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A mathematical model of determining the stress field and dilatant zones by geodetic data*

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An inverse problem to determine the location of acting equivalent forces to geodetic data on the surface of the Earth's crust is considered within the framework of the quasistatic model for the elastic isotropic half-space with the sources of point type. It uses the data of monitoring for deformations of areas of the Earth's crust surface. The determination of zones of possible dilatancy in the area of the source influence and search for the relation between such zones and the location and behavior of the anomalies – precursors of various nature – are discussed. Some results of numerical modeling are presented.

The process of accumulation of stresses and deformations is crucial in the preparation of earthquakes. It can be assumed in the multidisciplinary problem of earthquake prediction [1] that the appearance of anomalies – precursors of various nature (forces of gravity, ground water level, geochemical precursors, etc.) – is caused by this process in the area covering the epicenter of a future earthquake.

Now the data of monitoring of deformations of the Earth's crust are one of the main constituents of earthquake prediction. Geodetic measurements [2] are the main method to obtain these data and a basis for the prediction of the locations of earthquake preparation. This trend in earthquake prediction has received much attention in the last 20 years in China [3–5]. Both the data of the regional networks of monitoring (geodetic surveys) in large areas and observational data of the movements in the existing fractures [3, 4] are used. It is known that the data of deformations of the Earth's crust played an important role in the successful prediction of the 1975 Haicheng earthquake ($M = 7.3$). Confinement of the epicenter of a future earthquake to the area with high gradients of deformation observed many times [3] is an important rule.

It is necessary to use for earthquake prediction the quantitative model of deformation and stress of the Earth's crust. As the preparation process of strong earthquakes and accumulation of deformations and stresses is very slow (it takes years, tens of years, and more), it is natural to consider the

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deformation process as a quasistatic one. That is, the Earth's crust is in equilibrium in the given region at each moment of time, and this problem can be seen as a static problem of deformation of the elastic medium. In Russia, the static model describing earthquake preparation was used by I.P. Dobrovolsky [6]. Precursor deformations are observed at distances of hundreds of kilometers from the future epicenter [3–5]. In accordance with the Saint-Venant principle, replacement of loading in the place of its application by an equivalent system of forces gives a slightly different field of deformations at large distances. By using equivalent system of forces (fictition) it is possible to determine the stress field in the Earth's crust and then to find the dilatant zones.

Direct problem. Let the half-space $z \geq 0$ (the Cartesian coordinates $x = x_1$, $y = x_2$, $z = x_3$) with the free boundary $z = 0$ be filled by a homogeneous elastic isotropic medium with the shear modulus μ and the Poisson coefficient ν.

Let $\vec{u}^l(x,y,z) \equiv u^l\vec{i}_1 + v^l\vec{i}_2 + w^l\vec{i}_3$ be the displacement vector in the static problem on equilibrium of such medium due to a unit force applied at a point and acting along the positive direction of the axis x_l ($\vec{i}_l$ is the unit vector along the axis x_l) and applied at a point $M'(x',y',h')$. The explicit expression for $\vec{u}^l(u^l,v^l,w^l)$ and for the components of deformations e^l_{ik} and stresses σ^l_{ik} at any point x, y, z in terms of the algebraic functions of the quantities $x - x'$, $y - y'$, z, h', μ, ν were obtained by R. Mindlin [7].

Let us consider a more general point source $J_1(M')$, which is a combination of a simple force with the components F_l, three double forces without the moment and six double forces with the moment (source of the 1st type). The displacement vector $\vec{u}$ generated by such source has the form

$$\vec{u} \equiv \sum_{l=1}^{3} F_l\vec{u}^l + \sum_{l=1}^{3}\sum_{m=1}^{3} M_{lm}\frac{\partial \vec{u}^l}{\partial x_m} = u^l\vec{i}_1 + v^l\vec{i}_2 + w^l\vec{i}_3,$$

where the constants M_{lm} are the parameters of the double forces. The source of the second type J_2 is a combination of two sources of the first type $J_1(M')$ and $J_1(M'')$. In both cases, the solution to the direct problem of determination of u, v, w, e_{ik}, σ_{ik} (displacements, deformations, stresses) at the given parameters of the source can be computed by explicit formulae at any point x, y, z and, in particular, at the boundary of observations $z = 0$.

Inverse problem consists in determining the set of parameters x', y', h', F_l, M_{lm} in the case of J_1 and two such sets in the case of J_2 using the data of deformations on some set of points $M_i(x_i, y_i)$ in the given domain $\mathcal{D}$ at the boundary $z = 0$ ($i = 1, 2, \ldots, N$). The following characteristics of the deformation field which can be measured in the monitoring of earthquake precursors should be chosen as the data:

1. The vertical component of displacements w or the quantity $w + c$, where c is an unknown constant (it can be assumed that $c = -w_0$, where w_0 is the value of w at some fixed point M_0 in the domain $\mathcal{D}$ or outside of it). Let t_1 and $t_2 > t_1$ be two moments of time before an earthquake, $H_i(t_j)$ and $H_0(t_j)$ – absolute heights of the points M_i, M_0 at the epoch t_j. As a result of repeated leveling, the quantities $h_i(t_j) = H_i(t_j) - H_0(t_j)$ (relative heights of the points M_i at the epoch t_j) are measured and either the change in heights of the points M_i relative to M_0, $\Delta h_i = h_i(t_2) - h_i(t_1)$ (in mm), during the period $\Delta t = t_2 - t_1$ or the velocity of vertical displacements, $V_i = \Delta h_i/\Delta t$ (in mm/year), is determined. An example of such data for V_i in the Datong region (China) is given in [4]. Then the values $w_i = w(x_i, y_i, 0)$ are determined by the formula $w_i = \Delta h_i + \Delta h_0 - V_{\text{bgr}}\Delta t$, where $\Delta h_0 = H_0(t_2) - H_0(t_1)$, V_{bgr} is the velocity of vertical background motions in the domain $\mathcal{D}$ as a result of global tectonic processes. If one or both quantities Δh_0, V_{bgr} are unknown, we shall know w_i with an accuracy of the additive constant c. The value V_i during earthquake preparation is of the order of several mm/year.

2. Relative lengthening (deformations) of the linear horizontal elements $e_{xx} = \dfrac{\partial u}{\partial x}$ and $e_{yy} = \dfrac{\partial v}{\partial y}$ during the period Δt measured by the baseline method.

3. Tilts $\dfrac{\partial w}{\partial x}$ and $\dfrac{\partial w}{\partial y}$ of the horizontal elements.

4. Relative volume expansion $\theta = \operatorname{div} \vec{u}$.

It is important that the errors δX in the determination of these data X were at least 3–4 times smaller than the observed values X. As the values X, as a rule, increase with increasing Δt, the interval Δt between the moments of monitoring (observation) is bounded from below by the value δX. At the same time, from the point of view of the purposes of monitoring, the interval Δt must be sufficiently small. These contradictory requirements can be weakened, if more accurate measurement devices are applied.

The sets of points M_i for the data of the type 1–4 can coincide or not coincide; different combinations of these data are possible.

The coordinates of the sources and their parameters (the forces and the moments of forces) are found as a result of solving the inverse problem by minimizing the functional

$$\mathbb{G}(\vec{p}_1, \ldots, \vec{p}_m; \vec{\xi}_1, \ldots, \vec{\xi}_m) = \sum_{k=1}^{K} \left\| \left(\sum_{j=1}^{m} \mathbb{A}(\vec{\xi}_j, \vec{r}_k) \cdot \vec{p}_j \right) - \vec{q}_k \right\|^2 = \min, \quad (1)$$

where $m = i$ for the source of the type J_i; $\vec{\xi}_1, \ldots, \vec{\xi}_m$ are the sought for coordinates of the sources; $\vec{p}_1, \ldots, \vec{p}_m$ are the sought for forces and the moments of forces of the sources; K is the number of observation points; $\vec{r}_k$ are the coordinates of the k-th observation point; $\vec{q}_k$ is the vector of values of the experimental quantities (displacements, deformations) of dimension n given at the point $\vec{r}_k$; $\mathbb{A}$ is the $12 \times n$ matrix, the elements of which are expressed in terms of the quantities u_i^l and $\partial u_i^l / \partial x_\nu$.

The functional (1) represents the quadratic form relative to the whole set of the components of the vectors $\vec{p}_j$. This makes it possible to eliminate $\vec{p}_j$ from (1). As a result, we come to the minimization problem $\tilde{\mathbb{G}}(\vec{\xi}_1, \ldots, \vec{\xi}_m) = \min$. The nonlinearity of this problem in comparison to problem (1) increases, but the number of unknowns decreases. The possibility to compute partial derivatives in our case in the explicit form allows an effective use of the Newtonian methods with the help of the gradients and restrictions on the unknown values. The numerical experiments have shown that the solution to the problem with a smaller number of variables is economical and has an increased stability in comparison to the initial problem (1).

If there are data of monitoring of deformations for a sequence of the moments of time $t_1, t_2, \ldots, t_n$, the inverse problem is also solved in the regime of monitoring, the area of the source location (origin of a future earthquake) being determined more and more exactly step-by-step.

As the data of monitoring for the earthquakes which have already occurred are available, for example, in China [3], then we can test the effectiveness of this approach to long-range earthquake prediction by solving the inverse problem using these input data and comparing the results obtained with those which are known. These data can also be used to improve the quantitative model (to choose an appropriate combination of forces in the source giving an adequate picture of deformations in the direct problem, i. e., which is close to the picture observed).

Dilatant zones. After solving the inverse problem, we can compute the components of deformations and stresses at an arbitrary point of the volume containing the sources using the explicit formulae mentioned above. For the calculated stress field $\{\sigma_{ij}\}$ the spatial extent of the dilatant zone is found from the following condition:

$$\Phi_\sigma(\sigma_{ij}, \alpha, Y) = \tau - \alpha(p + \rho g z) - Y \geq 0,$$

where ρ is the density of the rock, g is the gravitational acceleration, z is the depth, α is the coefficient of internal friction, Y is the cohesion of the rock, $p = -\frac{1}{3}\sigma_{ij}\delta_{ij}$ is the pressure, τ is the shear stress intensity

$$\tau = \frac{\sqrt{3}}{2}\left[(\sigma_{11}-\sigma_{22})^2 + (\sigma_{22}-\sigma_{33})^2 + (\sigma_{33}-\sigma_{11})^2 + 6(\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2)\right]^{1/2}.$$

The influence of the dilatancy velocity Λ on the dynamics of the dilatant zone can be taken into account using the dilatancy relation between the volume plastic deformation $de^p = e^p_{ij}\delta_{ij}$ and the intensity of the plastic shear $d\gamma^p$:

$$\Phi_e(de^p_{ij}, \Lambda) = de^p - \frac{2}{\sqrt{3}}\Lambda \cdot d\gamma^p = 0,$$

where

$$d\gamma^p = \frac{1}{\sqrt{2}}\Big\{(de^p_{11} - de^p_{22})^2 + (de^p_{22} - de^p_{33})^2 + (de^p_{33} - de^p_{11})^2 + 6\left[(de^p_{12})^2 + (de^p_{23})^2 + (de^p_{31})^2\right]\Big\}^{1/2}.$$

The increments of the plastic deformation de^p_{ij} are related with the stressed state σ_{ij} in the following way:

$$de^p_{ij} = \left[\sigma_{ij} + \frac{2}{3}\Lambda\alpha H\delta_{ij} + \left(1 + \frac{2}{3}\Lambda\alpha\right)p\delta_{ij}\right] d\lambda,$$

where $d\lambda$ is the scalar function ($d\lambda > 0$ at $\Phi_\sigma = 0$ and the active loading, and $d\lambda = 0$ if there is the discharge $\Phi_\sigma < 0$ or $\Phi_\sigma = 0$, but the loading is neutral).

It will be considered that $\alpha = \alpha(\chi)$ and $\Lambda = \Lambda(\chi)$ are the functions of the parameter of state $\chi = e^p/e^{p^*}$, where p^* is such a deformation, at which the medium does not dilatate ($\Lambda(\chi = 1) = 0$); $e^{p^*} = f(p)$ is some function of pressure.

We assume that $\alpha = \alpha_0 + (\alpha_* - \alpha_0)\chi$, $\Lambda = b - \sqrt{(1 + 2b\alpha_0)} - (\alpha_* + 2b)\alpha$, where $\alpha_0 = \alpha(0)$, $\alpha_* = \alpha(1)$, $b = \sqrt{1 - \alpha_*}$.

Consequently, the dilatant zones can be found. Projecting them onto the daily surface $z = 0$, we can search for a connection between the location of dilatant zones and the distribution of the anomalies – precursors of different physical nature – using the data for the earthquakes which occurred. It can be assumed that all these anomalies are determined by the presence of dilatant zones. These zones can migrate with time in the area of the source influence and their behavior can turn out to be an important feature for prediction.

Numerical example. The geodetic data (the vertical components of displacements) for the Datong region before the large earthquake in October, 1989 were measured at the stations presented in Figure 1. These values range from approximately -30 to $+10$ mm.

The parameters (coordinates and force components) for two fictitious sources were found by solving numerically the inverse problem (1) using the least-squares method for the minimization of the error functional $\mathbb{G}$. The location of these sources is shown in Figure 2 (the black squares 1 and 2).

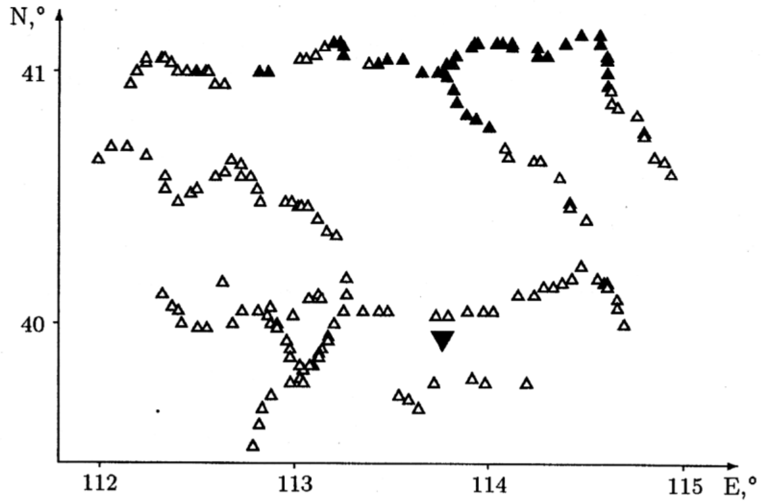

Figure 1. Stations and main epicenter locations: ▼ is the main epicenter, △ are stations with a negative displacement, and ▲ are stations with a positive one

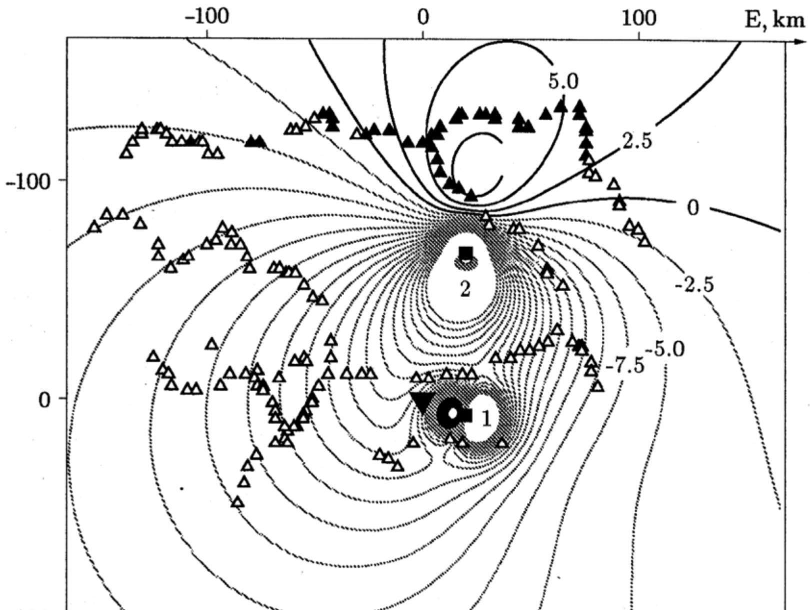

Figure 2. The isolines of the vertical component of the displacements field for the fictitious sources found (the epicenter is located at the center of the shown area): ■ are recovered (fiction) sources 1, 2; —— are negative levels (step 2.5 mm); **——** are positive levels (step 2.5 mm).

Solving the direct problem for the found fictitious sources, we can calculate the fields of displacements $\vec{u}$ and stresses $\{\sigma_{ij}\}$. For example, the vertical component of the displacements field on the daily surface is shown in Figure 2. The isoline picture for this field agrees well with the given geodetic data: the stations with negative (positive) values of vertical displacements are mostly located in areas with negative (positive) isoline levels.

Using the stress field $\{\sigma_{ij}\}$ found we can construct dilatant zones.

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