Bull. Nov. Comp. Center, Num. Model. in Atmosph., etc., 16 (2017), 13–19

Non-existence of the global solution of initial boundary value problem for the incompressible two-velocity medium equation

Kh.Kh. Imomnazarov, Sh.Kh. Imomnazarov, M.M. Mamatqulov

Abstract. An initial boundary value problem for systems of viscous two-fluid media with equilibrium of pressure phases is considered. Using the test function method proposed by S.I. Pohozhaev and E. Mitidieri, the effect of boundary and initial conditions on the appearance, time and rate of destruction of solutions of this problem is investigated.

Keywords: Two-velocity hydrodynamics, initial boundary value problem, destruction.

1. Introduction

The fundamental problem of modern hydrodynamics, inherited from classical fluid mechanics, remains the study of the dynamics and interaction of vortex structures. According to the figurative expression in Saffman's book [1], vortexes are "muscles and veins of hydrodynamics". Without idea of the mechanisms that determine their behavior, it is impossible to achieve progress in constructing dynamic models of turbulence, in solving stability, and control flow problems, in improving the wing aerodynamics and in developing many other sections of the hydrodynamic theory and technical applications.

One of the most important issues in the theory of nonlinear differential equations is the question of uniqueness, existence, and destruction of solutions. If there is a local solvability, a smooth solution to the evolution equations may not exist on the entire time axis and may collapse in finite time. Theoretical studies of the destruction phenomenon of the Cauchy problems for hydrodynamics models have begun since the last century [2, 3]. However, from a practical point of view, the formulation of a problem in an unbounded domain is complicated, and in the numerical simulation we have to consider the initial boundary value problem by adding the boundary conditions [4]. In this paper, we raise the question of the local solvability for systems of equations of a two-fluid medium with the equilibrium pressure phases and demonstrate the ability of the method offered by S.I. Pohozaev and E. Mitidieri to obtain sufficient conditions of the global unsolvability of possible supplementary problems for systems of the form [5–7]:

$$\operatorname{div}(\rho \boldsymbol{v}) = 0, \qquad \operatorname{div}(\tilde\rho \tilde{\boldsymbol{v}}) = 0, \tag{1}$$

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}, \nabla)\boldsymbol{v} = -\frac{\nabla p}{\bar\rho} + \nu\,\Delta \boldsymbol{v} + \frac{\tilde\rho}{2\bar\rho}\nabla(\tilde{\boldsymbol{v}} - \boldsymbol{v})^2, \tag{2}$$

$$\frac{\partial \tilde{\boldsymbol{v}}}{\partial t} + (\tilde{\boldsymbol{v}}, \nabla)\tilde{\boldsymbol{v}} = -\frac{\nabla p}{\bar\rho} + \tilde\nu\,\Delta \tilde{\boldsymbol{v}} - \frac{\rho}{2\bar\rho}\nabla(\tilde{\boldsymbol{v}} - \boldsymbol{v})^2, \tag{3}$$

where $\tilde{\boldsymbol{v}}$ and $\boldsymbol{v}$ are the velocity vectors of the subsystems of the components of the two-velocity continuum with the respective partial densities $\tilde\rho$ and ρ, ν, and $\tilde\nu$ are the respective kinematic viscosities, $\bar\rho = \tilde\rho + \rho$ is the total density of two-velocity continuum; $p = p(\bar\rho, (\tilde{\boldsymbol{v}} - \boldsymbol{v})^2)$ is the equation of state of the two-velocity continuum.

Systems of this type arise when describing the motion of mutual penetration of a less viscous fluid through a more viscous medium, as a kind of filtering process [5, 6]. Or by analogy with the Navier–Stokes equations, this model can be called a two-velocity Navier–Stokes equation or the two-velocity hydrodynamics.

To study the destruction of solutions, we use the method developed by S.I. Pohozaev, E. Mitidieri and V.A. Galaktionov and called the method of test functions, or the method of nonlinear capacity. For the detailed acquaintance with the possibilities of this method we refer to [8–13]. In particular, the use of the test functions of a special form in the hydrodynamics is discussed in [14–17].

2. Destruction of the solution to a system of the two-velocity hydrodynamics equations with one pressure

For the system of equations (1)–(3) in the cylindrical domain $\Omega = [0, R] \times [0, 2\pi] \times [0, L]$, we investigate the following initial boundary value problem with the following initial conditions

$$\tilde{\boldsymbol{v}}|_{t=0} = \tilde{\boldsymbol{v}}_0(\boldsymbol{x}), \quad \boldsymbol{v}|_{t=0} = \boldsymbol{v}_0(\boldsymbol{x}), \tag{4}$$

and the boundary conditions

$$\begin{gathered}\tilde v_z(r,\varphi,0) = 0, \quad \tilde v_r(R,\varphi,z) = \tilde v_r(0,\varphi,z) = 0,\\ -\int_0^{2\pi}\int_0^R \Big(\tilde v_z(r,\varphi,L) + L\frac{\partial \tilde v_z}{\partial z}(r,\varphi,0)\Big)d\varphi\,dr + \\ \int_0^{2\pi}\int_0^L (z-L)R\frac{\partial \tilde v_z}{\partial r}(R,\varphi,z)\,d\varphi\,dz = \tilde h(t),\end{gathered} \tag{5}$$

$$v_z(r,\varphi,0)=0,\quad v_r(R,\varphi,z)=v_r(0,\varphi,z)=0,$$

$$-\int_0^{2\pi}\int_0^R\Big(v_z(r,\varphi,L)+L\frac{\partial v_z}{\partial z}(r,\varphi,0)\Big)d\varphi\,dr+\int_0^{2\pi}\int_0^L(z-L)R\frac{\partial v_z}{\partial r}(R,\varphi,z)\,d\varphi\,dz=h(t),\tag{6}$$

where $h(t),\tilde h(t)\in C[0,\infty)$.

Let us choose the test function

$$\boldsymbol{u}=(0,0,z-L),\quad z\in[0,L].$$

By virtue of boundary conditions (5), (6) in the class $\boldsymbol{v}(\boldsymbol{x},t),\tilde{\boldsymbol{v}}(\boldsymbol{x},t)\in C^1((0,T];C^2(\bar\Omega)\times C^2(\bar\Omega)\times C^2(\bar\Omega))$, we have the following relations:

$$\begin{aligned}\int_\Omega\Delta\boldsymbol{v}\,\boldsymbol{u}\,d\boldsymbol{x}&=\int_\Omega\Big(\frac{1}{r}\frac{\partial}{\partial r}\Big(r\frac{\partial v_z}{\partial r}\Big)+\frac{1}{r^2}\frac{\partial^2 v_z}{\partial\varphi^2}+\frac{\partial^2 v_z}{\partial z^2}\Big)(z-L)r\,dr\,d\varphi\,dz\\&=\int_0^{2\pi}\int_0^L r\frac{\partial v_z}{\partial r}\Big|_0^R(z-L)\,d\varphi\,dz+\\&\quad\int_0^R\int_0^L\frac{1}{r}\frac{\partial v_z}{\partial\varphi}\Big|_0^{2\pi}(z-L)\,dr\,dz+\\&\quad\int_0^{2\pi}\int_0^R\Big((z-L)\frac{\partial v_z}{\partial z}-v_z\Big)\Big|_0^L d\varphi\,dr\\&=\int_0^{2\pi}\int_0^L R\frac{\partial v_z}{\partial r}(R,\varphi,z)(z-L)\,d\varphi\,dz-\\&\quad\int_0^{2\pi}\int_0^R\Big(v_z(r,\varphi,L)+L\frac{\partial v_z}{\partial z}(r,\varphi,0)\Big)d\varphi\,dr=h(t),\end{aligned}\tag{7}$$

$$\begin{aligned}\int_\Omega(\boldsymbol{v},\nabla)\boldsymbol{v}\,\boldsymbol{u}\,d\boldsymbol{x}&=\int_\Omega\Big(v_r\frac{\partial v_z}{\partial r}+\frac{1}{r}v_\varphi\frac{\partial v_z}{\partial\varphi}+v_z\frac{\partial v_z}{\partial z}\Big)(z-L)\,r\,dr\,d\varphi\,dz\\&=\int_\Omega\Big(\frac{\partial}{\partial r}(rv_rv_z)+\frac{\partial}{\partial\varphi}(v_\varphi v_z)+rv_z\frac{\partial v_z}{\partial z}\Big)(z-L)\,dr\,d\varphi\,dz-\\&\quad\int_\Omega\Big(\frac{\partial(rv_r)}{\partial r}+\frac{\partial v_\varphi}{\partial\varphi}\Big)v_z(z-L)\,dr\,d\varphi\,dz\\&=\int_0^{2\pi}\int_0^L rv_rv_z\Big|_0^R(z-L)\,d\varphi\,dz+\int_0^R\int_0^L v_\varphi v_z\Big|_0^{2\pi}(z-L)\,dr\,dz+\\&\quad\int_\Omega\Big(\frac{\partial v_z}{\partial z}-\frac{1}{r}\Big(\frac{\partial(rv_r)}{\partial r}+\frac{\partial v_\varphi}{\partial\varphi}\Big)\Big)v_z(z-L)r\,dr\,d\varphi\,dz\\&=\int_0^{2\pi}\int_0^R rv_z^2(z-L)\Big|_0^L dr\,d\varphi-\int_\Omega v_z^2r\,dr\,d\varphi\,dz\\&=-\int_\Omega v_z^2r\,dr\,d\varphi\,dz.\end{aligned}\tag{8}$$

Here we used the equality $\operatorname{div} \boldsymbol{v} = 0$ implying

$$\frac{\partial(rv_r)}{\partial r} + \frac{\partial v_\varphi}{\partial \varphi} = -r\frac{\partial v_z}{\partial z}.$$

We multiply both sides of equation (2) by ρ, equation (3) by $\tilde{\rho}$ and, as a result, obtain the law of conservation of momentum

$$\frac{\partial(\rho \boldsymbol{v} + \tilde{\rho}\tilde{\boldsymbol{v}})}{\partial t} + \rho\,(\boldsymbol{v},\nabla)\boldsymbol{v} + \tilde{\rho}\,(\tilde{\boldsymbol{v}},\nabla)\tilde{\boldsymbol{v}} = -\nabla p + \Delta(\rho\nu\boldsymbol{v} + \tilde{\rho}\tilde{\nu}\tilde{\boldsymbol{v}}). \tag{9}$$

Denote $P = p + \frac{\rho}{2}(\tilde{\boldsymbol{v}} - \boldsymbol{v})^2$. In terms of $\tilde{\boldsymbol{v}}$ and P, equation (3) has the form

$$\frac{\partial\tilde{\boldsymbol{v}}}{\partial t} + (\tilde{\boldsymbol{v}},\nabla)\tilde{\boldsymbol{v}} = -\frac{\nabla P}{\bar{\rho}} + \tilde{\nu}\,\Delta\tilde{\boldsymbol{v}}. \tag{10}$$

Taking into account the inequality

$$-\int_\Omega \nabla p\,\boldsymbol{u}\,d\boldsymbol{x} = \int_\Omega \frac{\partial p}{\partial z}(L-z)\,d\boldsymbol{x} = \int_\Omega p\,d\boldsymbol{x} + \int_0^R\int_0^{2\pi}(L-z)p\big|_0^L\,r\,dr\,d\varphi$$
$$\geq -\int_0^R\int_0^{2\pi} rL\,p(r,\varphi,0)\,dr\,d\varphi \equiv g(t),$$

where $\int_\Omega p\,d\boldsymbol{x} \geq 0$, $g(t) \in C[0,\infty)$, multiplying both sides of equations (9) and (10) by the test function, and using relations (7), (8), we obtain

$$\frac{d(\rho J + \tilde{\rho}\tilde{J})}{dt} \geq \rho\int_\Omega v_z^2\,d\boldsymbol{x} + \tilde{\rho}\int_\Omega \tilde{v}_z^2\,d\boldsymbol{x} + f_1(t), \tag{11}$$

$$\frac{d\tilde{J}}{dt} = \int_\Omega \tilde{v}_z^2\,d\boldsymbol{x} + f_2(t), \tag{12}$$

where

$$J = \int_\Omega (\boldsymbol{v}\,\boldsymbol{u})\,d\boldsymbol{x}, \qquad \tilde{J} = \int_\Omega (\tilde{\boldsymbol{v}}\,\boldsymbol{u})\,d\boldsymbol{x},$$
$$f_1(t) = g(t) + \rho\nu h(t) + \tilde{\rho}\tilde{\nu}\tilde{h}(t), \qquad f_2(t) = G(t) + \tilde{\nu}\tilde{h}(t),$$
$$G(t) = -\frac{1}{\bar{\rho}}\int_0^R\int_0^{2\pi} rL\,P(r,\varphi,0)\,dr\,d\varphi.$$

Further, the estimation proved in [16] for $\lambda \in (0,3)$ is written down as

$$\frac{(3-\lambda)^2}{L^{(6-\lambda)}}J^2 \leq 2\int_\Omega \tilde{v}_z^2\,d\boldsymbol{x}.$$

From (11), (12) one can obtain a system of differential inequalities

$$\frac{dJ}{dt} \geq k^2J^2 + f(t), \qquad \frac{d\tilde{J}}{dt} \geq k^2\tilde{J}^2 + f_2(t),$$

where

$$f(t)=\frac{f_1(t)-\tilde{\rho}f_2(t)}{\rho},\qquad k^2=\frac{(3-\lambda)^2}{2L^{(6-\lambda)}}.$$

Thus, according to [11, 16], we have established the validity of the following

Theorem. *The classical solution of problem* (1)–(6) *does not globally exist if the following conditions are fulfilled*:

1) *let* $f(t),\ f_2(t)\geq 0$, *then under the conditions* $J(0),\ \tilde{J}(0)>0$, *the lower estimations are valid*:

$$J(t)\geq\frac{J(0)}{1-J(0)k^2t},\qquad J(0)=\int_\Omega \boldsymbol{v}_0(\boldsymbol{x})\boldsymbol{u}\,d\boldsymbol{x},$$
$$\tilde{J}(t)\geq\frac{\tilde{J}(0)}{1-\tilde{J}(0)k^2t},\qquad \tilde{J}(0)=\int_\Omega \tilde{\boldsymbol{v}}_0(\boldsymbol{x})\boldsymbol{u}\,d\boldsymbol{x},$$

and there is an estimation for the time of destruction:

$$T_\infty\leq\frac{1}{k^2}\min\Big(\frac{1}{J(0)},\frac{1}{\tilde{J}(0)}\Big);$$

2) *let* $f(t)\geq a^2>0,\ f_2(t)\geq\tilde{a}^2>0$, *then*

$$J(t)\geq\frac{a}{k}\tan(akt+c_0),\qquad c_0=\arctan\Big(\frac{k\,J(0)}{a}\Big),$$
$$\tilde{J}(t)\geq\frac{\tilde{a}}{k}\tan(\tilde{a}kt+\tilde{c}_0),\qquad \tilde{c}_0=\arctan\Big(\frac{k\tilde{J}(0)}{\tilde{a}}\Big),$$

and the estimation for the time of destruction is

$$T_\infty\leq\frac{1}{k}\min\Big(\frac{\pi/2-c_0}{a},\frac{\pi/2-\tilde{c}_0}{\tilde{a}}\Big);$$

3) *let* $f(t)\geq -a^2,\ f_2(t)\geq-\tilde{a}^2$, *then under the conditions* $kJ(0)>|a|$, $k\tilde{J}(0)>|\tilde{a}|$ *the lower estimations are valid*:

$$J(t)\geq\frac{a}{k}\,\frac{1+c_0e^{2akt}}{1-c_0e^{2akt}},\qquad c_0=\frac{kJ(0)-a}{kJ(0)+a},$$
$$\tilde{J}(t)\geq\frac{\tilde{a}}{k}\,\frac{1+\tilde{c}_0e^{2\tilde{a}kt}}{1-\tilde{c}_0e^{2\tilde{a}kt}},\qquad \tilde{c}_0=\frac{k\tilde{J}(0)-\tilde{a}}{k\tilde{J}(0)+\tilde{a}},$$

and the estimation for the time of destruction is

$$T_\infty\leq\frac{1}{2\,k}\min\Big(\frac{1}{a}\ln\frac{kJ(0)+a}{kJ(0)-a},\frac{1}{\tilde{a}}\ln\frac{k\tilde{J}(0)+\tilde{a}}{k\tilde{J}(0)-\tilde{a}}\Big).$$

References

[1] Saffman P.G. Vortex Dynamics. — Cambridge: Cambridge Univ. Press, 1992.

[2] Constantin A., Escher J. Wave breaking for nonlinear nonlocal shallow water equations // Acta Math. — 1998. — Vol. 181, No. 2. — P. 229–243.

[3] Zhou Y. Wave breaking for a shallow water equation // Nonlinear Anal. Theory, Methods Appl. — 2004. — Vol. 57, No. 1. — P. 137–152.

[4] Bulatov O.V., Elizarova T.G. Regularized shallow water equations and an efficient method for numerical simulation of shallow water flows // Comput. Math. and Math. Phys. — 2011. — Vol. 51, No. 1. — P. 170–184 (In Russian).

[5] Dorovsky V.N., Perepechko Yu.V. Phenomenological description of two-velocity media with relaxing shear stresses // PMTF. — 1992. — No. 3. — P. 94–105 (In Russian).

[6] Dorovsky V.N., Perepechko Yu.V. The theory of partial melting // Geol. Geofiz. — 1989. — No. 9. — P. 56–64 (In Russian).

[7] Baishemirov Zh., Tang J.-G., Imomnazarov Kh., Mamatqulov M. Solving the problem of two viscous incompressible fluid media in the case of constant phase saturations // Open Eng. — 2016. — Vol. 6. — P. 742–745.

[8] Pokhozhaev S.I. Riemann quasi-invariants // Sb. Math. — 2011. — Vol. 202, No. 6. — P. 887–907 (In Russian).

[9] Pokhozhaev S.I. Weighted identities for the solutions of generalized Korteweg-de Vries equations // Math. Notes. — 2011. — Vol. 89, No. 3. — P. 382–396.

[10] Pokhozaev S.I. On the absence of global solutions of the Korteweg-de Vries equation // J. Math. Sci. — 2013. — Vol. 190, No. 1. — P. 147–156.

[11] Mitidieri E., Pokhozhaev S.I. A priori estimates and blow-up of solutions to nonlinear partial differential equations and inequalities // Proc. Steklov Inst. Math. — 2001. — Vol. 234. — P. 1–362.

[12] Galaktionov V.A., Mitidieri E., Pokhozhaev S.I. On global solutions and blow-up for Kuramoto–Sivashinsky–type models, and well-posed Burnett equations // Nonlinear Anal. — 2009. — Vol. 70, No. 8. — P. 2930–2952.

[13] Pokhozhaev S.I. On the nonexistence of global solutions for some initial-boundary value problems for the Korteweg-de Vries equation // Differ. Eq. — 2011. — Vol. 47, No. 4. — P. 488–493 (In Russian).

[14] Yushkov E.V. On destruction of a solution in systems of the Korteweg-de Vries type // TMF. — 2012. — Vol. 173, No. 2. — P. 197–206 (In Russian).

[15] Dobrokhotov S.Yu., Shafarevich A.I. On the behavior on infinity of the velocity field of incompressible fluid // Fluid-flow Mechanics. — 1996. — Vol. 31, No. 4. — P. 38–42 (In Russian).

[16] Yushkov E.V. On the blow-up of solutions of equations of hydrodynamic type under special boundary conditions // Differ. Eq. — 2012. — Vol. 48, No. 9. — P. 1234–1239 (In Russian).

[17] Lai S., Wu Y. Global solutions and blow-up phenomena to a shallow water equation // J. Differ. Eq. — 2010. — Vol. 249, No. 3. — P. 693–706.