-functions u and u^ρ. These equalities justify the use of the penalty method for solving problem (1.3), (1.4) on applying the domain decomposition method to problem (1.5), (1.6).

Finally, we give one well-known inequality that will repeatedly be used in the sequel. The estimate [19, p. 73]

$$\|v\|^2_{L_2(S)} \le c\left(\frac{1}{\delta}\|v\|^2_{L_2(\Omega_p)} + \delta|v|^2_{H^1(\Omega_p)}\right), \quad p = 1, 2, \tag{1.8}$$

holds for an arbitrary function $v \in H^1(\Omega_p)$ and an arbitrary $\delta > 0$; the number c is independent of δ and the function v.

2. Discretization and some inequalities

We introduce a certain notation connected with the approximation of problem (1.5), (1.6) by the finite element method. The terminology used henceforth adheres to the monograph [20]. We introduce regular systems of m-simplices $T_{h,p}$ in the subdomains Ω_p, $p = 1, 2$. In general, the set $T_h = T_{h,1} \cup T_{h,2}$ is not coordinated; i.e., the grid in Ω is composite. Using the sets $T_{h,p}$, we introduce basis systems of piecewise linear functions $\{\varphi_{p,i}(\bar{x})\}_{i=1}^{K_p}$. Here $\{\bar{x}_{p,i}\}_{i=1}^{K_p}$ is the set of all distinct vertices of the m-simplices in $T_{h,p}$ and $\varphi_{p,i}(\bar{x}_{p,j}) = \delta_{ij}$, where δ_{ij} is the Kronecker symbol. Observe that $K_p = O(h^{-m})$. Put $V_{h,p} = \operatorname{span}\{\varphi_{p,i}(\bar{x})\}_{i=1}^{K_p} \subset H^1(\Omega_p) \cap C(\overline{\Omega}_p)$. In line with [21], we introduce lumping linear operators

$$P_{h,p} : V_{h,p} \to L_{h,p},$$

where $L_{h,p} \subset L_2(\Omega_p)$, and

$$d_{h,p}(v, w) = (P_{h,p}v, P_{h,p}w), \quad v,\ w \in V_{h,p}, \quad p = 1, 2. \tag{2.1}$$

Bilinear forms (2.1) are continuous in $L_2(\Omega_p) \times L_2(\Omega_p)$ and $L_2(\Omega_p)$-elliptic.

We introduce the space $\widehat{V}_h = V'_{h,1} \times V'_{h,2}$ of vector-functions, where $V'_{h,p}$, $p = 1, 2$, are the spaces of functions in $V_{h,p}$ extended by zero beyond Ω_p. It is easy to see that $\widehat{V}_h \subset \widehat{H}^1 \cap \widehat{C}$.

To study convergence of the method proposed in the article, we need the Ritz projection of a solution of problem (1.5), (1.6) to the subspace $\widehat{V}_h$, given some $\widehat{H}^1$-elliptic bilinear form. Let λ be a positive number. Introduce the bilinear form

$$a_\lambda(u,v) = \lambda(u,v) + a(u,v) + \frac{1}{\rho}\int\limits_S (u_1 - u_2)(v_1 - v_2)\,ds. \tag{2.2}$$

Generally speaking, bilinear form (2.2) is not $\widehat{H}^1$-elliptic at $\lambda = 0$. We define the Ritz projection $w^\rho(t) \in \widehat{V}_h$ of the vector-function $u^\rho(t)$, given this bilinear form. Put $\varepsilon^\rho(t) = u^\rho(t) - w^\rho(t)$. The following assertion holds:

Lemma 2.1. *The inequality*

$$\|\varepsilon^\rho(t)\|_{\widehat{H}^1} \le ch\left(1 + \frac{h}{\rho}\right)^{1/2} \|u^\rho(t)\|_{\widehat{H}^2}, \quad t \in [t_0, t_*],$$

holds, where the number c is independent of h, ρ, and the vector-function $u^\rho(t)$.

A proof follows from standard estimates [20] and inequality (1.8) with $\delta = h$.

Remark 2.1. For $t \in [t_0, t_* - \tau]$, the vector-function $[w^\rho(t)]_\tau \in \widehat{V}_h$ is the Ritz projection of the vector-function $[u^\rho(t)]_\tau$; therefore, for $[\varepsilon^\rho(t)]_\tau$, some assertion holds analogous to Lemma 2.1 with the replacement of the norm $\|u^\rho(t)\|_{\widehat{H}^2}$ by the norm $\|[u^\rho(t)]_\tau\|_{\widehat{H}^2}$.

Henceforth we will use vector-matrix notation. Let $E^{(p)}$ denote Euclidean vector spaces of dimension K_p, $p = 1, 2$, and let $E = E^{(1)} \times E^{(2)}$. Denote by $\langle\cdot,\cdot\rangle_{(p)}$, $\|\cdot\|_{(p)}$ and $\langle\cdot,\cdot\rangle$, $\|\cdot\|$ the scalar product and the norm in $E^{(p)}$ and E. Put $\rho_{p,i}^2 = d_{h,p}(\varphi_{p,i}, \varphi_{p,i})$, $p = 1, 2$, $i = 1, \dots, K_p$. Further, let $u_p \in V_{h,p}$ and $\bar{u}_p \in E^{(p)}$; moreover, the components of the vector $\bar{u}_p$ are determined by the equalities $(\bar{u}_p)_i = \rho_{p,i} u(\bar{x}_{p,i})$, $i = 1, \dots, K_p$. The equalities give an isomorphism between the space $V_{h,p}$ of finite elements and the Euclidean space $E^{(p)}$. It is easy to see that the equality $d_{h,p}(u_p, v_p) = \langle\bar{u}_p, \bar{v}_p\rangle_{(p)}$ holds for all $u_p, v_p \in V_{h,p}$. Introduce square matrices A_p of order K_p, $p = 1, 2$, and $A_2^{(l)}$ of order K_2, $l = 1, \dots, m$, with entries $\frac{1}{\rho_{p,i}\rho_{p,j}} a_p(\varphi_{p,i}, \varphi_{p,j})$ and $\frac{1}{\rho_{2,i}\rho_{2,j}} a_2^{(l)}(\varphi_{2,i}, \varphi_{2,j})$ consequently. The equalities

$$\begin{aligned} a_p(u_p, v_p) &= \langle A_p \bar{u}_p, \bar{v}_p\rangle_{(p)}, \quad p = 1, 2, \\ a_2^{(l)}(u_2, v_2) &= \langle A_2^{(l)} \bar{u}_2, \bar{v}_2\rangle_{(2)}, \quad l = 1, \dots, m \end{aligned} \tag{2.3}$$

hold for all $u_p, v_p \in V_{h,p}$. It is obvious that $\sum\limits_{l=1}^{m} A_2^{(l)} = A_2$. The preceding equality and condition (1.2) readily imply nonnegativity of the matrices A_p. Observe that the kernel of A_p is one-dimensional: $\ker A_p = \{c\bar{e}_p\}$, where

$(\bar{e}_p)_i = \rho_{p,i}$. Now, introduce square matrices $B_{p,p}$ of order K_p, $p = 1,2$, with entries $\frac{1}{\rho_{p,i}\rho_{p,j}} \int_S \varphi_{p,i}\varphi_{p,j}\,ds$ and the $(K_1 \times K_2)$-matrix $B_{1,2}$ with entries $\frac{1}{\rho_{1,i}\rho_{2,j}} \int_S \varphi_{1,i}\varphi_{2,j}\,ds$, $i = 1,\ldots,K_1$, $j = 1,\ldots,K_2$. Observe that $\|\bar{u}_p\|_{B_{p,p}} = \|u_p\|_{L_2(S)}$.

We will present a succession of inequalities that will be of help in the analysis of stability and convergence for the decomposition method.

Lemma 2.2. *The inequality*

$$|\langle \bar{v}_1, B_{1,2}\bar{v}_2\rangle_{(1)}| \le \frac{1}{2\varepsilon}\|\bar{v}_1\|^2_{B_{1,1}} + \frac{\varepsilon}{2}\|\bar{v}_2\|^2_{B_{2,2}}$$

holds for every vector $\bar{v} \in E$ and every $\varepsilon > 0$.

From now on, we put $A_{p,\rho} = A_p + \frac{1}{\rho}B_{p,p}$, $p = 1,2,$, $A^{(l)}_{2,\rho} = A^{(l)}_2$, $l = 1,\ldots,m-1$, $A^{(m)}_{2,\rho} = A^{(m)}_2 + \frac{1}{\rho}B_{2,2}$.

Lemma 2.3. *Let $\rho \le 1/\lambda_2$, where λ_2 is the constant in condition* (1.4). *Then the matrices $A_{p,\rho}$ are positive definite.*

A proof follows from the generalized Friedrichs inequality [22, p. 129], condition (1.2), $L_2(\Omega_p)$-ellipticity of bilinear forms (2.1) and equalities (2.3).

Henceforth, we assume the condition $\rho \le 1/\lambda_2$ to be fulfilled. Lemma 2.3 implies existence for the matrix $A^{-1}_{1,\rho}$, which enables us to consider the Schur complement

$$\Lambda^{(m)}_{2,\rho} = A^{(m)}_{2,\rho} - \frac{1}{\rho^2}B^T_{1,2}A^{-1}_{1,\rho}B_{1,2} \tag{2.4}$$

of the matrix

$$A^{(m)}_\rho = \begin{pmatrix} A_{1,\rho} & -\frac{1}{\rho}B_{1,2} \\ -\frac{1}{\rho}B^T_{1,2} & A^{(m)}_{2,\rho} \end{pmatrix}.$$

Moreover, by Lemma 2.2 the inequality $\langle A^{(m)}_\rho \bar{v}, \bar{v}\rangle \ge \langle A^{(m)}_2 \bar{v}_2, \bar{v}_2\rangle_{(2)}$ holds for every $\bar{v} \in E$. On the other hand, the following equality

$$\langle \Lambda^{(m)}_{2,\rho}\bar{v}_2, \bar{v}_2\rangle_{(2)} = \langle A^{(m)}_\rho \bar{v}, \bar{v}\rangle$$

holds in the vector subspace $\{(\frac{1}{\rho}A^{-1}_{1,\rho}B_{1,2}\bar{v}_2, \bar{v}_2)^T,\ \bar{v}_2 \in E^{(2)}\}$ of E, that implies nonnegativity of the matrix $\Lambda^{(m)}_{2,\rho} - A^{(m)}_2$. Thereby, we established in particular that the matrix $\Lambda^{(m)}_{2,\rho}$ generates the seminorm $\|\bar{v}_2\|_{\Lambda^{(m)}_{2,\rho}}$ in $E^{(2)}$. The nonnegativity of the matrix $\Lambda^{(m)}_{2,\rho} - A^{(m)}_2$ and inequality (1.8) imply

Lemma 2.4. *The inequality*

$$\|\bar{v}_2\|^2_{B_{2,2}} \le c\Big(\frac{1}{\delta}\|\bar{v}_2\|^2_{(2)} + \delta\|\bar{v}_2\|^2_{\Lambda^{(m)}_{2,\rho}}\Big)$$

holds for every vector $\bar{v}_2 \in E^{(2)}$ and every $\delta > 0$, with the number c independent of h, δ, ρ, and the vector $\bar{v}_2$.

3. Description of the method

In this section we describe the domain decomposition method in vector-matrix form. Let N be a natural number; $\tau = (t_* - t_0)/N$; and $t_n = t_0 + n\tau$, $n = 1, \ldots, N$. Henceforth, let $\{\tau_k\}_{k=1}^{s}$ be a number sequence such that $\tau = \tau_1 + \ldots + \tau_s$ and $\tau_k > 0$. With notation of Section 2, we write down the method as follows:

$$\frac{\bar{u}_1^{n+k/s} - \bar{u}_1^{n+(k-1)/s}}{\tau_k} + A_{1,\rho}\bar{u}_1^{n+(k-1)/s} - \frac{1}{\rho}B_{1,2}\bar{u}_2^n = \bar{f}_1^n, \quad k = 1, \ldots, s, \tag{3.1}$$

$$\frac{\bar{u}_2^{n+l/m} - \bar{u}_1^{n+(l-1)/m}}{\tau} + A_{2,\rho}^{(l)}\bar{u}_2^{n+l/m} = \bar{0}, \quad l = 1, \ldots, m-1, \tag{3.2}$$

$$\frac{\bar{u}_2^{n+1} - \bar{u}_2^{n+(m-1)/m}}{\tau} + A_{2,\rho}^{(m)}\bar{u}_2^{n+1} - \frac{1}{\rho}B_{1,2}^T\bar{u}_1^{n,s} = \bar{f}_2^n, \tag{3.3}$$

$$(\bar{u}_p^0)_i = \rho_{p,i}u_{0,p}(\bar{x}_{p,i}), \quad i = 1, \ldots, K_p, \quad p = 1, 2, \tag{3.4}$$

where $\bar{u}_1^{n,s} = \sum\limits_{k=1}^{s} \frac{\tau_k}{\tau}\bar{u}_1^{n+(k-1)/s}$ and $\bar{f}_{p,i}^n = \rho_{p,i}^{-1}(f_p(t_n + (p-1)\tau), \varphi_{p,i})_p$. Here we assume $u_{0,p} \in H^2(\Omega_p)$. Equation (3.1) presents an explicit scheme with variable mesh-width in the subdomain Ω_1.

Eliminating the intermediate steps in the explicit scheme reduces formulas (3.1)–(3.3) to the next form:

$$\frac{\bar{u}_1^1 - \bar{u}_1^n}{\tau} + Q_s\Big[A_{1,\rho}\bar{u}_1^n - \frac{1}{\rho}B_{1,2}\bar{u}_2^n\Big] = Q_s\bar{f}_1^n, \tag{3.5}$$

$$\frac{\bar{u}_2^{n+l/m} - \bar{u}_1^{n+(l-1)/m}}{\tau} + A_{2,\rho}^{(l)}\bar{u}_2^{n+l/m} = \bar{0}, \quad l = 1, \ldots, m-1, \tag{3.6}$$

$$\begin{aligned}\frac{\bar{u}_2^{n+1} - \bar{u}_2^{n+(m-1)/m}}{\tau} &+ A_{2,\rho}^{(m)}\bar{u}_2^{n+1} - \frac{1}{\rho}B_{1,2}^T\Big[\frac{1}{\rho}R_sB_{1,2}\bar{u}_2^n + Q_s\bar{u}_1^n\Big] \\ &= \bar{f}_2^n + \frac{1}{\rho}B_{1,2}^TR_s\bar{f}_1^n,\end{aligned} \tag{3.7}$$

where $P_s = \prod\limits_{k=1}^{s}(I_1 - \tau_kA_{1,\rho})$, $Q_s = \frac{1}{\tau}A_{1,\rho}^{-1}(I_1 - P_s)$, $R_s = A_{1,\rho}^{-1}(I_1 - Q_s)$, $C_{2,\rho}^{(m)} = A_{2,\rho}^{(m)} - \frac{1}{\rho^2}B_{1,2}^TR_sB_{1,2}$, I_p are the identity matrices of order K_p.

The matrix P_s is easily seen to be a matrix polynomial of degree s in the matrix $A_{1,\rho}$. Furthermore, the properties of stability and convergence are governed by the choice of the sequence $\{\tau_k\}_{k=1}^s$. In the articles cited in the introduction, the answer to the question about expanding the stability region for explicit methods is based on the relation between the polynomial and Chebyshev's polynomials of the first kind. In our case (for the domain decomposition method) it is Lemma 4.1 that serves as a basis for studying convergence. The lemma takes place in case we consider Lanczos' polynomials [23]. The properties of the polinomials we need are given in [25].

4. Convergence theorems

Assume $\|A_{1,\rho}\| \leq \lambda_\rho$, where λ_ρ is some upper bound of the spectrum of the matrix $A_{1,\rho}$. Here and in the sequel, we consider the spectral norm of a matrix. According to [25], put

$$\sigma_s = \tau\lambda_\rho, \tag{4.1}$$

where $\sigma_s = \frac{(s+1)^2-1}{3}$. Condition (4.1) allows us to define some integer-valued parameter s. Let $\lambda_\rho = \min\{\tilde{\lambda}_\rho \mid \|A_{1,\rho}\| \leq \tilde{\lambda}_\rho,\, 3\tau\tilde{\lambda}_\rho+1 = k^2,\, k \text{ is an integer}\}$. Then we define the integer-valued parameter s by the formula

$$s = (3\tau\lambda_\rho + 1)^{1/2} - 1. \tag{4.2}$$

From [25] we take the formula for τ_k

$$\tau_k = \frac{\tau}{\sigma_s \sin^2 k\pi/(s+1)}, \quad k = 1, \ldots, s. \tag{4.3}$$

The properties of the matricies P_s, Q_s, R_s are formulated in [25, Lemma 5.1]. For the comfort reading we give it here:

Lemma 4.1. *The inequalities*

$$0 \leq \langle P_s\bar{v}, \bar{v}\rangle_{(1)} \leq \|\bar{v}\|_{(1)}^2, \qquad \frac{\nu_s}{\sigma_s}\|\bar{v}\|_{(1)}^2 \leq \langle Q_s\bar{v}, \bar{v}\rangle_{(1)} \leq \|\bar{v}\|_{(1)}^2,$$
$$0 \leq \langle R_s\bar{v}, \bar{v}\rangle_{(1)} \leq \tau\|\bar{v}\|_{(1)}^2, \quad 0 \leq \langle Q_s^{-1}R_s\bar{v}, \bar{v}\rangle_{(1)} \leq \nu_0\tau\|\bar{v}\|_{(1)}^2$$

hold for every vector $\bar{v} \in E^{(1)}$, where a number $\nu_s = 1 - (s+1)^{-2}$ and a number ν_0 is independent of s.

It is easy to observe that Lemmas 2.3 and 4.1 imply that the symmetric matrix $Q_sA_{1,\rho}$ is positive definite; i.e., the matrix generates a norm in the space $E^{(1)}$.

Assume $u^\rho(t)$ to be a solution to problem (1.5), (1.6) and let $w^\rho(t)$ be the Ritz projection of $u^\rho(t)$ to the subspace $\widehat{V}_h$ given bilinear form (2.2). Henceforth, we put $w_p^n = w_p^\rho(t_n)$, $p = 1,2$, $n = 0,\dots,N$, and denote by $\overline{w}_p^n$ the vectors with components $\rho_{p,i} w_p^n(\bar{x}_{p,i})$, $i = 1,\dots,K_p$. Introduce the following sequence of vectors in $E^{(p)}$: $\bar{\xi}_p^n = \bar{u}_p^n - \overline{w}_p^n$, and in $E^{(2)}$:

$$\bar{\xi}_2^{n+l/m} = \bar{u}_2^{n+l/m} - \overline{w}_2^n + \tau \bar{r}_2^{n+l/m}, \quad l = 1,\dots,m-1,$$

where

$$\bar{r}_2^{n+l/m} = \Pi_{h,2}\Big(\sum_{i=1}^{l} z_i(t_{n+1})\Big),$$

$$z_i(t_{n+1}) = -\frac{\partial}{\partial x_i}\Big[\lambda_{i,i}(\bar{x})\frac{\partial u_2^\rho}{\partial x_i}\Big](t), \quad n = 0,\dots,N.$$

Here $\Pi_{h,2}$ is the linear operator of piece-wise linear interpolation in the subdomain Ω_2 [20]. In accordance with (3.5)–(3.7), the equation in $\bar{\xi}^n$ takes the form

$$\frac{\bar{\xi}_1^{n+1} - \bar{\xi}_1^n}{\tau} + Q_s\Big[A_{1,\rho}\bar{\xi}_1^n - \frac{1}{\rho}B_{1,2}\bar{\xi}_2^n\Big] = \bar{g}_1^n, \tag{4.4}$$

$$\frac{\bar{\xi}_2^{n+l/m} - \bar{\xi}_2^{n+(l-1)/m}}{\tau} + A_{2,\rho}^{(l)}\bar{\xi}_2^{n+l/m} = \bar{g}_2^{n,l}, \quad l = 1,\dots,m-1, \tag{4.5}$$

$$\frac{\bar{\xi}_2^{n+1} - \bar{\xi}_2^{n+(m-1)/m}}{\tau} + A_{2,\rho}^{(m)}\bar{\xi}_2^{n+1} - \frac{1}{\rho}B_{1,2}^T\bar{\xi}_1^{n,s} = \bar{g}_2^{n,m}, \tag{4.6}$$

where

$$\bar{g}_1^n = Q_s\bar{z}_1^n - R_sA_{1,\rho}\overline{w}_{\tau,1}^n,$$

$$\bar{g}_2^{n,l} = \bar{r}_2^{n+l/m} - \bar{r}_2^{n+(l-1)/m} - A_{2,\rho}^{(l)}\overline{w}_2^n - \tau A_{2,\rho}^{(l)}\bar{r}_2^{n+l/m}, \quad l = 1,\dots,m-1,$$

$$\bar{g}_2^{n,m} = \bar{z}_2^n + \frac{1}{\rho}B_{1,2}^T R_s\bar{z}_1^n + \frac{1}{\rho}B_{1,2}^T(R_s - \tau E_1)\overline{w}_{\tau,1}^n + \sum_{l=1}^{m-1} A_{2,\rho}^{(l)}\overline{w}_2^n - \bar{r}_2^{n+(m-1)/m},$$

$$\bar{z}_1^n = \bar{f}_1^n - \overline{w}_{\tau,1}^n - A_{1,\rho}\overline{w}_1^n + \frac{1}{\rho}B_{1,2}\overline{w}_2^n,$$

$$\bar{z}_2^n = \bar{f}_2^n - \overline{w}_{\tau,2}^n - A_{2,\rho}\overline{w}_2^{n+1} + \frac{1}{\rho}B_{1,2}^T\overline{w}_1^{n+1}.$$

We use the notation $\overline{w}_{\tau,p}^n = \frac{1}{\tau}(\overline{w}_p^{n+1} - \overline{w}_p^n)$. Before formulating a theorem we assume smoothness conditions on the vector-function $u^\rho(t)$:

$$\lambda_{i,j} \in C(\overline{\Omega}_1) \cap C^3(\overline{\Omega}_2),$$

$$u^\rho \in C(t_0, t_*; \widehat{H}^4), \quad \frac{du^\rho}{dt} \in L_2(t_0, t_*; \widehat{H}_2), \quad \frac{d^2u^\rho}{dt^2} \in L_2(t_0, t_*; \widehat{L}_2). \tag{4.7}$$

Theorem 4.1. *Assume that conditions (4.7) are satisfied for problem (1.5), (1.6) with $\rho \le 1/\lambda_2$ (λ_2 is the ellipticity constant of (1.2)) and that, for solving the problem, method (3.1)–(3.4) is used with $h \le h_0$, $\tau \le \tau_0$, and the choice of the parameter s and the sequence $\{\tau_k\}_{k=1}^{s}$ made according to equalities (4.2) and (4.3). Then the following estimate holds:*

$$\max_{1\le n\le N} \|u^n - u^\rho(t_n)\|_{\widehat{L}_2} \le cM(u^\rho)\Big[h + \tau + \rho^{-1/2}(h^{3/2} + \tau) + \\ \rho^{-1}(h^{3/2} + \tau) + \rho^{-3/2}(h^3 + \tau^2)\Big],$$

where the numbers h_0, τ_0, and c are independent of the parameters ρ, h, τ, and s and the vector-function $u^\rho(t)$.

Proof. Multiply (4.4) by $2\tau\bar{\xi}_1^{n+1}$ to obtain

$$\|\bar{\xi}_1^{n+1}\|_{(1)}^2 + \|\bar{\zeta}_1^n\|_{P_s}^2 + \frac{\tau}{2}\|\bar{\eta}_1^n\|_{Q_sA_{1,\rho}}^2 + \frac{\tau}{2}\|\bar{\zeta}_1^n\|_{Q_sA_{1,\rho}}^2 - \\ \frac{2\tau}{\rho}\langle Q_sB_{1,2}\bar{\eta}_2^n, \bar{\eta}_1^n\rangle_{(1)} + \frac{2\tau}{\rho}\langle Q_sB_{1,2}\bar{\zeta}_2^n, \bar{\zeta}_1^n\rangle_{(1)} - \\ \frac{\tau}{2\rho}\langle Q_sB_{1,2}\bar{\eta}_2^n, \bar{\zeta}_1^n\rangle_{(1)} + \frac{\tau}{2\rho}\langle Q_sB_{1,2}\bar{\zeta}_2^n, \bar{\eta}_1^n\rangle_{(1)} = \|\bar{\xi}_1^n\|_{(1)}^2 + 2\tau\langle \bar{g}_1^n, \bar{\xi}_1^{n+1}\rangle_{(1)},$$

where $\bar{\eta}^n = \bar{\xi}^{n+1} + \bar{\xi}^n$, $\bar{\zeta}^n = \bar{\xi}^{n+1} - \bar{\xi}^n$. (4.5) multiply by $2\tau\bar{\xi}_2^{n+l/m}$, then

$$\|\bar{\xi}_2^{n+l/m}\|_{(2)}^2 + 2\tau\|\bar{\xi}_2^{n+l/m}\|_{A_{2,\rho}^{(l)}}^2 + \|\bar{\xi}_2^{n+l/m} - \bar{\xi}_2^{n+(l-1)/m}\|_{(2)}^2 \\ = \|\bar{\xi}_2^{n+(l-1)/m}\|_{(2)}^2 + 2\tau\langle \bar{g}_2^{n,l}, \bar{\xi}_2^{n+l/m}\rangle_{(2)}.$$

Multiply (4.6) by $2\tau\bar{\xi}_2^{n+1}$ and we arive at

$$\|\bar{\xi}_2^{n+1}\|_{(2)}^2 + 2\tau\|\bar{\xi}_2^{n+1}\|_{A_{2,\rho}^{(m)}}^2 + \|\bar{\xi}_2^{n+1} - \bar{\xi}_2^{n+(m-1)/m}\|_{(2)}^2 - \\ \frac{\tau}{2\rho^2}\langle B_{1,2}^T R_s B_{1,2}\bar{\eta}_2^n, \bar{\eta}_2^n\rangle_{(2)} + \frac{\tau}{2\rho^2}\langle B_{1,2}^T R_s B_{1,2}\bar{\zeta}_2^n, \bar{\zeta}_2^n\rangle_{(2)} - \\ \frac{2\tau}{\rho}\langle B_{1,2}^T Q_s\bar{\eta}_1^n, \bar{\eta}_2^n\rangle_{(2)} + \frac{2\tau}{\rho}\langle B_{1,2}^T Q_s\bar{\zeta}_1^n, \bar{\zeta}_2^n\rangle_{(2)} - \\ \frac{\tau}{2\rho}\langle B_{1,2}^T Q_s\bar{\eta}_1^n, \bar{\zeta}_2^n\rangle_{(2)} + \frac{\tau}{2\rho}\langle B_{1,2}^T Q_s\bar{\zeta}_1^n, \bar{\eta}_2^n\rangle_{(2)} \\ = \|\bar{\xi}_2^{n+(m-1)/m}\|_{(2)}^2 + 2\tau\langle \bar{g}_2^{n,m}, \bar{\xi}_2^{n+1}\rangle_{(2)}.$$

Take the sum of the equalities obtained. Using Lemma 4.1, we arrive at

$$\|\bar\xi^{n+1}\|_{\bar B}^2+2\tau\sum_{l=1}^{m-1}\|\bar\xi_2^{n+l/m}\|_{A_{2,\rho}^{(l)}}^2+\sum_{l=1}^{m}\|\bar\xi_2^{n+l/m}-\bar\xi_2^{n+(l-1)/m}\|_{(2)}^2+\frac{\tau}{2}\|\bar\eta_2^n\|_{\Lambda_{2,\rho}^{(m)}}^2+$$
$$\frac{\tau}{2}\|\bar\zeta_2^n\|_{\Lambda_{2,\rho}^{(m)}}^2+\frac{\tau}{2}\|\bar\varphi_1^n\|_{Q_sA_{1,\rho}}^2+\frac{\tau}{2}\|\bar\psi_1^n\|_{Q_sA_{1,\rho}}^2+\frac{\tau}{\rho^2}\|\bar\zeta_2^n\|_{B_{1,2}^TR_sB_{1,2}}^2$$
$$\le\|\bar\xi^n\|_{\bar B}^2+2\tau\langle\bar g_1^n,\bar\xi_1^{n+1}\rangle_{(1)}+2\tau\sum_{l=1}^{m}\langle\bar g_2^{n,l},\bar\xi_2^{n+l/m}\rangle_{(2)},\tag{4.8}$$

where $\|\bar\xi^n\|_{\bar B}^2=\|\bar\xi_1^n\|_{(1)}^2+\|\bar\xi_2^n\|_{(2)}^2+\tau\|\bar\xi_2^n\|_{A_{2,\rho}^{(m)}}^2$, $\bar\varphi_1^n=\bar\eta_1^n-\frac{1}{\rho}A_{1,\rho}^{-1}B_{1,2}\bar\eta_2^n$, $\bar\psi_1^n=\bar\zeta_1^n+\frac{1}{\rho}A_{1,\rho}^{-1}B_{1,2}\bar\zeta_2^n$.

Make the estimate of the scalar products:

$$2\tau\langle\bar g_1^n,\bar\xi_1^{n+1}\rangle_{(1)}+2\tau\sum_{l=1}^{m}\langle\bar g_2^{n,l},\bar\xi_2^{n+l/m}\rangle_{(2)}$$
$$=2\tau\langle\bar g^n,\bar\xi^{n+1}\rangle+2\tau^2\sum_{l=1}^{m-1}\langle A_{2,\rho}^{(l)}\bar r_2^{n+l/m},\bar\xi_2^{n+l/m}\rangle_{(2)}+$$
$$2\tau\sum_{l=1}^{m-1}\langle\bar r_2^{n+l/m}-\bar r_2^{n+(l-1)/m}-A_{2,\rho}^{(l)}\overline{w}_2^{n+1},\bar\xi_2^{n+l/m}\rangle_{(2)}+$$
$$2\tau^2\sum_{l=1}^{m-1}\langle A_{2,\rho}^{(l)}\overline{w}_{\tau,2}^n,\bar\xi_2^{n+l/m}\rangle_{(2)}+$$
$$2\tau\Big\langle\sum_{l=1}^{m-1}A_{2,\rho}^{(l)}\overline{w}_2^{n+1}-\bar r_2^{n+(m-1)/m},\bar\xi_2^{n+1}\Big\rangle_{(2)}.\tag{4.9}$$

Summand $2\tau\langle\bar g^n,\bar\xi^{n+1}\rangle$ can be evaluated like in [25]:

$$2\tau|\langle\bar g^n,\bar\xi^{n+1}\rangle|\le\varepsilon_1\tau\|\bar\xi^{n+1}\|_{\bar B}^2+\frac{\tau}{\varepsilon_1}(2\|\bar z_1^n\|_{(1)}^2+\|\bar z_2^n\|_{(2)}^2)+$$
$$\frac{\varepsilon_2\tau}{2}\|\bar\varphi_1^n\|_{Q_sA_{1,\rho}}^2+\frac{\varepsilon_2\tau}{2}\|\bar\psi_1^n\|_{Q_sA_{1,\rho}}^2+\frac{\nu_0\tau^3}{\varepsilon_2}\|\overline{w}_{\tau,1}^n\|_{A_{1,\rho}}^2+$$
$$4\varepsilon_3c\tau\|\bar\xi^{n+1}\|_{\bar B}^2+2\varepsilon_3c\tau\|\bar\xi^n\|_{\bar B}^2+\varepsilon_3c\tau\|\bar\eta_2^n\|_{\Lambda_{2,\rho}^{(m)}}^2+$$
$$2\varepsilon_3c\tau\|\bar\zeta_2^n\|_{\Lambda_{2,\rho}^{(m)}}^2+\frac{3\tau^3}{2\varepsilon_3\rho}\|\overline{w}_{\tau,1}^n\|_{A_{1,\rho}}^2.\tag{4.10}$$

Afterwards, using the piecewise linear interpolation theorem we obtain the estimate

$$2\tau^2\Big|\sum_{l=1}^{m-1}\langle A_{2,\rho}^{(l)}\bar r_2^{n+l/m},\bar\xi_2^{n+l/m}\rangle_{(2)}\Big|\le\tau^2\varepsilon_4\sum_{l=1}^{m-1}\|\bar\xi_2^{n+l/m}\|_{A_{2,\rho}^{(l)}}^2+\frac{cm\tau^2}{\varepsilon_4}\|u^\rho\|_{(*)}^2.\tag{4.11}$$

We use the notation $\|u^\rho\|_{(*)}^2=(\|u^\rho\|_{C(t_0,t_*;\widehat H^3)}^2+h^2\|u^\rho\|_{C(t_0,t_*;\widehat H^4)}^2)$. With the help of equality (2.2) and Lemma 2.1 we arrive at the estimate

$$2\tau\Big|\sum_{l=1}^{m-1}\langle\bar{r}_2^{n+l/m}-\bar{r}_2^{n+(l-1)/m}-A_{2,\rho}^{(l)}\overline{w}_2^{n+1},\bar{\xi}_2^{n+l/m}\rangle_{(2)}\Big|\le 2\varepsilon_5 cm\tau\|\bar{\xi}^{n+1}\|_{\bar{B}}^2+$$
$$2\varepsilon_5 m\tau\sum_{l=1}^{m-1}\|\bar{\xi}_2^{n+l/m}\|_{A_{2,\rho}^{(l)}}^2+\varepsilon_5\tau\sum_{l=1}^{m}\|\bar{\xi}_2^{n+l/m}-\bar{\xi}_2^{n+(l-1)/m}\|_{(2)}^2+$$
$$\frac{2cmh^2\tau(1+\frac{h}{\rho})}{\varepsilon_5}\|u^\rho\|_{C(t_0,t_*;\widehat{H}^2)}^2+\frac{2cmh^2\tau}{\varepsilon_5}\|u^\rho\|_{(*)}^2. \tag{4.12}$$

The obvious estimate of the next summand is

$$2\tau^2\Big|\sum_{l=1}^{m-1}\langle A_{2,\rho}^{(l)}\overline{w}_{\tau,2}^n,\bar{\xi}_2^{n+l/m}\rangle_{(2)}\Big|\le\frac{\tau^3}{\varepsilon_6}\|\overline{w}_{\tau,2}^n\|_{A_{2,\rho}}^2+\tau^2\varepsilon_6\sum_{l=1}^{m-1}\|\bar{\xi}_2^{n+l/m}\|_{A_{2,\rho}^{(l)}}^2. \tag{4.13}$$

Finally, the latter summand can be evaluated by the inequality (1.8) and Lemmas 2.1 and 4.1. The corresponding estimate looks like:

$$2\tau\Big|\langle\sum_{l=1}^{m-1}A_{2,\rho}^{(l)}\overline{w}_2^{n+1}-\bar{r}_2^{n+(m-1)/m},\bar{\xi}_2^{n+1}\rangle_{(2)}\Big|$$
$$\le\varepsilon_7 cm\tau\|\bar{\xi}^{n+1}\|_{\bar{B}}^2+\frac{1}{2}\varepsilon_7 c\tau\|\bar{\eta}_2^n\|_{\Lambda_{2,\rho}^{(m)}}^2+\frac{1}{2}\varepsilon_7 c\tau\|\bar{\zeta}_2^n\|_{\Lambda_{2,\rho}^{(m)}}^2+$$
$$\Big[\frac{2ch^2\tau}{\varepsilon_7}\Big(1+\frac{h}{\rho}+\frac{h}{\rho^2}\Big)\Big]\|u^\rho\|_{C(t_0,t_*;\widehat{H}^2)}^2+\frac{2cmh^2\tau}{\varepsilon_7}\|u^\rho\|_{(*)}^2. \tag{4.14}$$

When we obtain the latter estimate, we use the Nitsche's method. A posibility of the one is based on the convexity of the domain Ω, such as H^2-norm of the solution is estimated by L_2-norm of the right- hand side in this case [19]. That is why the set of domains under consideration was constricted.

In line with [25] we have the estimates:

$$\begin{aligned}
\|\bar{\xi}^0\|_{\bar{B}}^2 &\le ch^2\Big(1+\frac{h}{\rho}\Big)\Big(1+\frac{\tau}{\rho}\Big)\|u^\rho(t_0)\|_{\widehat{H}^2}^2,\\
\|\bar{z}^n\|^2 &\le c\Big\{h^2\Big(1+\frac{h}{\rho}\Big)\Big(\|u^\rho\|_{C(t_0,t_*;\widehat{H}^2)}^2+\frac{1}{\tau}\Big\|\frac{du^\rho}{dt}\Big\|_{L_2(t_n,t_{n+1};\widehat{H}^2)}^2\Big)+\\
&\qquad\tau\Big\|\frac{d^2u^\rho}{dt^2}\Big\|_{L_2(t_n,t_{n+1};\widehat{L}_2)}^2\Big\},\\
\|\overline{w}_{\tau,p}^{\,n}\|_{A_{1,\rho}}^2 &\le c\frac{1}{\tau\rho}\Big(\Big\|\frac{du^\rho}{dt}\Big\|_{L_2(t_n,t_{n+1};\widehat{H}^1)}^2+h^2\Big(1+\frac{h}{\rho}\Big)\Big\|\frac{du^\rho}{dt}\Big\|_{L_2(t_n,t_{n+1};\widehat{H}^2)}^2\Big).
\end{aligned} \tag{4.15}$$

Put $\varepsilon_1=\frac{1}{16}$, $\varepsilon_2=\frac{1}{2}$, $\varepsilon_3=\frac{1}{64c}$, $\varepsilon_4=\varepsilon_6=\frac{1}{2\tau}$, $\varepsilon_5=\frac{1}{32cm}$, $\varepsilon_7=\frac{1}{16cm}$. Assume $\tau\le 1/2$. By virtue of equalities (4.8)–(4.9), inequalities (4.10)–(4.15) and using the grid version of the Gronwall's lemma [24, p. 311], we obtain the estimate

$$\max_{1\le n\le N} \|u^n - u^\rho(t_n)\|^2_{\widehat{L}_2} \le cM^2(u^\rho)\Big[h^2 + \tau^2 + \rho^{-1}(h^3+\tau^2) + \rho^{-2}(h^3+\tau^2) + \rho^{-3}(h^6+\tau^4)\Big]. \quad \square$$

Remark 4.1. The inequality (4.8) yield the stability of the scheme (3.1)–(3.3) relative to initial data and right-hand sides.

Remark 4.2. At a fixed value of the parameter ρ, we obtain an error estimate of order $O(h+\tau)$.

So far, equalities (3.1)–(3.4) were treated as a method for solving the perturbed problem (1.5), (1.6). By analogy to [11], method (3.1)–(3.4) may be used as a method for solving the original problem (1.3), (1.4). In that event, inequalities (1.7) serve as a basis; i.e., we speak about the penalty method at small values of ρ. Let a solution $u(t)$ to problem (1.3), (1.4) satisfy the following conditions that are analogous to those of (4.7):

$$\lambda_{i,j} \in C(\overline{\Omega}_1) \cap C^3(\overline{\Omega}_2),$$
$$u \in C(t_0, t_*; \widehat{H}^4), \quad \frac{du}{dt} \in L_2(t_0, t_*; \widehat{H}_2), \quad \frac{d^2u}{dt^2} \in L_2(t_0, t_*; \widehat{L}_2). \tag{4.18}$$

Theorem 4.1 and inequalities (1.7) imply that the estimate

$$\|u^n - u(t_n)\|_{\widehat{L}_2} \le cM(u)\Big[h + \tau + \rho^{-1/2}(h^{3/2}+\tau) + \rho^{-1}(h^{3/2}+\tau) + \rho^{-3/2}(h^3+\tau^2) + \rho\Big]$$

holds for $\rho \le \min(\rho_0, 1/\lambda_2)$. The precise optimization of the right-hand side of the preceding inequality with respect to the parameter ρ is quite difficult (it leads to a fifth-order equation in $\rho^{-1/2}$). However, optimization of the expressions $\rho^{-1/2}(h^{3/2}+\tau)+\rho$, $\rho^{-1}(h^{3/2}+\tau)+\rho$, and $\rho^{-3/2}(h^3+\tau^2)+\rho$ is easily performable and yields $\rho_1 = c_1(h^{3/2}+\tau)^{2/3}$, $\rho_2 = c_2(h^{3/2}+\tau)^{1/2}$, and $\rho_3 = c_3(h^3+\tau^2)^{2/5}$ respectively. It is easy to see that the best estimate is attained at $\rho = \rho_2$. Thus we proved

Theorem 4.2. *Let conditions* (4.18) *be satisfied for problem* (1.3), (1.4) *and method* (3.1)–(3.4) *be used to solve the problem with* $h \le h_0$, $\tau \le \tau_0$, $\rho = c'(h^{3/2}+\tau)^{1/2} \le \min(\rho_0, 1/\lambda_2)$, *and the choice of the parameter* s *and the sequence* $\{\tau_k\}_{k=1}^{s}$ *make in accordance with equalities* (4.2) *and* (4.3). *Then the estimate*

$$\max_{1\le n\le N} \|u^n - u(t_n)\|_{\widehat{L}_2} \le cM(u)(h^{3/4} + \sqrt{\tau})$$

holds, where the numbers h_0, τ_0, c, *and* c' *are independent of the parameters* h, τ, *and* s *and the vector-function* $u(t)$.

Thus, as opposed to the scheme with constant mesh-width $\tau_k = \tau/s$ for which $s = O(\tau/h^2)$, the stability in our case is guaranteed at $s = O(\sqrt{\tau}/h)$.

5. Numerical experiments

Make some tests according to the method under consideration. Thus, we can confirm the fact that the estimate obtained in Theorem 4.2 can not be improved as regard with the power of τ. As for the power of h, the estimate is not optimal.

Let Ω be the square $(0,1)\times(0,1)$. Treat in Ω the parabolic Dirichlet problem:

$$\frac{\partial u}{\partial t} = \lambda_0\Big[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\Big], \quad (t,x,y) \in (0,1)\times\Omega,$$
$$u(t,x,y) = 0, \quad (t,x,y) \in (0,1)\times\partial\Omega,$$
$$u(0,x,y) = \sin(\pi x)\sin(\pi y), \quad (x,y)\in\Omega.$$

The solution to the problem is the function

$$u(t,x,y) = \exp(-2\lambda_0\pi^2 t)\sin(\pi x)\sin(\pi y).$$

Put $\lambda_0 = 0.05$. In this supposition the $\widehat{L}_2$-norm of the solution decreases approximately e times during the interval $t = 1$.

Let $\Omega_1 = (0,1)\times(0,3/8)$ and $\Omega_2 = (0,1)\times(3/8,1)$. We made tests on a uniform grid with $h = \tau$. The grid version of the $L_2(\Omega_p)$-error norm at the moment $t = 1$ we denote by

$$\varepsilon_p = h\Big[\sum_{i=1}^{K_p}\{u^N(\bar{x}_{p,i}) - u(\bar{x}_{p,i})\}^2\Big]^{1/2}, \quad p = 1,2,$$

where $N\tau = 1$. So, the $\widehat{L}_2$-error norm can be written as follows:

$$\varepsilon = \sqrt{\varepsilon_1^2 + \varepsilon_2^2}.$$

Let calculate parameter ρ by formula

$$\rho = c\sqrt{\tau + h^{3/2}},$$

where $c = 8$.

The results are given in the table. Variable interior mesh-widths τ_k calculate by formula (4.3). The parameter s was defined so, that the decrease s on 1 leads to the instability.

It is obvious that the reduction of the mesh-widths twice implies the decrease of the $\widehat{L}_2$-error norm roughly in $\sqrt{2}$ times and the increase of the parameter s – in approximately $\sqrt{2}$ times.

$\tau = h$	s	ε	$\tau = h$	s	ε	$\tau = h$	s	ε
2^{-3}	1	0.024006	2^{-4}	2	5.03349702	2^{-5}	4	$2.827 * 10^{18}$
	2	0.023098		3	0.015347		5	0.010599
	3	0.022990		4	0.015328		6	0.010598
2^{-6}	7	$3.492 * 10^{33}$	2^{-7}	11	2.19232914	2^{-8}	16	overflow
	8	0.0074342		12	0.0052428		17	0.0037044
	9	0.0074340		13	0.0052428		18	0.0037044

In conclusion, we make some remarks. We does not address the question about the influence of round-off errors. As for the subdomains in which the implicit scheme is used, the decomposition itself diminishes the dimension of the arising algebraic systems, which increases the stability of algorithms for solving them. The question about the influence of the round-off errors in implementation of the explicit scheme relates to the well-studied problem of permutating the roots of polynomials [24]. At this juncture, the use of the Lanczos polynomials in place of the Chebyshev polynomials improves the situation due to the presence of multiple roots. This is experimentally confirmed by numerical calculations in which, for s near 30, the different ordering of the roots of polynomials is practically negligible. For the Chebyshev polynomials and the same values of s, the different ordering essentially influences stability [24].

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