Bull. Nov. Comp. Center, Num. Anal., 7 (1996), 1–6

Numerical solution of one-dimensional Focker–Plank–Kolmogorov equation*

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The paper considers the algorithm of numerical solution of the Focker–Plank–Kolmogorov equation for the probability density of a solution of a stochastic differential equation. Its solution is approximated by cubic splines on the time-dependent moving grid.

For a numerical analysis of solutions of nonlinear stochastic differential equations (SDE) the method of statistic simulation of trajectories is usually successfully used [1]. But even in the case of one dimension there are examples of SDE for which numerical estimations made by this method are wrong. It arises, as a rule, from a small probability of big deviations of the SDE solution when it is necessary to simulate a huge number of trajectories to get a sufficient accuracy of an estimation. One of the ways out in this situation is a numerical solution of the Focker–Plank–Kolmogorov equation for the probability density of the SDE solution. It is a parabolic partial differential equation.

To solve one-dimensional parabolic partial differential equations, effective package of programs has been developed [2]. However, in the algorithms used in [2] a vital defect is that ends of a space grid are constant while time changing and cannot take an infinite value. But a probability density of a stochastic process can have an unlimited support on space, and its support can move significantly on space while time changing.

In article [3] the algorithm of solution of the one-dimensional Focker–Plank–Kolmogorov equation in the case of a constant noise intensity was given. In this paper we supply the algorithm in the general case.

Consider the one-dimensional autonomous SDE in the sense of Ito:

$$dy(t) = f(y(t))\,dt + \sigma(y(t))\,dw(t), \quad 0 \le t \le t_{\text{end}}, \qquad y(0) = y_0, \qquad (1)$$

where $w(t)$ is a standard Wiener process, y_0 is a random variable with the probability density $p_0(y)$; $f(y)$, $\sigma(y)$ are sufficiently smooth functions and, besides, we assume that

$$\operatorname{Dom} f \cap \operatorname{Dom} \sigma = [y_L, y_R], \quad -\infty \le y_L < y_R \le +\infty,$$
$$\sigma(y) \ne 0 \quad \text{for } y_L < y < y_R.$$

*Supported by the Russian Foundation of Basic Research under Grant 94–01–00074.

Here Dom is a domain of a function. We also assume the functions $f(y)$, $\sigma(y)$ be chosen such that the solution of (1) exists and is unique [4].

It is known (see, for example, [4]) that the probability density $p(t,y)$ of the stochastic process $y(t)$ satisfies the following initial-boundary problem:

$$\frac{\partial p(t,y)}{\partial t} = -\frac{\partial}{\partial y}(f(y)p(t,y)) + \frac{1}{2}\frac{\partial^2}{\partial y^2}(\sigma^2(y)p(t,y)), \qquad (2)$$
$$0 < t < t_{\text{end}}, \quad y_L < y < y_R,$$
$$p(0,y) = p_0(y), \quad y_L < y < y_R, \qquad (3)$$
$$p(t,y_L) = p(t,y_R) = 0, \quad 0 < t < t_{\text{end}}. \qquad (4)$$

Besides,

$$p(t,y) \geq 0 \quad \text{for } 0 < t < t_{\text{end}},\ y_L < y < y_R, \qquad (5)$$
$$\int_{y_L}^{y_R} p(t,y)dy = 1 \quad \text{for } 0 < t < t_{\text{end}}. \qquad (6)$$

Equation (2) is called the Focker–Plank–Kolmogorov equation.

Let us make the substition in (2)–(4)

$$u(t,y) = \ln p(t,y), \qquad (7)$$

and let $b(y) = 0.5\sigma(y)^2$. After the substitution we obtain the following problem:

$$\frac{\partial u}{\partial t} = -\left(\frac{\partial f}{\partial y} + f\frac{\partial u}{\partial y}\right) + \frac{\partial^2 b}{\partial y^2} + 2\frac{\partial b}{\partial y}\frac{\partial u}{\partial y} + b\frac{\partial^2 u}{\partial y^2} + b\left(\frac{\partial u}{\partial y}\right)^2, \qquad (8)$$
$$0 < t < t_{\text{end}}, \quad y_L < y < y_R,$$
$$u(0,y) = \ln p_0(y), \quad y_L < y < y_R, \qquad (9)$$
$$u(t,y_L) = u(t,y_R) = -\infty, \quad 0 < t < t_{\text{end}}. \qquad (10)$$

Substition (7) allows to keep no care about the positivity of $p(t,y) = \exp u(t,y)$ while numerical solving (8).

We define a mesh on the interval $[0, t_{\text{end}}]:\ 0 = t_0 < t_1 < \ldots < t_K = t_{\text{end}}$. Let us define a time-dependent uniform moving grid on y with a constant number of points N

$$\begin{aligned}
&y_1(t) = \max\{e(t_s) - C\sqrt{v(t_s)}, y_L\},\\
&y_N(t) = \min\{e(t_s) + C\sqrt{v(t_s)}, y_R\},\\
&h(t) = \frac{y_N(t) - y_1(t)}{N-1},\\
&y_i(t) = y_1(t) + (i-1)h(t), \quad i = 2,\ldots,N-1,\\
&t \in [t_s, t_{s+1}), \quad s = 0,\ldots,K-1,
\end{aligned} \qquad (11)$$

where $C > 0$ is a positive constant, $e(t)$, $v(t)$ are the mean and, respectively, the variancy of $y(t)$ is the following:

$$e(t) = \int_{y_L}^{y_R} y \exp u(t,y)\, dy, \qquad v(t) = \int_{y_L}^{y_R} (y - e(t))^2 \exp u(t,y)\, dy.$$

Let $S(t,y)$ be a cubic spline on y of the $C^2(-\infty,+\infty)$ class with the grid points $y_i(t)$, $i = 1,\ldots,N$ that approximates $u(t,y)$ for every t. We introduce the notations:

$$u_i(t) = S(t, y_i(t)), \quad m_i(t) = \frac{\partial S}{\partial y}(t, y_i(t)), \quad M_i(t) = \frac{\partial^2 S}{\partial y^2}(t, y_i(t)),$$
$$i = 1,\ldots,N.$$

Further we shall omit the argument t in some notations. Let us put

$$S(t,y) = u_1 + m_1(y - y_1) + \frac{M_1}{2}(y - y_1)^2 \quad \text{for } y < y_1; \tag{12}$$

$$S(t,y) = u_N + m_N(y - y_N) + \frac{M_N}{2}(y - y_N)^2 \quad \text{for } y > y_N. \tag{13}$$

We can write for $y \in [y_i, y_{i+1}]$, $i = 1,\ldots,N-1$ according to [5]:

$$\begin{aligned} S(t,y) &= u_i(1-\Delta_i)^2(1+2\Delta_i) + u_{i+1}\Delta_i^2(3-2\Delta_i) + \\ &\quad hm_i\Delta_i(1-\Delta_i)^2 - hm_{i+1}\Delta_i^2(1-\Delta_i), \end{aligned} \tag{14}$$

where $\Delta_i = h^{-1}\cdot(y - y_i)$.

In conditions of a continuity of the spline first derivative [5]

$$\begin{aligned} 2m_1 + m_2 &= 3\frac{u_2 - u_1}{h} - \frac{h}{2}M_1, \\ m_{i-1} + 4m_i + m_{i+1} &= 3\frac{u_{i+1} - u_{i-1}}{h}, \qquad i = 2,\ldots,N-1, \\ m_{N-1} + 2m_N &= 3\frac{u_N - u_{N-1}}{h} + \frac{h}{2}M_N, \end{aligned}$$

we put $M_1 = \dfrac{m_2 - m_1}{h}$, $M_N = \dfrac{m_N - m_{N-1}}{h}$. Then we obtain the system of linear algebraic equations for every t:

$$\begin{pmatrix} 1 & 1 & & & \\ 0.5 & 2 & 0.5 & & \\ & \ddots & \ddots & \ddots & \\ & & 0.5 & 2 & 0.5 \\ & & & 1 & 1 \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \\ \vdots \\ m_N \end{pmatrix} = \frac{1}{h} \begin{pmatrix} -2 & 2 & & & \\ -1.5 & 0 & 1.5 & & \\ & \ddots & \ddots & \ddots & \\ & & -1.5 & 0 & 1.5 \\ & & & -2 & 2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{pmatrix}. \tag{15}$$

We demand $S(t,y)$ satisfy equation (8) on every interval $[t_s, t_{s+1})$, $s = 0,\ldots,K-1$, at the grid points $y_i(t)$, $i = 1,\ldots,N$. Then

$$M_i = \frac{1}{b(y_i)}\Big(\frac{du_i}{dt} + \frac{\partial f}{\partial y}(y_i) + f(y_i)m_i - 2\frac{\partial b}{\partial y}(y_i)m_i - \frac{\partial^2 b}{\partial y^2}(y_i)\Big) - m_i{}^2.$$

If we substitute M_i into conditions of a continuity of the spline second derivative [5], then, after simple transformations, we obtain the system of ordinary differential equations on each of the intervals $[t_s, t_{s+1})$, $s = 0, \ldots, K-1$:

$$\frac{du}{dt} = \frac{3}{h^2}BCu + D\omega + \frac{3}{h}B\theta - \alpha + 2\beta + \gamma - \delta, \tag{16}$$

where $B = DA^{-1}$, $D = \mathrm{diag}(b(y_1), b(y_2), \ldots, b(y_N))$,

$$A = \begin{pmatrix} 1 & 0.5 & & & \\ 0.5 & 2 & 0.5 & & \\ & \ddots & \ddots & \ddots & \\ & & 0.5 & 2 & 0.5 \\ & & & 0.5 & 1 \end{pmatrix}, \qquad C = \begin{pmatrix} -1 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -1 \end{pmatrix},$$

$$u = [u_1, \ldots, u_N]^T, \qquad \frac{du}{dt} = \Big[\frac{du_1}{dt}, \ldots, \frac{du_N}{dt}\Big]^T,$$

$$\alpha = [f(y_1)m_1, \ldots, f(y_N)m_N]^T, \qquad \beta = \Big[\frac{\partial b}{\partial y}(y_1)m_1, \ldots, \frac{\partial b}{\partial y}(y_N)m_N\Big]^T,$$

$$\gamma = \Big[\frac{\partial^2 b}{\partial y^2}(y_1), \ldots, \frac{\partial^2 b}{\partial y^2}(y_N)\Big]^T, \qquad \delta = \Big[\frac{\partial f}{\partial y}(y_1), \ldots, \frac{\partial f}{\partial y}(y_N)\Big]^T,$$

$$\theta = [-m_1, 0, \ldots, 0, m_N]^T, \qquad \omega = [m_1^2, \ldots, m_N^2]^T.$$

The numerical solution of (8)–(10) will be performed by the following scheme.

0. Let $s = 0$. Calculate the mean $e(0)$ and the variance $v(0)$ of the random variable y_0. By formulae (11) find the values of the grid points $y_i = y_i(0)$, $i = 1, \ldots, N$. Calculate the initial values of the spline and its derivative:

$$u_i = \ln p_0(y_i), \quad m_i = \frac{\partial(\ln p_0)}{\partial y}(y_i), \quad i = 1, \ldots, N.$$

1. By using the initial values u_i and m_i, $i = 1, \ldots, N$ calculate the solution of (16) at the point $t = t_{s+1}$ by any numerical method [1].
2. Check the execution of normalization (6), calculate

$$I = \int_{y_L}^{y_R} \exp S(\tau, y)\, dy \approx \frac{h}{2}\sum_{i=1}^{N-1} \exp(u_i) + \exp(u_{i+1}).$$

In the case of a considerable violation of normalization (6) (i.e., $I \neq 1$) we need to increase the number of grid points N or to decrease the stepsize from t_s to t_{s+1}. If the normalization is sufficient, then we can find $p(t_{s+1}, y) = \exp S(t_{s+1}, y)$.

3. Calculate the mean $e(t_{s+1})$ and the variance $v(t_{s+1})$:

$$e(t_{s+1}) \approx \frac{h}{2} \sum_{i=1}^{N-1} y_i \exp(u_i) + y_{i+1} \exp(u_{i+1}),$$

$$v(t_{s+1}) \approx \frac{h}{2} \sum_{i=1}^{N-1} (y_i - e(t_{s+1}))^2 \exp(u_i) + (y_{i+1} - e(t_{s+1}))^2 \exp(u_{i+1}).$$

By formulae (11) find the values of the new grid points y_i, $i = 1, \ldots, N$ at $t = t_{s+1}$.

4. Calculate the values of the spline on the new grid by using the formulae (12), (13) and (14). Find the values of the spline derivative on the new grid by solving (15). Increase s by 1. Switch to the calculation of the density at the next time point by using the received values u_i, m_i, $i = 1, \ldots, N$ as the initial ones (i.e., switch to item 1).

Now we suggest the results of numerical calculations of a solution of the Focker–Plank–Kolmogorov equation for two SDE in the sense of Ito [1]. In tests the number of grid points $N = 20$ and $C = 3$ in formulae (11). A numerical solution of ODE system (16) was made by the explicit Euler method with the stepsize 10^{-3}.

$$1) \qquad dy = -(y-1)\,dt + \sqrt{2y}\,dw(t), \quad t \in [0,1], \tag{17}$$

where y_0 is the normal random variable with the mean 2 and the variance 0.25. For this equation $y_L = 0$, $y_R = +\infty$. A solution of SDE (17) has an exponential stationary distribution with the density $p(y) = e^{-y}$, $y > 0$.

Figure 1 shows a dynamics of a transformation on the whole time interval of the calculated density $p(t, y)$ from the normal density at the begining to the exponential one at the end.

$$2) \qquad dy = -(y-0.5)\,dt + \sqrt{y(1-y)}\,dw(t), \quad t \in [0,1], \tag{18}$$

where y_0 is the normal random variable with the mean 0.5 and the variance 0.0225. For this equation $y_L = 0$, $y_R = 1$. A solution of SDE (18) has a uniform stationary distribution with the density $p(y) = 1$, $0 < y < 1$.

Figure 2 shows a dynamics of a transformation on the whole time interval of the calculated density $p(t, y)$ from the normal density at the begining to the uniform one at the end.

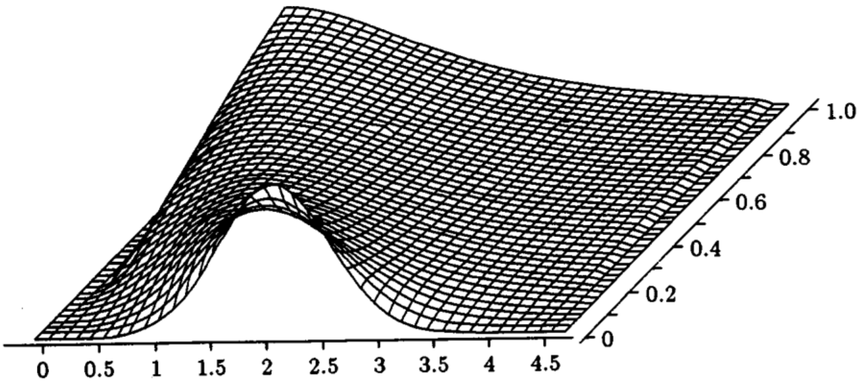

Figure 1. Solution of Focker–Plank–Kolmogorov equation for SDE (17)

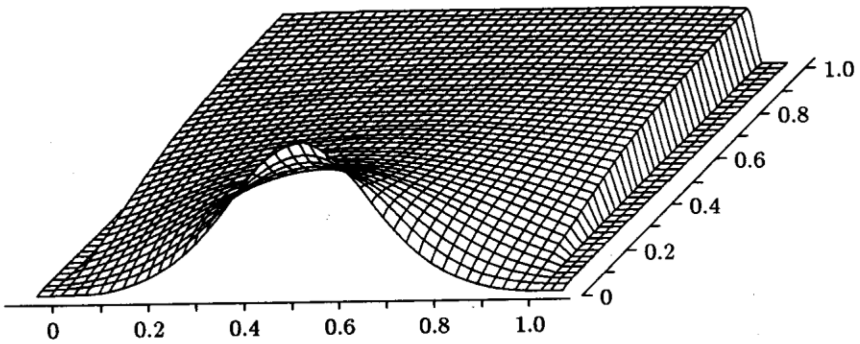

Figure 2. Solution of Focker–Plank–Kolmogorov equation for SDE (18)

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