Bull. Nov. Comp. Center, , 41 (2017), 1–12
© 2017 NCC Publisher

The notion of N-density for relational structures*

E.N. Bozhenkova

Abstract. The intention of this paper is to extend classical results concerning the relationships between K- and N-density to their generalizations and modifications in the framework of the class of relational structures with distinct, irreflexive relations on countable sets of systems events.

Keywords: Relational structure, N-density, K-density, concurrency.

1. Introduction

To analyze concurrent and distributed systems, a variety of models has been proposed. Different relations between actions are used for systems behaviour modelling. Causality, concurrency and conflict are the most fundamental ones.

In posets [5], behaviour is modelled by occurrences of events, which are partially ordered. Such events are considered as causally dependent. The absence of causality supposes concurrency between events. In other models [16], causality and conflict (mutual exclusion) are investigated. Three relations — causality, conflict and concurrency — are presented in the context of event structures [9, 17, 27]. In some models, conflict has the forward hereditary property, in others it is not necessarily symmetric [6, 13].

For unification of possible aspects of relations, we use a model of relational structures — a set of elements with a number of different relations on it. The authors of papers [11, 12, 14, 15] have proposed a subclass of the model where the general causal concurrent behavior is represented by a pair of relations instead of just one, as in the standard partial order approach. So, causality can be represented by either a partial order and irreflexive weak causality or a symmetric and irreflexive mutex relation (non-simultaneity) and irreflexive weak causality.

Concurrency axioms (including K-density, N-density, etc.), proposed by Petri for combinatorial models, are investigated in the context of many existing models. K-density is based on the idea that at any time instant, any sequential subprocess of a concurrent structure must be in some state or changing its state. N-density can be viewed as a sort of local density. It

*Partially supported by DFG (Project CAVER, Grant BE 1267/14-1).

has turned out that concurrency axioms allow avoiding inconsistency between syntactic and semantic representations of concurrent processes and excluding unreasonable processes represented by the concurrent structures.

The properties and interrelations between concurrency axioms were studied for causal nets [3, 4], posets [5, 8, 18] and event structures [23, 24]. Furthermore, the analysis techniques for concurrent and distributed systems that are grounded on the behavioural relations have become popular recently [1, 2, 21].

In paper [25], generalization of K-density was defined and alternative characterization was given in the context of relational structures.

Plünnecke [20] considered N-density for posets as a local K-density. Boudol [7] considered freeness from N-shaped structures to describe labelled event structures which could be expressed by terms of sequential and parallel compositions. It has turned out that this is a subclass of $N-$ and triangle-free labelled event structures.

Here we consider generalizations of some variations of the notion of N-density, their interrelations with K-density in the context of relational structures.

2. Preliminary

In this section, we consider a model of relational structures and its properties. First, we introduce some notions and notations which will be useful throughout the text.

Given a set X and a relation $R \subseteq X \times X$,

- R is *cyclic* iff there exists a sequence of distinct elements $x_1, \ldots, x_k \in X$ $(k > 1)$ such that $x_j \; R \; x_{j+1}$ $(1 \leqslant j \leqslant k-1)$ and $x_k \; R \; x_1$,
- R is *acyclic* iff it is not cyclic,
- R is *asymmetric* iff $(x \; R \; x') \Rightarrow \neg(x' \; R \; x)$, for all $x, x' \in X$,
- R is *antisymmetric* iff $(x \; R \; x') \wedge (x \neq x') \Rightarrow \neg(x' \; R \; x)$, for all $x, x' \in X$,
- R is *symmetric* iff $(x \; R \; x') \iff (x' \; R \; x)$, for all $x, x' \in X$,
- R is *transitive* iff $(x \; R \; x') \wedge (x' \; R \; x'') \Rightarrow (x \; R \; x'')$, for all $x, x', x'' \in X$,
- R is *irreflexive* iff $\neg(x \; R \; x)$, for all $x \in R$,
- $R^\alpha = R \cup \mathrm{id}_X$ (the reflexive closure of R),
- $R^\beta = R \cup R^{-1}$ (the symmetric closure of R),
- $R^\gamma = R^\beta \cup \mathrm{id}_X$ (the reflexive and symmetric closure of R),
- $R^\delta = (R \backslash \mathrm{id}_X) \backslash (R \backslash \mathrm{id}_X)^2$ (the irreflexive, intransitive relation), if R is a transitive relation, and $R^\delta = R$, otherwise,

Notice that a relation is asymmetric iff it is both antisymmetric and irreflexive; a transitive relation is asymmetric iff it is irreflexive; if a relation R is irreflexive and transitive, then it is acyclic and antisymmetric, i.e. a (strict) partial order, and, moreover, R^δ is the immediate predecessor relation. Given the elements $x_1, x_2 \in X$ and subsets $A \subseteq X' \subseteq X$, let $[x_1\ R\ x_2] = \{x \in X \mid x_1\ R^\alpha\ x\ R^\alpha\ x_2\}$, ${}^R A = \{x' \in X \mid \exists x \in A : (x'\ R^\alpha\ x)\}$, and A is a *(maximal) R-clique of* X' iff A is a (maximal) set containing only the pairwise $R \cup \mathrm{id}_{X'}$-related elements of X'.

Definition 1. A *relational structure* is a tuple $S = (E, V_1, \ldots, V_n)$ $(n \geqslant 1)$, where

- E is a countable set of elements,
- $V_1, \ldots, V_n \subseteq E \times E$ are irreflexive relations such that
  - $\bigcup_{1 \leqslant i \leqslant n} V_i^\beta = (E \times E) \setminus \mathrm{id}_E$, where id_E is identity on E,
  - $V_i^\beta \cap V_j^\beta = \varnothing$, for all $1 \leqslant i \neq j \leqslant n$.

From now on, we shall use P, Q, and R to denote the unions of the form $\bigcup_{i \in \mathcal{V}} V_i$ and call them *connectives* of S. Here $\mathcal{V} \subseteq \{i \mid 1 \leqslant i \leqslant n\}$.

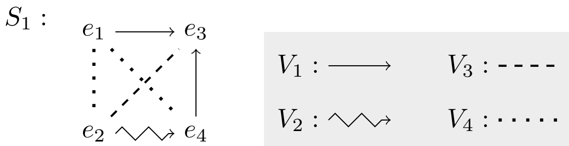

Figure 1.

.

A simple example of a relational structure with four relations is shown in Figure 1. Assume that V_1 is an irreflexive and transitive relation (a strict partial order), V_2 is an asymmetric relation, and V_3 and V_4 are irreflexive and symmetric relations. We can interpret the relation V_1 as causality dependence, V_2 as asymmetric conflict [19, 6], V_3 as synchronous concurrency (simultaneity), and V_4 as asynchronous concurrency (independence).

Consider two auxiliary properties of relational structures which will be useful in further considerations. We shall call a relational structure S with its connective P:

- *P-transitive (irreflexive, symmetric, respectively)* iff P is transitive (irreflexive, symmetric, respectively);
- *P-finite* iff any P-clique of E is finite.

3. N-freeness properties

One of the reasons, Petri [18] has introduced "concurrency axioms" is that not all correct formal models are suitable for the purpose of an adequate representation of concurrent and distributed processes.

The notion of K-density requires non-empty intersections of maximal w.r.t. causality sets with maximal w.r.t. concurrency sets in the context of causal nets. Later some modifications and extensions of this notion for models with different basic relations were proposed: L-density [16] on acyclic nets, R-density [23] and M-density [24] on event structures, B-density [10] in the framework of occurrence nets.

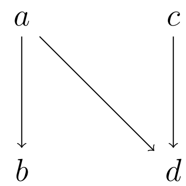

Figure 2.

As known, N-shaped structures (as shown in Figure 2) cannot be expressed in a process algebra as a composition of sequenatial and parallel operators. To avoid such structures, in [7] the property of N-freeness for labelled event structures was formulated. For posets [20], N-density is considered as a local K-density, i.e. every four-element N-shaped subposet can be extended to a K-dense subposet by addition of one point. The authors of paper [22] have obtained an algorithm of construction of a minimal N-free extension for posets by adding dummy points.

Our aim is to give generalizations of different approaches to the notion of N-density in the setting of relational structures and compare them.

A generalization of the K-density property for relational structures was given in [25].

Definition 2. Given a relational structure S and a maximal $(P \cup Q)$-clique $\widetilde{E}$ of E,

- $\widetilde{E}$ is K_{PQ}*-dense* iff for any maximal P-clique E' of $\widetilde{E}$ and for any maximal Q-clique E'' of $\widetilde{E}$, $E' \cap E''$ is a (unique) maximal $(P \cap Q)$-clique of $\widetilde{E}$,
- S is K_{PQ}*-dense* iff any maximal $(P \cup Q)$-clique $\widetilde{E}$ of E is K_{PQ}-dense.

Clearly, K-density of a Winskel's prime event structure is K_{V_1,V_2}-density of the corresponding relational structure with V_1 being partial order causality, V_2 — symmetric concurrency and V_3 — symmetric conflict.

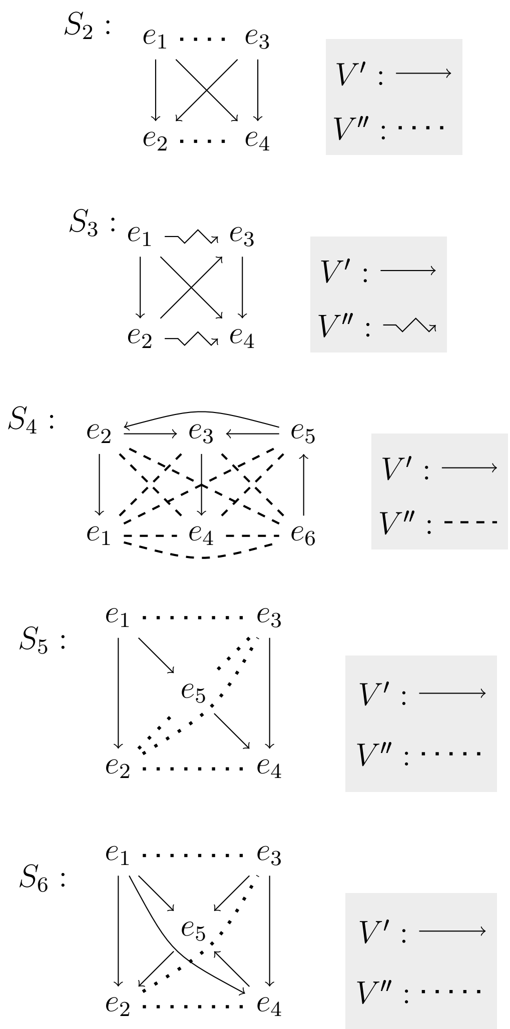

Figure 3.

One way to define the N-freeness property for relational structures is to consider four elements with relations between them.

Definition 3. A relational structure S with distinct connectives P and Q is called $\bowtie_{PQ}$*-dense*, iff in any maximal $(P \cup Q)$-clique $\widetilde{E}$ of E, whenever $(e_0\ P\ e_1\ Q\ e_2)$ and $(e_0\ Q\ e_3\ P\ e_2)$, then $(e_0\ P^{\delta}\ e_2) \Longrightarrow (e_3\ P\ e_1)$ for all distinct elements $e_0, e_1, e_2, e_3 \in \widetilde{E}$.

Consider the relational structures $S_2 - S_4$ shown in Figure 3. The relational structure $S_2 = (E_2, V', V'')$ with the transitive or symmetric relation V' and the symmetric relation V'' is $\bowtie_{V'V''}$-dense and $K_{V'V''}$-dense.

The relational structure $S_3 = (E_3, V', V'')$ with the non-transitive and non-symmetric relation V' is $K_{V'V''}$-dense, but not $\bowtie_{V'V''}$-dense. In fact, in the maximal $(V' \cup V'')$-clique $\{e_1, \ldots, e_4\}$ of E there are distinct elements e_1, e_2, e_3, e_4 such that $(e_1\ V'\ e_2\ V''\ e_4)$, $(e_1\ V''\ e_3\ V'\ e_4)$, and $(e_1\ V'^{\delta}\ e_4)$ but $\neg(e_3\ V'\ e_2)$.

The relational structure $S_4 = (E_4, V', V'')$ with the non-transitive and non-symmetric relation V' and the symmetric relation V'' is $\bowtie_{V'V''}$-dense but not $K_{V'V''}$-dense.

So, we see, that existence of $\bowtie$-density gives N-freeness in relational structures, but interconnection with the K-density property is weak. Let us consider another variant of the definition of N-density, where we take into account the symmetric closure of relations.

Definition 4. A relational structure S with distinct connectives P and Q is called ${\bowtie'}_{PQ}$*-dense*, iff in any maximal $(P \cup Q)$-clique $\widetilde{E}$ of E, whenever $(e_0\ P^{\beta}\ e_1\ Q^{\beta}\ e_2)$ and $(e_0\ Q^{\beta}\ e_3\ P^{\beta}\ e_2)$, then $(e_0\ P^{\beta}\ e_2) \Longrightarrow (e_3\ P^{\beta}\ e_1)$ for all distinct elements $e_0, e_1, e_2, e_3 \in \widetilde{E}$.

The relational structures S_2 and S_3 are simple examples of ${\bowtie'}_{V'V''}$-dense ones.

The relational structure S_4 is not ${\bowtie'}_{V'V''}$-dense.

An example of $K_{V'V''}$-dense but not $\bowtie'_{V'V''}$-dense relational structure is represented by $S_5 = (E_5, V', V'')$ with the transitive relation V' and the symmetric relation V''. In the maximal $(V' \cup V'')$-clique $\{e_1, \ldots, e_5\}$ of E_5, there are distinct elements e_1, e_2, e_3, e_4 such that $e_1\ V'^{\beta}\ e_2\ V''^{\beta}\ e_4$, $e_1\ V''^{\beta}\ e_3\ V'^{\beta}\ e_4$, and $e_1\ V'^{\beta}\ e_4$ but $\neg(e_3\ V''^{\beta} e_2)$.

The next proposition establishes a relationship between $\bowtie'$-density and K-density.

Proposition 1. *Let S be a P- or Q-finite relational structure with distinct connectives P and Q. Then*

$$S \text{ is } {\bowtie'}_{PQ}\text{-dense} \Longrightarrow S \text{ is } K_{PQ}\text{-dense}.$$

Proof. Suppose S is not K_{PQ}-dense. Then there exists a maximal $(P \cup Q)$-clique $\widetilde{E}$ of E with a maximal P-clique B and a maximal Q-clique C s.t. $\mid B \cap C \mid = 0$.

From the definition of the maximal P-clique B and he maximal Q-clique C, we obtain an auxiliary lemma.

Lemma A. $\forall b \in B\ \exists c \in C : b\ P^{\beta}\ c$ and $\forall c \in C\ \exists b \in B : c\ Q^{\beta}\ b$.

Take $b_1 \in B$. By Lemma A, for b_1 there exists $c_1 \in C$ s.t. $b_1\ P^{\beta}\ c_1$, for c_1 there exists $b_2 \in B$ s.t. $c_1\ Q^{\beta}\ b_2$.

By repeating infinitely many times these steps, we obtain infinite sequences $b_1, b_2, \ldots$ from B and $c_1, c_2, \ldots$ from C s.t. $b_i\ P^{\beta}\ c_i\ Q^{\beta}\ b_{i+1}$ for

$1 \leqslant i$. From $\bowtie'_{PQ}$-density, we obtain $b_j\ Q^\beta\ c_i$, $c_j\ P^\beta\ b_i$, $b_j \neq b_i$, $c_j \neq c_i$, for $1 \leqslant i < j$

So, we come to a contradiction to either P- or Q-finitness of S.

□

In the following variant of the definition of N-freeness, we will combine the symmetric closure and requirement of existence of the immediate relation.

Definition 5. A relational structure S with distinct connectives P and Q is called $\bowtie^\beta_{PQ}$*-dense*, iff in any maximal $(P \cup Q)$-clique $\widetilde{E}$ of E, whenever $(e_0\ P^\beta\ e_1\ Q^\beta\ e_2)$ and $(e_0\ Q^\beta\ e_3\ P^\beta\ e_2)$, then $(e_0\ P^{\delta\beta}\ e_2) \Longrightarrow (e_3\ P^\beta\ e_1)$ for all distinct elements $e_0, e_1, e_2, e_3 \in \widetilde{E}$.

It is easy to see that S_2, S_3 and S_5 are $\bowtie^\beta_{V'V''}$-dense.

The relational structure S_4 is not $\bowtie^\beta_{V'V''}$-dense. Indeed, in the maximal $(V' \cup V'')$-clique $\{e_1, \ldots, e_6\}$ of E_4 there are elements e_1, e_2, e_3, e_4 such that $(e_2\ (V')^\beta\ e_1\ (V'')^\beta\ e_3)$, $(e_2\ (V'')^\beta\ e_4\ (V')^\beta\ e_3)$, and $(e_2\ ((V')^\delta)^\beta\ e_3)$ but $\neg(e_4\ (V')^\beta\ e_1)$.

For further consideration of interconnections between $\bowtie^\beta$-density and K-density, we will remind important notions in the concurrency theory — the properties of discreteness and combinatority.

Definition 6. Given a relational structure S and a maximal $(P \cup Q)$-clique $\widetilde{E}$ of E with distinct connectives P and Q

- $\widetilde{E}$ is PQ*-combinatorial* iff $\mid [e_1\ P\ e_2] \cap E' \mid < \infty$ for some maximal P-clique E' of $\widetilde{E}$ and for all $e_1, e_2 \in \widetilde{E}$;
- $\widetilde{E}$ is PQ*-discrete* iff $\mid [e_1\ P\ e_2] \cap E' \mid < \infty$ for all maximal P-cliques E' of $\widetilde{E}$ and for all $e_1, e_2 \in \widetilde{E}$;
- S is PQ*-combinatorial* (PQ*-discrete*, respectively) iff any maximal $(P \cup Q)$-clique $\widetilde{E}$ of E is PQ-combinatorial (PQ-discrete, respectively).

The relationships between $\bowtie^\beta$-density and K-density in the context of relational structures was shown in paper [25].

Theorem 1. *Let S be a P- or Q-finite relational structure with distinct connectives P and Q. Then,*

(i) S is K_{PQ}-dense and P is transitive $\Longrightarrow$ S is $\bowtie^\beta_{PQ}$-dense.

(ii) S is K_{PQ}-dense $\Longleftarrow$ S is $\bowtie^\beta_{PQ}$-dense and PQ-discrete.

The main idea behind the generalization of the notion of N-density for relational structures is that the property of N-density is considerd as a local K-density.

Definition 7. A relational structure S with distinct connectives P and Q is called N_{PQ}-*dense*, iff in any maximal $(P \cup Q)$-clique $\widetilde{E}$ of E for all distinct elements $e_0, e_1, e_2, e_3 \in \widetilde{E}$, if $(e_0\ P^\beta\ e_1\ Q^\beta\ e_2)$, $(e_0\ Q^\beta\ e_3\ P^\beta\ e_2)$, $(e_0\ P^\beta\ e_2)$, and $(e_3\ Q^\beta\ e_1)$ then $\exists e \in \widetilde{E}$ such that $\{e, e_0, e_2\}$ is a P-clique and $\{e, e_3, e_1\}$ is a Q-clique of $\widetilde{E}$, for all distinct elements $e_0, e_1, e_2, e_3 \in \widetilde{E}$.

The relational structures S_4 and S_5 are $N_{V'V''}$-dense. The relational structure $S_6 = (E_6, V', V'')$ with the non-transitive relation V' and the symmetric relation V'' is not $N_{V'V''}$-dense because in the maximal $(V' \cup V'')$-clique $\{e_1, \ldots, e_5\}$ of E_6, there are distinct elements e_1, e_2, e_3, e_4 such that $e_1\ V'\ e_2\ V''\ e_4$, $e_1\ V''\ e_3\ V'\ e_4$, and $e_1\ V'\ e_4$, $\{e_1, e_4, e_5)$ is V'-clique of E_6, but $\{e_2, e_3, e_5)$ is not V''-clique of E_6.

The relationships between N-density and K-density in the context of relational structures were shown in paper [26].

Theorem 2. *Given a relational structure with distinct connectives P and Q,*

(i) S is K_{PQ}-dense $\Longrightarrow$ S is N_{PQ}-dense,

(ii) S is K_{PQ}-dense $\Longleftarrow$ S is N_{PQ}-dense, PQ-combinatorial, P-transitive and Q-finite.

The following results establish the relationships between the properties of N-freeness and N-density defined so far.

Proposition 2. *Let S be a relational structure with distinct connectives P and Q, and let P be a transitive or symmetric connective. Then,*

(i) S is $\bowtie^\beta_{PQ}$-dense $\Longrightarrow$ S is $\bowtie_{PQ}$-dense,

(ii) S is $\bowtie^\beta_{PQ}$-dense $\Longleftarrow$ S is $\bowtie_{PQ}$-dense, Q is symmetric.

Proof.
(i) Consider an arbitrary maximal $(P \cup Q)$-clique $\widetilde{E}$ of E. Let e_0, e_1, e_2, e_3 be distinct events from $\widetilde{E}$, s.t. $(e_0\ P\ e_1\ Q\ e_2)$, $(e_0\ Q\ e_3\ P\ e_2)$ and $e_0\ P^\delta\ e_2$. It is obvious that $(e_0\ P^\beta\ e_1\ Q^\beta\ e_2)$, $(e_0\ Q^\beta\ e_3\ P^\beta\ e_2)$ and $e_0\ (P^\delta)^\beta\ e_2$. By $\bowtie^\beta_{PQ}$-density of E, we obtain $e_3\ P^\beta\ e_1$. If P is a symmetric connective, we are done.

Consider a case when P is a transitive connective. Suppose $e_1\ P\ e_3$. Then we have $e_0\ P\ e_3$ by transitivity of P. So, we come to a contradiction to $P^\beta \cap Q^\beta = \varnothing$. So, it holds that $e_3\ P\ e_1$, and we have $\widetilde{E}$ is $\bowtie_{PQ}$-dense.

(ii) Consider an arbitrary maximal $(P \cup Q)$-clique $\widetilde{E}$ of E. Let e_0, e_1, e_2, e_3 be distinct events from $\widetilde{E}$, s.t. $(e_0\ P^\beta\ e_1\ Q^\beta\ e_2)$, $(e_0\ Q^\beta\ e_3\ P^\beta\ e_2)$ and $e_0\ (P^\delta)^\beta\ e_2$. By symmetry of Q, it is obvious that $e_1\ Q\ e_2$ and $e_0\ Q\ e_3$.

If P is a symmetric connective also, we have $(e_0\ P\ e_1\ Q\ e_2)$, $(e_0\ Q\ e_3\ P\ e_2)$ and $e_0\ (P^\delta)\ e_2$. Then, by $\bowtie_{PQ}$-density of $\widetilde{E}$, we obtain $e_1\ P\ e_3$, i.e. $e_3\ P^\beta\ e_1$ and $\widetilde{E}$ is $\bowtie^\beta_{PQ}$-dense.

Consider a case P is a transitive connective. Consider possible cases which follow from $e_0\ P^\beta\ e_1$.

1. If $e_0\ P\ e_1$, then according to transitivity of P and the property $P^\beta \cap Q^\beta = \varnothing$, we obtain $e_0\ P^\delta\ e_2$. In the similar way, we get $e_3\ P\ e_2$.

   Then, by $\bowtie_{PQ}$-density of $\widetilde{E}$, it holds that $e_3\ P\ e_1$, i.e. $e_3\ P^\beta\ e_1$. So, $\widetilde{E}$ is $\bowtie^\beta_{PQ}$-dense.
2. If $e_1\ P\ e_0$, then in the way similar to case 1, we get $e_2\ P\ e_0$, $e_2\ P\ e_3$. So, $(e_2\ P\ e_3\ Q\ e_0)$, $(e_2\ Q\ e_1\ P\ e_0)$ and $e_2\ (P^\delta)\ e_0$.

   Then, by $\bowtie_{PQ}$-density of $\widetilde{E}$, $e_1\ P\ e_3$, i.e. $e_3\ P^\beta\ e_1$. So, $\widetilde{E}$ is $\bowtie^\beta_{PQ}$-dense.

□

Immediately from the definitions of R^δ, $\bowtie'_{PQ}$- and $\bowtie^\beta_{PQ}$-density, we obtain

Proposition 3. *Let S be a relational structure with distinct connectives P and Q. Then,*

(i) S is $\bowtie^\beta_{PQ}$-dense and P is non-transitive $\Longrightarrow$ S is $\bowtie'_{PQ}$-dense,

(ii) S is $\bowtie^\beta_{PQ}$-dense $\Longleftarrow$ S is $\bowtie'_{PQ}$-dense.

□

Proposition 4. *Let S be a relational structure with distinct connectives P and Q. Then,*

(i) S is N_{PQ}-dense and P is transitive $\Longrightarrow$ S is $\bowtie^\beta_{PQ}$-dense.

(ii) S is N_{PQ}-dense $\Longleftarrow$ S is $\bowtie^\beta_{PQ}$-dense and PQ-discrete.

Proof. (i) Let $\widetilde{E}$ be an arbitrary maximal $(P \cup Q)$-clique of E. Take distinct elements e_0, e_1, e_2, e_3 of $\widetilde{E}$ such that $(e_0\ P^\beta\ e_1\ Q^\beta\ e_2)$, $(e_0\ Q^\beta\ e_3\ P^\beta\ e_2)$ and $(e_0\ P^{\delta\beta}\ e_2)$. Let us show that $e_3\ P^\beta\ e_1$.

Suppose $e_3\ Q^\beta\ e_1$. Then, by N_{PQ}-density of $\widetilde{E}$, $\exists e \in \widetilde{E}$ such that $\{e, e_0, e_2\}$ is a P-clique and $\{e, e_3, e_1\}$ is a Q-clique of $\widetilde{E}$.

W.l.o.g. we suppose $e_0\ P\ e_1$, $e_3\ P\ e_2$ and $(e_0\ P^\delta\ e_2)$.

By transitivity of P, it is impossible that $e \ P \ e_0$ and $e_2 \ P \ e$. So, $e_0 \ P \ e \ P \ e_2$. We have a contradiction to $e_0 \ P^{\delta\beta} \ e_2$. Hence, $e_3 \ P^{\beta} \ e_1$ and S is $\bowtie^{\beta}_{PQ}$-dense.

(ii) Let $\widetilde{E}$ be an arbitrary maximal $(P \cup Q)$-clique of E. Suppose there are distinct elements e_0, e_1, e_2, e_3 of $\widetilde{E}$ such that $(e_0 \ P^{\beta} \ e_1 \ Q^{\beta} \ e_2)$, $(e_0 \ Q^{\beta} \ e_3 \ P^{\beta} \ e_2)$, $(e_0 \ P^{\beta} \ e_2)$, and $(e_3 \ Q^{\beta} \ e_1)$.

1) If P is non-transitive, then $e_0 \ (P^{\delta})^{\beta} \ e_2$. Then $(e_3 \ Q^{\beta} \ e_1)$ contradicts the definition of $\bowtie^{\beta}_{PQ}$-density. So, for arbitrary four distinct elements of $\widetilde{E}$, it does not hold $(e_0 \ P^{\beta} \ e_1 \ Q^{\beta} \ e_2)$, $(e_0 \ Q^{\beta} \ e_3 \ P^{\beta} \ e_2)$, $(e_0 \ P^{\beta} \ e_2)$, and $(e_3 \ Q^{\beta} \ e_1)$. So, S is N_{PQ}-dense.

2) If P is transitive, w.l.o.g. we suppose $e_0 \ P \ e_1$, $e_3 \ P \ e_2$ and $(e_0 \ P^{\delta} \ e_2)$.

If $e_0 \ P^{\delta} \ e_2$, we obtain a contradiction in the way similar to case (1). If $e_0 \ P \ e_2$, by PQ-discretness, there are $e'_i, 1 \leqslant i \leqslant k < \infty$ such that $e_0 \ P^{\delta} \ e'_1 \ P^{\delta} \ e'_2 \ldots P^{\delta} \ e'_k \ P^{\delta} \ e_2$. W.l.o.g. we can suppose $k = 1$. By transitivity of P, it is impossible that $e'_1 \ P \ e_3$ and $e_1 \ P \ e'_1$.

Consider two admissible cases. If $e'_1 \ P \ e_1$, then, by transitivity of P, it is impossible that $e_3 \ P \ e'_1$. Hence, $e_3 \ Q^{\beta} \ e'_1$. So, we obtain a contradiction to the definition of $\bowtie^{\beta}_{PQ}$-density. If $e_3 \ P \ e'_1$, then we can get a contradiction in a similar way.

So, for a transitive P, it holds that S is N_{PQ}-dense. $\square$

References

[1] Bernardinello L., Ferigato C., Haar S., Pomello L. Closed sets in occurrence nets with conflicts // Fundamenta Informaticae. – 2014. – Vol. 133, No. 4. – P. 323–344.

[2] Bernardinello L., Pomello L., Rombola S. Orthomodular algebraic lattices related to combinatorial posets // CEUR-WS.ORG. – 2014. – Vol. 1231. – P. 241–245.

[3] Best E. The relative strenghth of K-density// Lect. Notes Comp. Sci. – 1980. – Vol. 84. – P. 261–276.

[4] Best E. A theorem on the characteristics of non-sequential processes// Fundamenta Informaticae. – 1980. – Vol. 3. – P. 77–94.

[5] Best E., Fernandez C. Nonsequential processes — a Petri net view // EATCS Monographs on Theoretical Comput. Sci. – Springer-Verlag. – 1988. – Vol. 13.

[6] Baldan P., Corradini A., Montanari U. Contextual Petri nets, asymmetric event structures, and processes// Information and Computation. – 2001. – Vol. 171, No. 1. – P. 1–49.

[7] Boudol G., Castellani I. Concurrency and atomicity//Theoretical Comput. Sci. – 1988. – Vol. 59. – P. 25–84.

[8] Fernandez C. Non-sequential processes// Lect. Notes Comp. Sci. – 1986. – Vol. 254. – P. 95–116.

[9] van Glabbeek R.J., Plotkin G.D. Configuration structures, event structures and Petri nets.// Theoretical Comput. Sci. – 2009. – Vol. 410, No. 41. – P. 4111–4159.

[10] Haar S. Occurrence Net Logics// Fundamenta Informaticae. – 2000. – Vol. 43, No. 1-4. – P. 105–127.

[11] Janicki R., Koutny M. Invariants and paradigms of concurrency theory// Lect. Notes Comp. Sci. – 1991. – Vol. 506. – P. 59–74.

[12] Juhas G., Lorenz R., Mauser S. Synchronous + Concurrent = Earlier Than + Not Later Than// Proc. of ACSD06, IEEE Press, New York. – 2006. – P. 261–272.

[13] Katoen J.-P. Quantitative and qualitative extensions of event structures: PhD Thesis. – Twente University, 1996.

[14] Kleijn J., Koutny M. Process semantics of general inhibitor nets// Inform. and Comput. – 2004. – Vol. 190. – P. 18–69.

[15] Kleijn J., Koutny M. Mutex Causality in Processes and Traces of General Elementary Nets// Fundamenta Informaticae. – 2013. – Vol. 122, No. 1-2. – P. 119–146.

[16] Kotov V., Cherkasova L. On structural properties of generalized processes// Lect. Notes Comp. Sci. – 1984. – Vol. 188. – P. 288–306.

[17] Nielsen M., Plotkin G., Winskel G. Petri nets, event structures and domains// Theoretical Comput. Sci. – 1981. – Vol. 13. – P. 85–108.

[18] Petri C.A. Concurrency// Lect. Notes Comp. Sci. – 1980. – Vol. 84. – P. 251–260.

[19] Pinna G. M., Poigné A. On the nature of events: another perspective in concurrency// Theoretical Comput. Sci. – 1995. – Vol. 138. – P. 425–454.

[20] Plünnecke H. K-density, N-density and finiteness properties// Lect. Notes Comp. Sci. – 1984. – Vol. 188. – P. 392–412.

[21] Polyvyanyy A., Weidlich M., Conforti R., La Rosa M., ter Hofstede A.H.M. The 4C Spectrum of Fundamental Behavioral Relations for Concurrent Systems// Lect. Notes Comp. Sci. – 2014. – Vol. 8489. – P. 210–232.

[22] Pouzet M., Zaguia N. N-free extensions of posets. Note on a theorem of P.A. Grillet// Contributions to Discrete Mathematics. – 2006. – Vol. 1, No. 1. – P. 80–87.

[23] Virbitskaite I. Some characteristics of nondeterministic processes// Parallel Processing Letters. – 1993. – Vol. 3, No. 1. – P. 99–106.

[24] Bozhenkova E.N., Virbitskaite I.B. Unified Characterization of Some Properties of Event Structures // Proc. Intern. Conf. CONPAR 94 – VAPP VI. – Linz, Austria, September 1994.–P.29–32 – (RISC-Linz Rep. Ser. – 1994. – No 94-48).

[25] Virbitskaite I., Bozhenkova E., Erofeev E. Space-Time Viewpoints for Concurrent Processes Represented by Relational Structures// CEUR Workshop Proceedings. – 2015. – Vol. 1492. – P. 222–233.

[26] Bozhenkova E., Virbitskaite I. Some Space-Time Characteristics of Relational Structures// The Fourth Symp. RuFiDim. Proc.– 2017.–P.27–38.–(TUCS Lecture Notes; Vol. 26.).

[27] Winskel G. Events in computation: PhD Thesis University of Edinburgh, 1980.