Bull. Nov. Comp. Center, Math. Model. in Geoph., 6 (2000), 125–130

The non-uniqueness of solution to the inverse problem of scattering

G.M. Tsibulchik

The inverse problem of wave scattering on inhomogeneities of a medium within the framework of a scalar wave equation is considered. The scattered field in such a model can be described in two ways: either with the help of the surface distribution of secondary sources, or by using volume distribution. These two ways of description completely coincide in the outer domain and at the boundary. They are, however, different inside the inhomogeneity: the volume sources reconstruct the "refracted" field existing inside the inhomogeneity, whereas the Kirchhof description yields zero values there.

From the point of view of the inverse problem, in which the observation data can be gathered only in the outer domain, this leads to the following: it is insufficient to have the full knowledge of the scattered field everywhere in the outer domain to distinguish between volume scattering and surface scattering. It is important to note that this non-uniqueness of the solution is fundamental and is not due to the monochrome character of the wave process under consideration; it is also not eliminated by observation systems with multiple overlapping.

Let us consider the following problem statement in the form of a scalar wave equation as the basic model for description of the wave propagation process:

$$\Big(\Delta - \frac{1}{c^2(\boldsymbol{x})}\frac{\partial^2}{\partial t^2}\Big)u(\boldsymbol{x},t) = \delta(\boldsymbol{x}-\boldsymbol{x}_0)\delta(t), \quad t \in \mathbb{R}^1, \;\; \boldsymbol{x} \in \mathbb{R}^3, \qquad (1)$$
$$u(\boldsymbol{x},t) \equiv 0, \quad t < 0, \;\; \boldsymbol{x} \in \mathbb{R}^3.$$

The model under consideration is convenient due to its "utmost" simplicity: the medium is characterized only by one parameter, namely, the wave propagation velocity $c(\boldsymbol{x})$. In what follows it will, however, be shown that corresponding formulations of the inverse problem on the determination of the velocity $c(\boldsymbol{x})$ are not simple and, in the general case of piecewise continuous distribution of $c(\boldsymbol{x})$, remain unsolved up to now.

We assume, in the subsequent discussion, that the velocity distribution can be represented as follows:

$$c(\boldsymbol{x}) = c_0 + c_1(\boldsymbol{x}), \qquad (2)$$

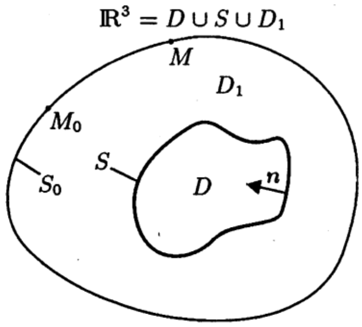
Here $c_0 = \text{const}$ (reference medium), and the support $\operatorname{supp} c_1(\boldsymbol{x})$ is finite and occupies a domain D bounded by a surface S. In short, in this case statement

(1) is the problem of scattering (diffraction) of waves on the inclusion D. So far, we do not impose any restrictions to the velocity distribution $c_1(\boldsymbol{x})$, except for the requirement of finiteness of its support $\operatorname{supp} c_1$. The domain D is assumed to be multiply connected, and the transition from c_0 to c_1 is jumpwise. In this case, the surface S is an interface of the first kind. Additionally to (1), we require that the following fitting conditions of the "rigid contact" type be satisfied at this interface:

$$[u](\boldsymbol{x}, t) = 0, \quad [\partial_n u](\boldsymbol{x}, t) = 0 \quad \text{for} \quad \boldsymbol{x} \in S, \ t > 0. \tag{3}$$

Here the square brackets denote, as usual, the function jump as it passes through the surface S.

It is known [1–3] that problem (1), taking into account (3), is a closed statement, its solution is unique, and can serve as a starting point for further investigations. The figure shows the main configuration of the data of the problem, in which the point source M_0 (with coordinates $\boldsymbol{x}_0$) is fixed, and point of the receiver M (with coordinates $\boldsymbol{x}$) sweeps a closed surface S_0 that contains the velocity anomaly D. The inverse problem consists in reconstruction of the velocity anomaly $c_1(\boldsymbol{x})$ by using the field $u(\boldsymbol{x}, t; \boldsymbol{x}_0)$ observed at points of the surface S_0. The surface S_0, which carries the data of observations, is virtual in the sense that only the trace of the wave process is fixed at its points. Then the wave "runs further" in the infinite space R^3. These observation conditions, which are not real (in reality, observations are performed at a physical boundary of the "free surface" type; besides, observations on a closed surface are usually not possible in geophysics), were chosen deliberately. This was done to make clear the conditions of solvability of the inverse problem under consideration and to exclude from the reasoning such speculations as "lighting up from below" or "lighting up from the side", etc., that appeared in the literature [5].



The next step is to represent the full field $u(\boldsymbol{x}, t; \boldsymbol{x}_0)$ as the sum of two terms:

$$u(\boldsymbol{x}, t; \boldsymbol{x}_0) = u_0(\boldsymbol{x}, t; \boldsymbol{x}_0) + u_s(\boldsymbol{x}, t; \boldsymbol{x}_0). \tag{4}$$

The first of these terms, u_0, is the initial field produced by the point source acting at the point M_0. The field propagates in the reference medium c_0 (without the inclusion D). The other term, u_s, denotes the wave field scattered by the inhomogeneity D. Note here that such splitting of the full field is especially convenient in the consideration of the inverse problem, as it will be shown below. This splitting is acceptable, since the incident wave u_0 is a short pulse not overlapping in time with the scattered waves in seismograms. The initial field u_0 does not carry any information about the anomaly D. Therefore, subtraction of u_0 from the full seismogram leaves the scattered field component of the field u_s on this seismogram. It serves as basic data in solving the inverse problem.

Formally, the terms in (4) are determined by the following conditions:

$$\begin{aligned}&\Big(\Delta - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\Big)u_0(\boldsymbol{x},t;\boldsymbol{x}_0) = \delta(\boldsymbol{x}-\boldsymbol{x}_0)\delta(t), \quad t\in\mathbb{R}^1, \ \boldsymbol{x}\in\mathbb{R}^3,\\ &u_0(\boldsymbol{x},t;\boldsymbol{x}_0)\equiv 0, \quad t<0, \ \boldsymbol{x}\in\mathbb{R}^3;\end{aligned} \tag{5}$$

$$\begin{aligned}&\Big(\Delta - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\Big)u_s(\boldsymbol{x},t;\boldsymbol{x}_0) = m(\boldsymbol{x})\ddot{u}(\boldsymbol{x},t;\boldsymbol{x}_0), \quad t\in\mathbb{R}^1, \ \boldsymbol{x}\in\mathbb{R}^3,\\ &u_s(\boldsymbol{x},t;\boldsymbol{x}_0)\equiv 0, \quad t<0, \ \boldsymbol{x}\in\mathbb{R}^3.\end{aligned} \tag{6}$$

Here the field symbols with points over them denote partial derivatives with respect to time, and the function

$$m(\boldsymbol{x}) = c^{-2}(\boldsymbol{x}) - c_0^2 \tag{7}$$

describes the velocity anomaly that occupies the domain D, i.e.,

$$\bar{D} = \operatorname{supp} m(\boldsymbol{x}).$$

It should be emphasized that the system of equations (5)–(6) is totally equivalent to problem (1). At the same time, as noted above, it is more convenient for analysis of the inverse problem: relation (6) shows that the scattered field u_s is generated by the secondary sources induced in the domain of inhomogeneity D by the incident signal u_0.

Further analysis is performed in a frequency ω–domain. For this, we define the Fourier transforms by the following relation:

$$U(\boldsymbol{x},\omega) = \int_{-\infty}^{\infty} e^{-i\omega t}u(\boldsymbol{x};t)\,dt, \qquad u(\boldsymbol{x};t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{+i\omega t}U(\boldsymbol{x};\omega)\,d\omega. \tag{8}$$

In this case, the wave operator (1) is replaced by a Helmholtz equation, and problems (5)–(6) are replaced by the following statement:

$$(\Delta + k_0^2)U_0(\boldsymbol{x},\boldsymbol{x}_0;\omega) = \delta(\boldsymbol{x}-\boldsymbol{x}_0), \quad \boldsymbol{x}\in\mathbb{R}^3, \tag{9}$$

$$(\Delta + k_0^2)U_s(\boldsymbol{x},\boldsymbol{x}_0;\omega) = -\omega^2 m(\boldsymbol{x})U(\boldsymbol{x},\boldsymbol{x}_0;\omega), \quad \boldsymbol{x}\in\mathbb{R}^3. \tag{10}$$

Here $k_0 = \omega/c_0$ is the wave number, and the zero initial data in (5) and (6) are replaced by the following so-called "radiation conditions at infinity":

waves far from the inhomogeneity and the radiation source must asymptotically have the form of diverging spherical waves.

In accordance with equality (9), the fundamental solution of the Helmholtz equation completely coincides (at the same "radiation condition at infinity") with the incident field u_0, which goes through the medium. It has the following form:

$$G(\boldsymbol{x}_0, \boldsymbol{x}; \omega) = U_0(\boldsymbol{x}, \boldsymbol{x}_0; \omega) = -\frac{1}{4\pi}\frac{\exp(-ik_0 r)}{r}, \tag{11}$$

where $r = |\boldsymbol{x} - \boldsymbol{x}_0|$.

The scattered field u_s, considered as a solution to problem (10), is determined by the volume potential for $\boldsymbol{x} \in R^3$ and $\boldsymbol{x}_0 \in S_0$:

$$U_s(\boldsymbol{x}, \boldsymbol{x}_0; \omega) = \frac{\omega^2}{4\pi}\iiint\limits_D \frac{\exp(-ik_0|\boldsymbol{x} - \boldsymbol{\xi}|)}{|\boldsymbol{x} - \boldsymbol{\xi}|} m(\boldsymbol{\xi}) U(\boldsymbol{\xi}, \boldsymbol{x}_0; \omega)\, dV_\xi. \tag{12}$$

For the points $\boldsymbol{x}$ that lies in the domain outside to $\bar{D}$, i.e., $\boldsymbol{x} \in \mathbb{R}^3 \setminus \bar{D}$, expression (12) determines the wave field scattered by the inhomogeneity D into the external medium. For the points $\boldsymbol{x}$, however, that lie inside D there exists only one "refracted" field U (individually, the terms U_0 and U_s inside D do not have a physical meaning). This field satisfies, in accordance with (4), the following integral equation of the Fredholm type of the second kind for $\boldsymbol{x} \in D$ and $\boldsymbol{x}_0 \in S_0$:

$$U(\boldsymbol{x}, \boldsymbol{x}_0; \omega) = U_0(\boldsymbol{x}, \boldsymbol{x}_0; \omega) + \frac{\omega^2}{4\pi}\iiint\limits_D \frac{\exp(-ik_0|\boldsymbol{x} - \boldsymbol{\xi}|)}{|\boldsymbol{x} - \boldsymbol{\xi}|} m(\boldsymbol{\xi}) U(\boldsymbol{\xi}, \boldsymbol{x}_0; \omega)\, dV_\xi. \tag{13}$$

Due to the properties of smoothness of the volume potential [1, 2] in the case of a jumpwise velocity transition $c(\boldsymbol{x})$ at the boundary of inhomogeneity S, conditions of "rigid contact" of the type (3) (continuity of the field u and its normal derivative $\partial_n u$) are satisfied.

Applying Green's formula to the outside domain, one can obtain the following expression for the same scattered field:

$$\begin{aligned} U_s(\boldsymbol{x}, \boldsymbol{x}_0; \omega) &= \frac{1}{4\pi}\iint\limits_S \Big\{ \frac{\exp(-ik_0|\boldsymbol{x} - \boldsymbol{\xi}|)}{|\boldsymbol{x} - \boldsymbol{\xi}|} \partial_n U(\boldsymbol{\xi}, \boldsymbol{x}_0; \omega) - \\ &\qquad U(\boldsymbol{\xi}, \boldsymbol{x}_0; \omega) \partial_n \Big(\frac{\exp(-ik_0|\boldsymbol{x} - \boldsymbol{\xi}|)}{|\boldsymbol{x} - \boldsymbol{\xi}|} \Big) \Big\}\, dS_\xi, \\ &\qquad \boldsymbol{x} \in \mathbb{R}^3 \setminus \bar{D}, \quad x_0 \in S_0; \\ U_s(\boldsymbol{x}, \boldsymbol{x}_0; \omega) &= 0, \quad \boldsymbol{x} \in D, \ \boldsymbol{x}_0 \in S_0. \end{aligned} \tag{14}$$

Here $\boldsymbol{n}$ is the unit normal to S directed from the outside into D (see the figure).

Using surface delta-functions to describe the density of the secondary sources, which are distributed over S in the form of simple and double layers, we can write the following conditions, which are satisfied by U_s from (14), in the uniform form (see [1]):

$$
\begin{aligned}
&(\Delta + k_0^2)U_s(\boldsymbol{x}, \boldsymbol{x}_0; \omega) = -\partial_n U \delta_s - \frac{\partial}{\partial n}(U\delta_s), \quad \boldsymbol{x} \in \mathbb{R}^3, \ \boldsymbol{x}_0 \in S_0, \\
&U_s(\boldsymbol{x}, \boldsymbol{x}_0; \omega) = 0, \quad \boldsymbol{x} \in D, \ \boldsymbol{x}_0 \in S_0.
\end{aligned} \tag{15}
$$

Here "radiation conditions at infinity" are assumed to be satisfied.

If we follow the approach of the direct problem of scattering (diffraction), we should again use Green's formula, only we apply it to the internal domain D characterized by the velocity distribution c_1. Then we should match the appearing Kirchhof integrals of the type (14), in accordance with the boundary conditions (3). As a result, we obtain a system of two boundary integral equations for the determination of the field U and its normal derivative $\partial_n U$ on S. Note once again that this is the solution scheme of the direct problem of scattering based on the integral Kirchhof representation. In this paper, however, we are interested in the inverse problem, in which observation data can be obtained only in the outer domain $D_1 = \mathbb{R}^3 \setminus \bar{D}$ (see the surface S_0 in the figure), and there is no information about the "refracted" field existing inside D. A comparison of (10) and (15) shows that the knowledge of the scattered field U_s everywhere in D_1 turns out to be insufficient to distinguish between the volume scattering in (12) and the surface scattering in (14): both representations coincide in D_1 and on S, and differ in D.

It is important to note that the ambiguity of the solution mentioned above is not associated with the monochrome character of the wave process under consideration (a similar Green's formula is valid also in the time representation of the field); it is also not eliminated by systems of multiple overlapping. This is, in our opinion, the main challenge of the inverse problem of scattering, which many researchers failed to see.

In an explicit form, this difference in the description of the scattered field manifests itself further in an approximate solution of the inverse problem of scattering, when the following linearization of the problem is used: the full field U under the sign of the integrals (12) and (14) is replaced by the incident field U_0 as a result of the first iteration in the solution of the corresponding integral equation.

To be more exact, the following approximate boundary conditions (see [4]) are used in the case of equation (14):

$$
U(\boldsymbol{x}; \omega) = (1 + K)U_0(\boldsymbol{x}; \omega), \quad \partial_n U(\boldsymbol{x}; \omega) = (1 - K)\partial_n U_0(\boldsymbol{x}; \omega), \quad \text{for } \boldsymbol{x} \in S,
$$

where K is the reflection coefficient.

Summing up, the approach of (14) leads to a high-frequency Kirchhof representation of the scattered field which corresponds locally to a plane-wave description of the process of wave reflection within the framework of the ray method, whereas the approach of (12) gives a low-frequency Born representation of the scattered field. It is important that in both cases the linearized statement of the inverse problem has uniqueness of the solution, and this solution can be constructed in different ways. It should be noted that the Kirchhof approach (14) can be used in algorithms of migration types that are widely used in the practice of processing of seismic prospecting data; the Born description (12), however, is now being introduced into this practice, and will find its place in the interpretation space of seismics.

References

[1] Vladimirov V.S. Equations of Mathematical Physics. – Moscow: Nauka, 1971.

[2] Koshlyakov N.S., Gliner E.B., Smirnov M.M. Differential Equations of Mathematical Physics. – Moscow: Fizmatgiz, 1962.

[3] Petrashen' G.I. Fundamentals of the mathematical theory of elastic wave propagation // Questions of the Dynamic Theory of Seismic Wave Propagation. – Leningrad: Nauka, 1978. – Issue 18. – P. 3–248.

[4] Bass F.G., Fuks I.M. Wave Scattering on a Statistically Uneven Surface. – Moscow: Nauka, 1972.

[5] Mora P. Inversion = migration + tomography // Geophysics. – 1989. – Vol. 54, №. 12. – P. 1575–1586.