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Hamiltonian systems with continuous spectrum and vortex structures in rotating fluid*

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The problem of small oscillations of rotating inviscid incompressible fluid is considered. The linearization of the Euler equation in the rotating coordinate system on the solution, which corresponds to the rotation of the fluid as a rigid body around z-axis, gives the system of differential equations, first obtained by H. Poincaré [1]. The systematic investigation of the systems of equations of this new type began with the work of S.L. Sobolev [2].

The characteristics of spectra in this problem were investigated in detail in the case when the velocity $V = (u, v, w)$ and the pressure P depend only on two spatial variables x,z. We consider the case when the fluid domain is an endless cylinder Q with the ruling parallel to the axis y and the convex domain Ω bounded by the smooth contour Γ in the base. Without loss of generality we can assume that the angle velocity is $\mathbf{k} = (0, 0, 1/2)$. It was found [3, 4], that as a result of variations of Γ the corresponding self-adjoint operators may exhibit absolutely continuous bands in their spectra; and a singular continuous spectrum may arise as well [5, 6]. The aim of this report is to describe the motion of fluid particles and the evolution of vortex structures which correspond to the intervals of a continuous spectrum.

Under the assumptions mentioned above we obtain the system of differential equations ($(x, z) \in \Omega$, $t \in \mathbf{R}$):

$$\frac{\partial u}{\partial t} = v - \frac{\partial P}{\partial x}, \qquad \frac{\partial v}{\partial t} = -u, \qquad \frac{\partial w}{\partial t} = -\frac{\partial P}{\partial z}, \tag{1}$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0, \tag{2}$$

with the initial data

$$V|_{t=0} = V_0(x, z), \qquad (x, z) \in \Omega, \tag{3}$$

and the boundary condition on Γ

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$$(un_1 + wn_3)|_\Gamma = 0, \tag{4}$$

where $\mathbf{n} = (n_1, n_3)$ is a normal vector on Γ in the plane x,z. The energy conservation law holds for any solution of the problem (1)–(4) (see [4]):

$$\mathcal{E}(t) = \int_\Omega (|u|^2 + |v|^2 + |w|^2)\, dxdz = \text{const}. \tag{5}$$

The set of all stationary solutions of the problem (1)–(4) is described by the formulae $\widetilde{V} = (0, \tilde{v}(x), 0)$, $\widetilde{P}(x) = \int \tilde{v}(x)\, dx$, where $\tilde{v}(x)$ is an arbitrary smooth function.

The stream function Ψ for the components u, w of V is defined by the equations

$$u = -\frac{\partial \Psi}{\partial z}, \qquad w = \frac{\partial \Psi}{\partial x}. \tag{6}$$

It follows from (2), (4) that for any smooth solution V, P this function exists for all t in $\overline{\Omega} = \Omega \cup \Gamma$, and we can assume that $\Psi|_\Gamma = 0$.

Lemma 1. *Let $V(x, z, t)$, $P(x, z, t)$ be an arbitrary smooth solution of the problem* (1)–(4). *Then the corresponding stream function Ψ is the solution of the following problem*

$$\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial z^2}\right) + \frac{\partial^2 \Psi}{\partial z^2} = 0, \quad (x, z, t) \in \Omega \times \mathbf{R}, \tag{7}$$

$$\Psi|_\Gamma = 0, \tag{8}$$

$$\Psi|_{t=0} = \Psi_0(x, z), \qquad \left.\frac{\partial \Psi}{\partial t}\right|_{t=0} = \Psi_1(x, z), \tag{9}$$

where the initial data Ψ_0, Ψ_1 are defined uniquely by V_0. Conversely, if the smooth function $\Psi(x, z, t)$ satisfies (7)–(9), *then the set of solutions of* (1)–(4)

$$V = V_0(x, z, t) + \widetilde{V}(x), \quad P = P_0(x, z, t) + \widetilde{P}(x), \tag{10}$$

corresponds to Ψ. In (10) *V_0, P_0 is a particular solution and $\widetilde{V}$, $\widetilde{P}$ is an arbitrary stationary solution.*

For the problem (7)–(9) it turned out natural to consider an operator A defined on smooth functions Ψ in the Hilbert space $\overset{\circ}{W}{}^1_2(\Omega)$ as the solution of the problem

$$\Delta A\Psi = -\Psi_{zz}, \qquad A\Psi|_\Gamma = 0, \tag{11}$$

where $\Delta = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial z^2}$. This operator is self-adjoint and bounded in the inner product

$$\langle \Psi, \Phi \rangle = \int_\Omega (\Psi_x \overline{\Phi}_x + \Psi_z \overline{\Phi}_z)\, dxdy. \tag{12}$$

Using A the problem (7)–(9) may be written in the following form

$$\Psi'' = A\Psi, \qquad \Psi(0) = \Psi_0, \qquad \Psi'(0) = \Psi_1, \tag{13}$$

where $\Psi(\cdot,\cdot,t)$ is considered as a function of t with the values in $\overset{\circ}{W}{}^1_2(\Omega)$. The family of invariant subspaces H_N^k of the operator A in $\overset{\circ}{W}{}^1_2(\Omega)$ is constructed in [6] for $N \geq 2$, $k = 1, \ldots, N-1$, $GQD(N,k) = 1$. If the initial data Ψ_0, Ψ_1 are chosen in H_N^k, then the solution of (13) is represented in the following form (see [6]):

$$\Psi(x,z,t) \;=\; \int_{s_1}^{s_2} \phi(x,z,s) \left(h_0(s) \cos\omega(s)t + h_1(s) \frac{\sin\omega(s)t}{\omega(s)} \right) ds, \tag{14}$$

where $\phi(x,z,s)$ is the family of generalized eigenfunctions of the operator A in $L_2(\Omega)$, s is a parameter, $\omega(s)$ is the function, defined by the choice of invariant subspace H_N^k and by the geometry of the domain Ω, $h_0(s)$ and $h_1(s)$ are defined by the initial data Ψ_0, Ψ_1. The operator A has a continuous spectrum on H_N^k if $\omega(s)$ is not constant. The solutions (14) may be written in a special form with the help of a change of variables.

Theorem 1. *Let $\overline{\Omega}_1$ be a unit disk $\{x^2 + z^2 \leq 1\}$ and let us define the coordinates*

$$\xi = x\sin(k\pi/2N) + z\cos(k\pi/2N), \quad \eta = -x\sin(k\pi/2N) + z\cos(k\pi/2N) \tag{15}$$

in $\overline{\Omega}_1$. There exists a diffeomorphism $F : \overline{\Omega} \to \overline{\Omega}_1$, which is defined by the functions $\xi = \xi(x,z)$, $\eta = \eta(x,z)$, such that:

i) *coordinate lines* $\xi = \text{const}$, $\eta = \text{const}$ *are the segments of straight lines in $\overline{\Omega}$;*

ii) *any solution* (14) *of the problem* (7)–(9) *with the initial data Ψ_0, Ψ_1 chosen in the subspace H_N^k is represented as*

$$\Psi(x,z,t) = H_1(\xi(x,z),t) - H_2(\eta(x,z),t), \tag{16}$$

where $H_1(\xi,t)$, $H_2(\eta,t)$ are defined by the choice of Ψ_0, Ψ_1.

Let $\Psi(x,z,t)$ be any solution of (7)–(9), and let $V(x,z,t)$ be the velocity field corresponding to the chosen function Ψ by Lemma 1. The motion of fluid particles in this case is described by differential equations (a dot denotes the derivative with respect to t):

$$\dot{x} = -\Psi'_z(x,z,t), \qquad \dot{z} = \Psi'_x(x,z,t), \qquad \dot{y} = v(x,z,t). \tag{17}$$

The evolution of two coordinates x, z is governed by a time-dependent Hamiltonian system. If $(x(0), z(0)) = (x_0, z_0) \in \overline{\Omega}$, then $(x(t), z(t)) \in \overline{\Omega}$ for all t for any solution of this system. This fact follows from the condition $\Psi|_\Gamma = 0$. The third equation in (17) has a family of solutions

$$y(t) = \int_0^t v(x(s), z(s), s)\, ds + y_0,$$

which corresponds to a given solution of the first two equations. The structure of system (17) is typical for any motion of incompressible fluid, which is invariant under the action of a one-parameter symmetry group (see [7]).

We now consider the dynamical system in $\overline{Q} = \overline{\Omega} \times \mathbf{R}$, which defines for a given t the stream lines of fluid motion in Q:

$$\frac{dx}{d\tau} = -\frac{\partial \Psi}{\partial z}(x, z, (t)), \qquad \frac{dz}{d\tau} = \frac{\partial \Psi}{\partial x}(x, z, (t)), \tag{18}$$

$$\frac{dy}{d\tau} = v(x, z, (t)), \tag{19}$$

where notation (t) means that t is fixed. Let $\mathcal{D}_2(t)$ and $\mathcal{D}_3(t)$ be the dynamical systems in $\overline{\Omega}$ and $\overline{Q}$ respectively, defined by the equations (18) and (18)–(19). The variable τ plays the role of auxiliary "time" and acts as a parameter for the stream lines in $\overline{Q}$ and for their projections onto $\overline{\Omega}$. Thus two families of dynamical systems $\mathcal{D}_2(t)$ and $\mathcal{D}_3(t)$ in $\overline{\Omega}$ and $\overline{Q}$ correspond to a given solution $V(x, z, t)$ of the problem (1)–(4).

Let $\mathcal{K}(\Psi, t) = \{(x, z) \in \overline{\Omega} \,|\, \nabla_{x,z}\Psi = 0\}$ be the set of all critical points of the function $\Psi(x, z, (t))$ in $\overline{\Omega}$ (the stationary points of $\mathcal{D}_2(t)$). A level line $\{\Psi(x, z, (t)) = c\}$ is called *regular* if $\mathcal{K} \cap \{\Psi = c\} = \emptyset$. We define $\Delta_2(x, z, (t))$ to be a Hessian $\Psi_{xx}\Psi_{zz} - (\Psi_{xz})^2$ of the function $\Psi(x, z, (t))$. The set $\mathcal{K}(\Psi, t)$ is decomposed into three disjoint subsets $\mathcal{K}_h$, $\mathcal{K}_e$, and $\mathcal{K}_d$ of hyperbolic $(\Delta_2 < 0)$, elliptic $(\Delta_2 > 0)$ and degenerate $(\Delta_2 = 0)$ points respectively. Note that $\mathcal{K}_e$ consists only of the strong local extremums of the function $\Psi(x, z, (t))$, therefore the condition $\Psi|_\Gamma = 0$ implies that $(\mathcal{K} \cap \Gamma) \subset (\mathcal{K}_h \cup \mathcal{K}_d)$.

If $(x_0, z_0) \in \mathcal{K}_e(\Psi, t)$, then the level lines $\{\Psi(x, z, (t)) = c\} = \gamma(c)$ are regular in a neighbourhood of this point and define closed orbits of the dynamical system $\mathcal{D}_2(t)$. Let $(x(\tau), z(\tau))$ be the periodic solution of (18), corresponding to the orbit $\gamma(c)$, and let $T(c)$ be it's period. Denote

$$\Delta y(c) = \int_0^{T(c)} v(x(\tau), z(\tau), (t))\, d\tau,$$

then the orbits of the dynamical system $\mathcal{D}_3(t)$ belonging to the cylinder $\gamma(c) \times \{y\}$ are the "screw lines" with the step $\Delta y(c)$ for $\Delta y(c) \neq 0$, and the closed curves for $\Delta y(c) = 0$.

Let $\mathcal{U}(x_0, z_0, (t))$ be the largest neighbourhood of the point $(x_0, z_0) \in \mathcal{K}_e$, consisting of the regular level lines $\gamma(c)$ plus the given point (x_0, z_0). It is clear that $\mathcal{U}(x_0, z_0, (t))$ is an invariant subset of $\overline{\Omega}$ under the system $\mathcal{D}_2(t)$. If we linearize system (18) near the stationary point $(x_0, y_0) \in \mathcal{K}_e$, then the corresponding matrix has eigenvalues $\pm i\ae$, where $\ae = \ae(x_0, z_0, (t)) = \sqrt{\Delta_2(x_0, z_0, (t))}$. We can define vortex structures, corresponding to every point $(x_0, z_0) \in \mathcal{K}_e(\Psi, t)$ (see [8]).

- Suppose $(x_0, z_0) \in \mathcal{K}_e(\Psi, t)$, and let $\mathcal{U}(x_0, z_0, (t))$ be the above-mentioned neighbourhood, invariant under $\mathcal{D}_2(t)$, of the point (x_0, z_0). We call the set of all orbits (stream lines) of the dynamical system $\mathcal{D}_3(t)$ belonging to $\mathcal{U} \times \{y\}$ *a swirl*, the point (x_0, z_0) – *the center of the swirl*, the straight line $(x_0, z_0) \times \{y\}$ – *the (instantaneous) axis of rotation*, $\ae(x_0, z_0, (t))$ – *the velocity of rotation of the swirl*, and the neighbourhood $\mathcal{U}$ – *the domain of influence of the swirl.*

The set $\mathcal{K}(\Psi, t)$ in $\overline{\Omega}$ is defined for a given t by the system

$$\Psi_x(x, z, t) = 0, \qquad \Psi_z(x, z, t) = 0. \tag{20}$$

If the coordinates $(x_0(t_0), z_0(t_0))$ of the center of one of the swirls has been found for $t = t_0$, then the condition $\Delta_2(x_0, z_0, t_0) > 0$ implies that for some $\varepsilon > 0$ the smooth solution $(x_0(t), z_0(t))$ of the system (20) is defined for all $|t - t_0| < \varepsilon$. This solution describes the motion of the center of the swirl in Ω. Respectively, the velocity of rotation $\ae(t) = \ae(x_0(t), z_0(t), t)$ is a smooth function of t on the interval $|t - t_0| < \varepsilon$. The moments of arising and disappearance of swirls are connected with the bifurcations in the family of dynamical systems $\mathcal{D}_2(t)$ when t is increasing. It follows from the definition of $\mathcal{U}$ that it's boundary $\partial\mathcal{U}$ lies in some level set $\{\Psi(x, z, (t)) = c\}$, containing at least one hyperbolic or degenerate point from the set $\mathcal{K}_h \cup \mathcal{K}_d$.

Theorem 2. *Let $V(x, z, t)$ be a velocity field in $\mathcal{Q}$ defined by Lemma 1 for a given smooth solution $\Psi(x, z, t)$ of the problem* (7)–(9) *with the initial data from the subspace H^k_N. Then in the coordinates (ξ, η, y) $((\xi, \eta) \in \overline{\Omega}_1, y \in \mathbf{R})$ (see Theorem 1 for the definition of ξ, η) the system* (17) *for fluid particles motion takes the following form*

$$\dot{\xi} = \frac{1}{J_1(\xi, \eta)} h_2(\eta) \sin(\omega_2(\eta)t + \vartheta_2(\eta)), \tag{21}$$

$$\dot{\eta} = \frac{1}{J_1(\xi, \eta)} h_1(\xi) \sin(\omega_1(\xi)t + \vartheta_1(\xi)), \tag{22}$$

$$\dot{y} = \tilde{v}(\xi, \eta, t), \tag{23}$$

where $J_1(\xi, \eta)$ is the Jacobian of the inverse map $F^{-1}\colon (\xi, \eta) \to (x(\xi, \eta), z(\xi, \eta))$, $\tilde{v}(\xi, \eta, t) = v(x(\xi, \eta), z(\xi, \eta), t)$. The functions $\omega_1(\xi)$, $\omega_2(\eta)$ are *defined by the geometry of the domain* Ω *and satisfy the condition* $\omega_1(\xi)|_\Gamma = \omega_2(\eta)|_\Gamma$. *The functions* $h_1(\xi)$, $h_2(\eta)$, $\vartheta_1(\xi)$, $\vartheta_2(\eta)$ *are defined by the choice of* $\Psi_0, \Psi_1 \in H_N^k$ *in* (9).

To illustrate the effects which correspond to a continuous spectrum we consider in $Q_1 = \overline{\Omega}_1 \times \{y\}$ the system of differential equations similar to (21)–(23)

$$\dot{\xi} = -\eta \sin(1+\varepsilon\eta^2)t, \tag{24}$$

$$\dot{\eta} = \xi \sin(1+\varepsilon(1-\xi^2))t, \tag{25}$$

$$\dot{y} = \eta\cos(1+\varepsilon\eta^2)t + \xi\cos(1+\varepsilon(1-\xi^2))t, \tag{26}$$

where $\varepsilon > 0$; and $\xi = (x+z)/\sqrt{2}$, $\eta = (x-z)/\sqrt{2}$ are new coordinates in the unit disk Ω_1 of the plane x, z.

Suppose the right-hand sides of equations (24)–(26) are components of the velocity vector $V(\xi,\eta,t)$. It can be easily verified that the conditions $V \cdot \mathbf{n}|_{\partial Q_1} = 0$, $\operatorname{div} V = 0$ are satisfied; and the energy conservation law (5) holds. Moreover, if we consider $V(\cdot,\cdot,t)$ as a function of t with the values in the Hilbert space $H_0 = \{U(\xi,\eta) = (u_1,u_2,u_3) | (\xi,\eta) \in \overline{\Omega}_1\}$ with the norm

$$\|U\|^2 = \int_{\Omega_1} (|u_1|^2 + |u_2|^2 + |u_3|^2)\, d\xi\, d\eta,$$

then $V(\cdot,\cdot,t)$ has a continuous energy spectrum and there are no limit points for $V(\cdot,\cdot,t)$ in H_0 when $t \to \pm\infty$, because $\|V(\cdot,\cdot,t)\| = \text{const}$ and $\text{w-lim}_{t\to\pm\infty} V(\cdot,\cdot,t) = 0$ in H_0. Let $\omega_1(\xi) = \varepsilon(1-\xi^2)$, $\omega_2(\eta) = 1+\varepsilon\eta^2$, then Hamiltonian for the first two equations is

$$\Psi(\xi,\eta,t) = \int_0^\xi s\sin\omega_1(s)t\,ds + \int_0^\eta s\sin\omega_2(s)t\,ds. \tag{27}$$

For a given t the critical points of $\Psi(\xi,\eta,t)$ are defined by the system

$$\xi\sin(\omega_1(\xi)t) = 0, \qquad \eta\sin(\omega_2(\eta)t) = 0. \tag{28}$$

Let $\overline{\Omega}_1^+$ be the intersection of $\overline{\Omega}_1$ with the quadrant $\{\xi > 0, \eta > 0\}$. and let $\omega_1(\xi) = \varepsilon(1-\xi^2)$, $\omega_2(\eta) = 1+\varepsilon\eta^2$. Then (28) reduce in $\overline{\Omega}_1^+$ to the system

$$\omega_1(\xi)t \equiv (1+\varepsilon(1-\xi^2))t = jn\pi, \qquad \omega_2(\eta)t \equiv (1+\varepsilon\eta^2)t = jm\pi, \tag{29}$$

where $j,m,n \in \mathrm{N}$, $\mathrm{GCD}(m,n) = 1$; and the following inequalities hold

$$\frac{1}{1+\varepsilon} \le \frac{m}{n} \le 1+\varepsilon.$$

Equations (29) imply that the critical points of $\Psi(\xi,\eta,t)$ in $\overline{\Omega}_1^+$ localize on arcs γ_n^m of ellipses

$$\varepsilon(m\xi^2 + n\eta^2) = m(1+\varepsilon) - n.$$

Let $(\xi_j(t), \eta_j(t)) \in \gamma_n^m$ be a solution of (29). By direct checking one can easily conclude that this solution is defined for $t \in (t_1^j, t_2^j)$, where

$$t_1^j = \frac{jm\pi}{1+\varepsilon}, \qquad \xi_j(t_1^j) = 0, \qquad \eta_j(t_1^j) = \sqrt{\frac{m(1+\varepsilon)-n}{\varepsilon n}},$$

$$t_2^j = jm\pi, \qquad \xi_j(t_2^j) = \sqrt{\frac{m(1+\varepsilon)-n}{\varepsilon m}}, \qquad \eta_j(t_2^j) = 0.$$

Intervals $\Delta_j = \dfrac{jm\pi\varepsilon}{1+\varepsilon}$ – the intervals of existence for these solutions – increase with j. Calculating the values of $\Psi_{\xi\xi}$ and $\Psi\eta\eta$ we obtain:

$$\Psi_{\xi\xi}(\xi_j(t), \eta_j(t), t) = -2\varepsilon\xi_j^2(t)(-1)^{jn}t = -2j(-1)^{jn}\frac{n\pi\xi_j^2(t)}{\omega_1(\xi_j(t))},$$

$$\Psi_{\eta\eta}(\xi_j(t), \eta_j(t), t) = 2\varepsilon\eta_j^2(t)(-1)^{jm}t = 2j(-1)^{jm}\frac{n\pi\eta_j^2(t)}{\omega_2(\eta_j(t))}.$$

The point $(\xi_j(t), \eta_j(t))$ is hyperbolic for all j if $m+n = 2l$. In the case $m+n = 2l+1$ this point is hyperbolic for $j = 2k$ and elliptic for $j = 2k+1$. The velocity of rotation of the swirl with the center in the elliptic point $(\xi_j(t), \eta_j(t))$ is

$$æ_j(t) = æ(\xi_j(t), \eta_j(t)) = j\varepsilon\pi\xi_j(t)\eta_j(t)\sqrt{\frac{nm}{\omega_1(\xi_j(t))\omega_2(\eta_j(t))}}. \tag{30}$$

The velocities of rotation of swirls passing through the given point $(\xi_0, \eta_0) \in \gamma_n^m$ form an arithmetic progression. We define the velocity of motion of the swirl along the line γ_n^m as $U_j(t) = (\xi_j'(t), \eta_j'(t))$. One can obtain from (29) that

$$\xi_j'(t) = \frac{\omega_1(\xi_j(t))}{2\varepsilon t\xi_j(t)} = \frac{\omega_1^2(\xi_j(t))}{2j\varepsilon n\pi\xi_j(t)}, \quad \eta_j'(t) = -\frac{\omega_2(\eta_j(t))}{2\varepsilon t\eta_j(t)} = \frac{\omega_2^2(\eta_j(t))}{2j\varepsilon m\pi\eta_j(t)}. \tag{31}$$

It follows from (30)–(31) that

$$æ_j(t) \cdot |U_j(t)| = \sqrt{\frac{m\omega_1^4(\xi_j(t))\eta_j^2(t) + n\omega_2^4(\eta_j(t))\xi_j^2(t)}{mn}}.$$

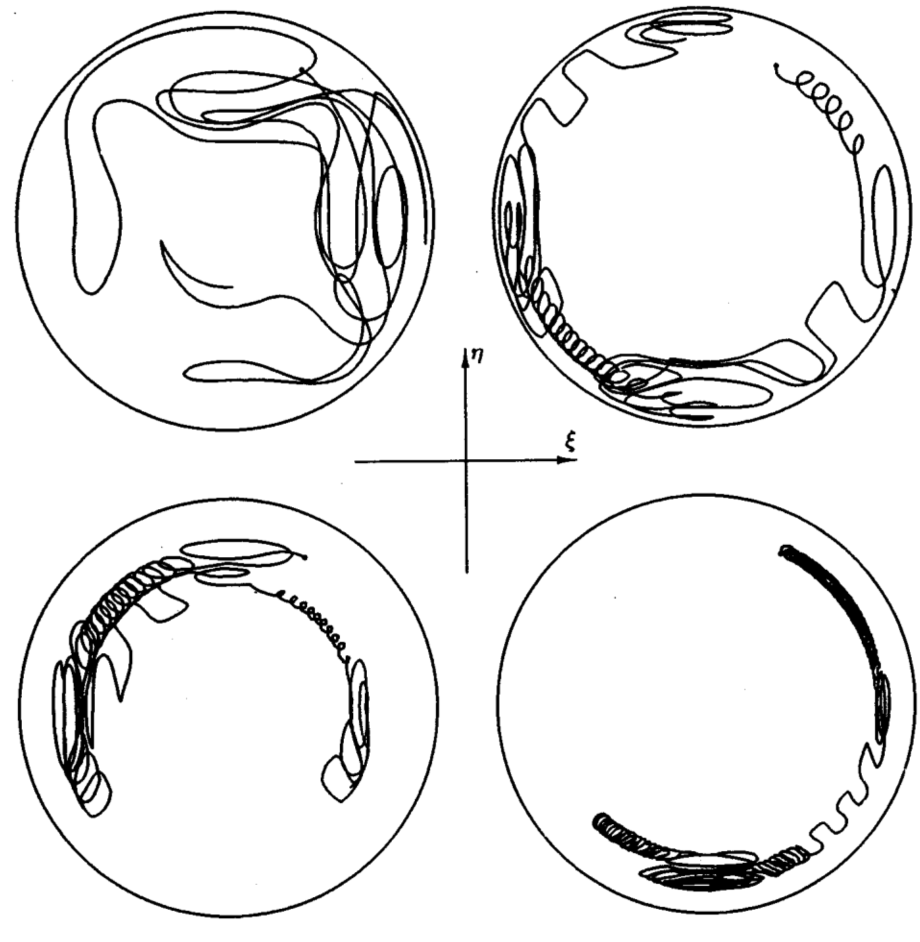
This means that for all swirls with the centers $(\xi_j(t), \eta_j(t)) = (\xi_0, \eta_0) \in \gamma_n^m$ passing through the given point on γ_n^m the value $æ_j(t) \cdot |U_j(t)|$ does not depend on j.

It is easy to verify that the total number $N(t)$ of critical points of the Hamiltonian function $\Psi(\xi, \eta, t)$ in $\overline{\Omega}_1$ increases with t; and the following estimate for large values of t holds

$$N(t) \geq \delta t^2, \qquad (\delta = \delta(\varepsilon) > 0).$$

The diameters of the domains of influence of the swirls are decreasing when j is increasing.

The specific process of arising, evolution and disappearance of the vortex structures is described by the system (24)–(26). The number of vortex structures increases in time and their scale decreases in the case of continuous spectrum for all systems of the type (21)–(23). This effect may be considered as one of the mathematical models of development of turbulence in the rotating ideal fluid and short-term atmospheric vortex phenomena (as tornado, for example).



The projection of trajectories of fluid particles onto plain ξ, η for given initial data $\xi(t_0) = 0.38$, $\eta(t_0) = 0.73$, $\varepsilon = 0.1$ and time interval $[t_0, t_0 + 50]$: a) $t_0 = 21.6$; b) $t_0 = 113.5$; c) $t_0 = 221.3$; d) $t_0 = 345.1$.

The illustration of the fluid particles motion corresponding to the system (24)–(26) is given in the figure. The large scale motion for small values of t transforms into microoscillations of fluid particles for large values of t.

References

[1] H. Poincaré, *Sur l'équilibre d'une masse fluide animée d'un mouvement de rotation*, Acta Mathematica, **7**, 1885, 259–380.

[2] S.L. Sobolev, *On a new problem of mathematical physics*, Izv. Akad. Nauk SSSR, Ser. Mat., **18**, No. 1, 1954, 3–50 (in Russian).

[3] R.A. Aleksandryan, *Spectral properties of operators induced by systems of differential equations of the Sobolev Type*, Proc. Moscow Math. Soc., **9**, 1960, 455–506 (in Russian).

[4] T.I. Zelenyak, *Selected Questions of the Qualitative Theory of Partial Differential Equations*, Izdat. Novosibirsk. Gos. Univ., Novosibirsk, 1970 (in Russian).

[5] M.V. Fokin, *The existance of a singular spectrum and the asymptotic behavior of solutions in a problem of S.L. Sobolev*, Russian Acad. Sci. Dokl. Math., **48**, No. 3, 1994, 529–533.

[6] M.V. Fokin, *Existence of the singular spectrum and asymptotic behavior of solutions in Sobolev's problem*. I, II, Siberian Adv. in Math., **4**, No. 1, 1994, 18–51; **4**,1994, No. 2, 16–53.

[7] I. Mezić and S. Wiggins, *On the integrability and pertubation of three-dimensional fluid flowes with symmetry*, Journ. of Nonlinear Science, **4**, No. 2, 1994, 157–194.

[8] M.V. Fokin, *Hamiltonian systems and small oscillations of rotating ideal fluid*, Preprint No. 20, Sobolev Institite of Mathematics, 1995 (in Russian).