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Restoration of tsunami incident by remote measurements of the surface water oscillations*

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Abstract. The inversion problem to infer the initial sea perturbation is considered as a usual ill-posed problem of the hydrodynamic inversion of tsunami tide-gage records. The ill-posed inverse reconstruction problem is regularized by means of the least square inversion using the truncated SVD approach. Numerical experiments are presented for the model bottom relief having some basic morphological features typical of the island arc regions.

1. Introduction

During the past few years, the tsunami events that occurred in the Pacific and the Indian Oceans had caused to turn to on the inverse tsunami problem. In the present paper, we use an inverse method, already described in the previous publication [4]. We use the inverse method, for which the data space consists of the water level oscillations given by a set of receivers, and the space of models is represented by a linear combination of the given basic functions.

Mathematically, this problem is formulated as inverse problem of mathematical physics for restoration of the initial water displacement in the source area by the water level oscillations observed at a number of points distributed in the ocean. The possibility to obtain a unique solution exists (see [1]) only when the function of a source allows factorization, i.e., the dependence on time and spatial variables is separated. We assume the time dependence to be described by the Heavyside function. The forward problem, i.e., the calculation of synthetic tide-gage records from the initial water elevation field is based on a linear shallow-water system of differential equations in the rectangular coordinates. This system is approximated by the explicit-implicit finite difference scheme on a uniform rectangular grid, so that a system of the linear algebraic equations is obtained. The ill-posed inverse restoration problem is regularized by means of the least square inversion using the truncated SVD approach. In this method, the inverse operator is regularized with the help of its restriction on the subspace spanned on a finite sample of the first right singular vectors (see [5]). The so-called r-solution (see [2]) is a result of the numerical process. Its quality depends

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on the number of receivers and their disposition. The quality of the solution obtained is evaluated as relative errors (in L_2-norm) in restoration of the source function. The results are fairly satisfactory if the receivers have a good azimuthal coverage with respect to source area.

2. Statement of the problem

Let us neglect the curvature of the Earth. Let z-axis be directed downwards with depth. The plane $\{z = 0\}$ corresponds to an undisturbed water surface. Since the tsunami wave in the ocean is a long gravitational wave with a small amplitude, its propagation can be described by the shallow water equation:

$$W_{tt} = \operatorname{div}(h(x, y) \operatorname{grad} W) + f_{tt}(t, x, y) \tag{1}$$

with the initial conditions

$$W|_{t=0} = 0, \quad W_t|_{t=0} = 0 \tag{2}$$

and the boundary conditions

$$\frac{\partial W}{\partial n}\Big|_{\Gamma} = 0, \tag{3}$$

where $W(x, y, t)$ is water elevation above the undisturbed state, $h(x, y)$ is the depth of the ocean, $f(x, y, t)$ describes the movement of the bottom in the tsunami area.

In this paper, we consider a depth as function $h(x, y)$. Then the velocity of the tsunami wave propagation is also described as $c(x, y) = \sqrt{gh}$. We assume that the function $f(x, y, t)$ describing the movement of the bottom in the tsunami area, can be factorized in the following way: $f(x, y, t) = \varepsilon(t) \cdot \varphi(x, y)$, where $\varepsilon(t)$ is the Heavyside function.

We solve the problem in the domain Φ with piecewise-linear inner and outer boundaries. Let us assume that Φ is a subdomain of the rectangle $\Pi = \{x_0 \le x \le x_M;\ y_0 \le y \le y_N\}$. The uniform rectangular grid is defined in Π, but in fact, part of grid points correspond to dry land. So, the difference scheme employs only grid points disposed in Φ. The letter Ω denotes a subdominant of Φ, which is assumed to be a support of the tsunami center (the target object domain).

Our problem now consists in recovering the bottom movement $\varphi(x, y)$ in the domain Ω, when the given data are the water elevation $W_0(x, y, t)$ in a certain set of the receivers $\{M_p = (\bar{x}_p, \bar{y}_p),\ p = \overline{1, P}\}$, disposed in the ocean:

$$W_0(x, y; t)|_{M_p} = w_{0p}(t), \quad 0 \le t \le T.$$

Let us assume that the function $\varphi(x, y)$ is of the class $W_2^1(\Phi)$. Thus, the solution of problem (1)–(3) is now reduced to the following vector equation:

$$\mathcal{A}\langle\varphi(x,y)\rangle = \bar{W}_0(t); \tag{4}$$

where $\bar{W}_0(t) = (w_{01}(t), w_{02}(t), \ldots, w_{0p}(t))^T$. Following [3], we can consider $\mathcal{A} : W_2^1(\Phi) \to L_2((0,L) \times (0,T))$. As was done in [4], it is possible to prove that this operator is a compact one, so it does not possess a bounded inverse. The numerical solution to equation (4) includes its regularization using the SVD-decomposition of the operator $\mathcal{A}$ that results in the construction of r-solution (see [4]). The algorithm used in this way and substantiation of using this approach are described in detail in [5]. As it was shown in [4], it is possible to prove that the operator $\mathcal{A}$ is a compact one, so it does not possess a bounded inverse. This implies that any attempt to numerically solve equation (4) should be followed by some regularization procedure. In this paper, the regularization is performed by means of the truncated singular value decomposition (SVD) of the operator $\mathcal{A}$, which is realized as its narrowing on the subspace, spanned on a finite sample of r first right singular vectors (see [5]). The so-called "r-solution" is constructed ([2]) as a result of the numerical process.

Any compact operator possesses a singular system $\{s_j, \vec{u_j}, \vec{v_j}\}$, i.e., a sample of singular values $\{s_j \geq 0,\ s_1 \geq s_2 \geq \ldots\}$ and pairs of left $\{\vec{v_j}\}$ and right $\{\vec{u_j}\}$ singular vectors. The ill-conditionality of operator equation (4) with the compact operator $\mathcal{A}$ is due to the fact that $s_j \to 0$ when $j \to \infty$. So, if the right-hand side $W_0(\bar{x}_p, \bar{y}_p, t)$ is in some way perturbed, then its insignificant perturbations could result in rather a large perturbation of the solution. It should be noted that the perturbation of the operator also leads to the solution instability.

3. Discretization of the problem

Let us assume that the domain Ω is $[x_1, x_M] \times [y_1, y_N]$ rectangle. The unknown function $\varphi(x,y)$ will be sought for in the form of a series of spatial harmonics with unknown coefficients a_{mn}:

$$\varphi(x,y) = \sum_{m=1}^{M}\sum_{n=1}^{N} a_{mn} \sin\frac{m\pi(x - x_c)}{x_M - x_1} \sin\frac{n\pi(y - y_c)}{y_N - y_1}, \tag{5}$$

where x_c and y_c are central points of the corresponding intervals. System of equations (4) is approximated by the explicit-implicit finite difference scheme on the uniform rectangular grid based on the four-point pattern. The scheme is of second order of accuracy with respect to spatial variables and of first order with respect to time. The arrival of the wave to the coast is not considered in this work; it is assumed to arrive along the normal vector.

4. Numerical experiments

As a model of the initial water displacement, we used the displacement representing the bottom deformation due to the typical tsunamigenic earthquakes with reverse dip-slip or low-angle trust mechanisms. A series of calculations was carried out by the method proposed and were aimed at recovering the unknown function $\varphi(x, y)$. In all these calculations, the conditioning number of r-solution equals the value cond $= 10$.

The domain Ω is $[150, 250] \times [50, 150]$ rectangle, the center point of the tsunami source at $(150, 100)$ (all the sizes are assumed to be measured in kilometers). The observed data concerning the form of the arrived wave were simulated as a result of solution to direct problem (1)–(3), perturbed by the background noise, i.e., a high-frequency disturbance. All the experiments presented here were carried out with the disturbance rate of 5 % of a maximum amplitude of a signal over all the receivers. It is necessary to recognize that the results obtained strongly depend on the presence of disturbance due to the ill-conditionality of the problem. However, since a tsunami wave is considered to be of essentially lower frequency as compared to the background noise, it is reasonable enough to apply the frequency filtration of the observed signal.

In this work, the filtration is done by the method described in [6]. Figure 1 represents the bottom topography. Figure 2 shows natural logarithms of singular values of matrix A in logarithmic scale. A sharp decrease in the singular values when their number increases is typical for all the calculations. The influence of the conditioning number is essential, too. However, strong oscillations appear in the solution when the conditioning number increasing (cond > 10). This is typical for all ill-posed problems. The recovered tsunami wave form are represented in Figure 3 for seven receivers.

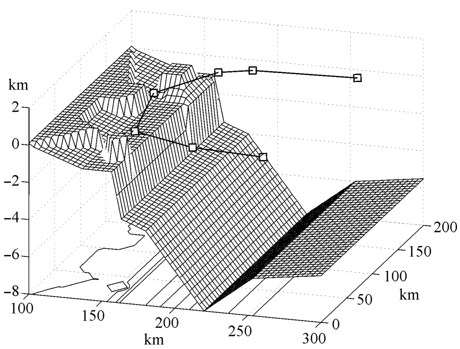

Figure 1

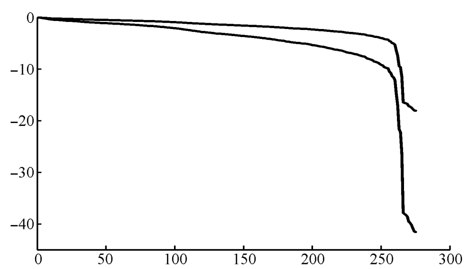

Figure 2

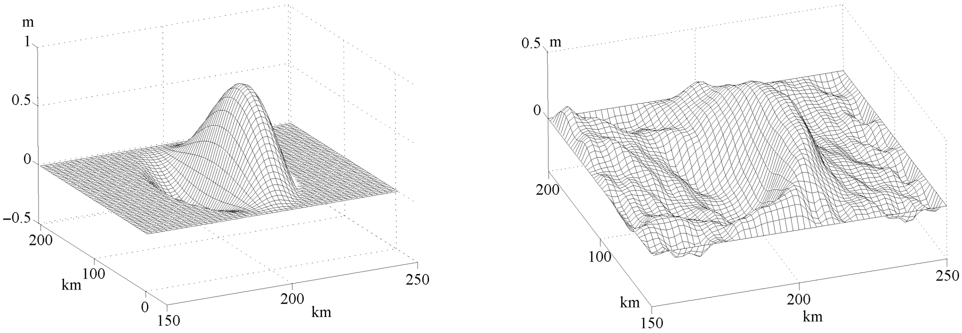

Figure 3. Initial and recovered tsunami waves

5. Conclusion

This paper represents the algorithm of the solution of the inverse problem dealing with an arbitrary bottom topography with inner and outer boundaries. We have shown that to attain a reasonable quality of the source restoration in this case we need, at least, seven records distributed over the space domain, and their azimuthal coverage plays the key role in obtaining the satisfactory results of inversion.

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