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Parallel algorithms for options pricing by Monte Carlo method*

S.S. Artemiev, S.A. Gusev, O.G. Monakhov

Problems of modeling and calculations of dynamics of the price of options by Monte Carlo method on parallel processors are considered. Technique of calculation of some factors, enabling to investigate change of the price of options and to evaluate possible consequences of the made bargains is described. Numerical calculations show the speed up in many cases is close to linear function of number of processors.

1. Introduction

Monte Carlo method is a convenient means of numerical modeling of forward financial operations [1]. In particular, all calculations connected with options can be executed by this method. The main difficulty arising by use of Monte Carlo method is it's executing time. If calculations with high accuracy are needed, one has to simulate 10^5–10^8 of trajectories of solutions of stochastic differential equations (SDE's), that requires transition from PC to more high-speed computers. It is the most convenient for these purposes to use parallel processors, because statistical algorithms can be parallelized by a natural way. Algorithms for computation of derivatives from option price with respect to parameters for options are indicated in this paper.

In the paper algorithms for computing derivatives from premium of an option on parameters for options of american style are described, that is very difficult to do using ordinary PC.

The values of premium of option are calculated by Monte Carlo method at the nodes of grids given on closed intervals of values of appropriate parameters. The approximate values of derivatives from the premium of the option are received as a result of differentiation of appropriate interpolational cubic splines.

Main and the most labour-consuming part of the used algorithms is the modeling of large number of trajectories of SDE's, i.e., decision of a plenty of the same type independent problems. Therefore the part of the parallel program, connected with simulating of trajectories can be effectively executed on a network of parallel processors. The numerical examples indicated at the end of the paper show, that acceleration, considered as the relation of a

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time of computation on one processor to a time of computation on several processors, close to linear function from quantity of processors.

2. Pricing of options

An option has a right either to buy or to sell a specified amount or value of a particular underlying interest at a fixed exercise price by exercising the option before its specified expiration date. An option which gives a right to buy is a *call* option, and an option which gives a right to sell is a *put* option. We consider here only the options of call style.

Options are currently available covering four types of underlying interests: equity securities, stock indexes, government debt securities, and foreign currencies. The options are of the european style to a fixed date of execution and of the american style, which up to a fixed extreme date of the expiration of the contract can be presented to execution at any moment. In this paper pricing for the options of american style only are considered, because they have not the exact formulas such as the Black–Scholes [2] type for theoptions of european style and therefore Monte Carlo method is the unique way of computations.

The market cost of an option is determined as a result of auction tenders on optional exchange. The price, on which agree buyer and seller of the option, refers to as the premium. So the named fair cost of an option is a theoretically justified floor price, at which the subscriber of the option can, using a hedging strategy, supply by a guaranteed image the optional payments, irrespective of a casual condition of the prices of underlying interest at the market.

Characteristic parameters of an option are:

Pr – premium of option,

K – exercise price,

T – expiration time,

σ – volatility of the underlying interest,

r – annual riscless interest rate,

S_0 – the price of the underlying interest at the conclusion of the contract.

Hereinafter for simplicity we consider the share without payment of dividends as the unit of the underlying interest.

The model of price dynamics of the share is usually set as the SDE in the Ito sense

$$dS_t = rS_t\,dt + \sigma S_t\,dw(t), \quad S(0) = S_0, \tag{1}$$

where r and σ are the real parameters, w is the standard Wiener process.

The exact solution of (1) is the following:

$$S_t = S_0 \exp\left(\left(r - \frac{\sigma^2}{2}\right)t + \sigma w(t)\right). \tag{2}$$

The movement of the share price can be set by the recurrent formula

$$S_{n+1} = S_n \exp\left(\left(r - \frac{\sigma^2}{2}\right)h + \sigma\sqrt{h}\zeta_n\right), \tag{3}$$

where S_n is the value of share price at $t = t_n$, h – the step of uniform grid on $[0, T]$, $\{\zeta_n\}$ – the sequence of normal random variables having zero mathematical expectation and unit dispersion.

The premium of a standard call option of american style can be calculated as

$$Pr = \max_{0 \le t \le T} \langle e^{-rt}(S_t - K)^+ \rangle, \tag{4}$$

where the angle brackets mean mathematical expectation. There are every day changes of prices of underlying interest of options at the market. As a result prices of options changes accordingly. The factor *delta* represents the ratio of the change of the option price to change of the price of the interest:

$$\delta = \frac{\partial Pr(S_0, K, T, r, \sigma)}{\partial S_0}.$$

Except the factor δ there are the other factors connected with the option premium. They are: $\Gamma = \frac{\partial^2 Pr}{\partial S_0^2}$ – the rate of change of the factor δ as S_0 change, $\omega = \frac{\partial Pr}{\partial \sigma}$ – the rate of change of the premium as σ change and $\rho = \frac{\partial Pr}{\partial r}$ – the premium rate change of the premium as r change.

3. Numerical algorithms

In this section the algorithms for computation of the factors δ, Γ, ω, ρ for the options of american style are considered. These computations have been got by using a parallel computer system MVS-100, on the base of eight parallel processors Intel860.

At first we consider calculation of the factor δ. Let set $D = [S_0^{\min}, S_0^{\max}]$ a set of values of initial price of underlying interest and an uniform grid $G_1 = \{s_i\}_{i=1}^n$ on it. Let denote Pr_i the premium of option corresponding to initial price of underlying interest s_i according to formula (4).

We also set a grid $G_2 = \{t_i\}_{i=1}^{n+1}$ for construction of an interpolational cubic spline as the linear combination of normalized B-splines of fourth order

$$t_1 = t_2 = t_3 = t_4 = S_0^{\min},$$
$$s_1 = S_0^{\min} < s_2 < s_3 = t_5 < \ldots < s_{n-2} = t_n < s_{n-1} < s_n = t_{n+1} = S_0^{\max}. \tag{5}$$

The linear combination of normalized B-splines of fourth order

$$Sp(S_0) = \sum_{i=1}^{n} \alpha_i B_i(S_0), \quad S_0 \in [S_0^{\min}, S_0^{\max}],$$

may be obtained by the solution of the linear system

$$\sum \alpha_k B_k(s_i) = Pr_i, \quad i = 1, \ldots, n.$$

Note, the choice of knots (5) corresponds to boundary conditions which mean continuity of third derivative of the spline at s_2 and s_{n-1}. As a result the approximate function $\delta(S_0)$ is $\delta(S_0) = \frac{dSp(S_0)}{ds}$.

In [3] valuation of an approximation error of interpolating spline derivative is indicated, when interpolated function on a grid is given not exactly. In our case this valuation is

$$\left\| \delta(S_0) - \frac{dSp(S_0)}{ds} \right\|_D \leq c_1 \Delta^2 + \varepsilon c_2 \Delta^{-1}, \tag{6}$$

where $\Delta = s_{i+1} - s_i$ is a step of grid G_1, ε is a computation error of Pr_i, $\|f(s)\|_D = \max_{s \in D} \|f(s)\|$.

From the condition of a minimum of the right-hand side of (6) at the given ε the step $\Delta = (\frac{\varepsilon c_2}{2c_1})^{1/3}$ is received, and the valuation of an error is equal to $(1 + 2^{1/3}) c_1^{1/3} c_2^{2/3} \varepsilon^{2/3}$.

In this case the most labour-consuming part of the program is that, which is connected in calculation for each node of the grid G_1 values of premium according to formulas (4). Here it is necessary to simulate the given number of trajectories of the random process S_t determined by equation (1) on the segment $[0, T]$ with the initial values $S_0 = s_i$. Simultaneously these trajectories S_t are used for the calculation of Pr_i. As far as the simulating of trajectories of the specified random processes for each node s_i can be executed independently, this part of the program may be effectively realized on parallel processors.

As our algorithms contain many calculations of the same type it is convenient to parallelize them like a processor farm (master–slave method).

Thus the common structure of the program for calculation of the factor δ consists in the following. Let us have in the system P parallel processors with the numbers $0, 1, \ldots, P-1$ connected among themselves by channels. Processor with number 0 we name as *main*, and other processors – as *auxiliary*. The main processor executes input of the initial information, preparation and transfer on communication channels tasks for auxiliary processors.

Now we distribute all quantity of the knots n between processors whenever possible so that the loading of processors will be uniform. The following way of distribution of loading of processors is offered. If n is divisible by P, each processor makes computation of values of option premium for n/P of nodes. If n is not divisible by P, then each auxiliary processor must calculate values of premium for $[n/P]+1$ nodes, and the main processor remains to make computations for $n-([n/P]+1)(P-1)$ nodes. Such distribution of nodes among processors guarantees, a number of nodes, intended to a main processor not more than any that of auxiliary processors.

Computations of each of the factors Γ, ω, ρ are executed by analogy with computation of the factor δ, except for factor Γ we need to calculate the second derivative of interpolating spline.

4. Numerical results

In this section examples of computations of the factors δ, Γ, ω, ρ are indicated.

Example 1. **Computation of the factors δ, Γ.** The floor price of the basic share – 40, the maximum price of the basic share – 50, the exercise price – 45, the expiration time of option – 0.5 years, the annual riscless interest rate – 5%, the volatility of the underlying interest – 4%, number of simulated trajectories – $3 \cdot 10^6$. In Figures 1, 2 the schedules of dependence of the factors δ, Γ on a given interval of the initial price of the underlying interest are indicated.

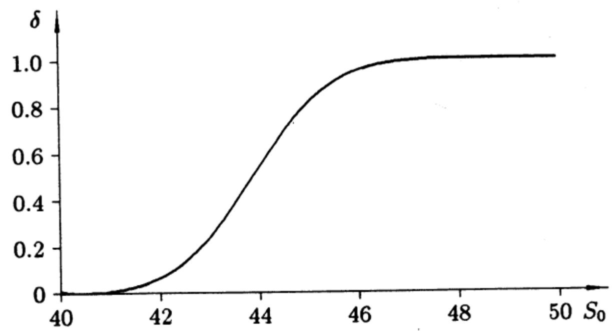

Figure 1. Computations of the factor δ

Example 2. **Computation of the factor ω.** The Initial price of the underlying interest – 40, the exercise price – 42, the expiration time – 0.5 years, the minimal size of volatility – 5%, the maximal size of volatility – 10%, annual riscless interest rate – 15%, a number of simulated trajectories – 10^6. In Figure 3 the received schedule of the factor ω is indicated.

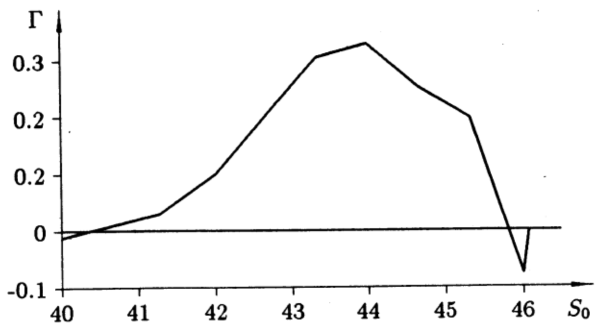

Figure 2. Computations of the factor Γ

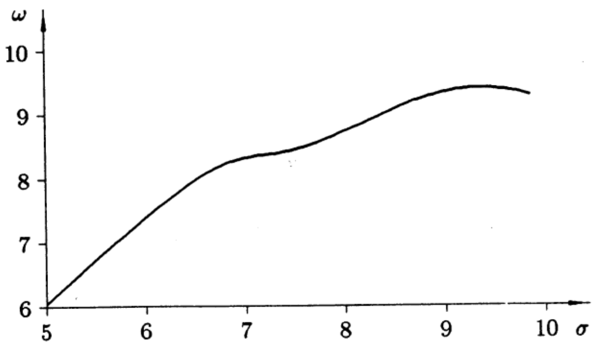

Figure 3. Computations of the factor ω

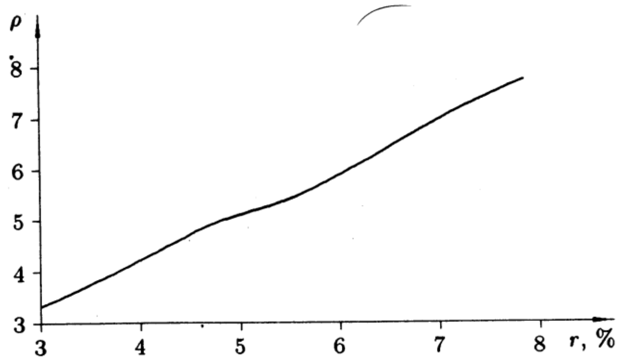

Figure 4. Computations of the factor ρ

Example 3. Computation of the factor ρ. The Initial price of the basic share – 40, the exercise price – 42, the expiration time – 0.5 year, the minimum annual riscless interest rate – 2%, maximum annual riscless interest rate – 8%, volatility of the underlying interest – 5%, a number of simulated trajectories – 10^6. In Figure 4 the received schedule of the factor ρ is indicated.

In the table below the values of speed up, received as a result of computations of the factors δ, Γ, ω, ρ are demonstrated. The speed up here is the ratio of calculation time for one processor to calculation time for i ($i = 2, \ldots, 8$) processors.

factor	Number of processors						
	2	3	4	5	6	7	8
δ	2.02	3.06	3.85	4.70	5.53	7.23	8.07
Γ	2.02	3.06	4.37	5.85	6.12	8.72	9.43
ω	1.91	3.02	4.05	5.31	6.06	7.12	8.26
ρ	2.00	3.02	4.23	5.21	6.13	7.14	8.14

It is easy to understand that for considered parallel programs at the increase of number of simulated trajectories only expenditures connected with direct simulating of trajectories increase. Therefore at uniform loading of all processors it is possible to expect the speed up will come nearer to linear function of number of processors. The adduced numerical calculations also confirm it.

References

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